Application of the multiscale singular perturbation method to nonparaxial beam propagations in free space

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1. INTRODUCTION

It is well known that the paraxial wave equation gives an accurate description for wave beams near the axis as long as the beam width $w_0$ remains larger than the radiation wavelength $\lambda$ throughout the propagation. However, many light sources, such as some solid-state lasers or semiconductor injection lasers, generate wide-angle beams for which the paraxial approximation is not applicable. The importance of the physical model of focused laser fields has been clearly demonstrated in direct laser-cable. The importance of the physical model of focused laser beams for which the paraxial approximation is not applicable must be considered for semiconductor injection lasers, which generate wide-angle beams in the far-field region. For weakly nonparaxial beams, our correction solutions can be expressed in a very simple form that is proved to be exactly consistent with the solutions obtained by Cao and Deng [J. Opt. Soc. Am. A 15, 1144 (1998)]. In addition, the lowest-order correction to the paraxial approximation can be found to be in good agreement with the results of Lax et al. [Phys. Rev. A 11, 1965 (1975)] and Seshadri [J. Opt. Soc. Am. A 19, 2134 (2002)]. © 2007 Optical Society of America

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the vector electromagnetic field of Gaussian, flattened Gaussian, and annular Gaussian laser modes by using the angular spectrum method. The solutions obtained by Borghi and Santarsiero [19] and by Sepke and Umstadter [5–8] are suitable for linearly polarized beams. Here, we deal with the propagation problem of a general nonparaxial light beam whose initial field is of arbitrary symmetry by using the multiple-scale method, one of the singular perturbation methods [30], which was used to derive the evolution equations of optical pulses in the femtosecond regime [31,32]. This method can be used to deal with linearly polarized beams, radially polarized beams, etc.

The paper is organized as follows: Section 2 is devoted to a concise summary of base equations. In Section 3, the multiple-scale method of the singular perturbation methods is applied to dealing with the propagation problem of a general nonparaxial light beam. Accordingly, two new equations are derived for transverse and longitudinal electric fields even in the case of tightly focused nonparaxial laser beams. Some applications of this model are given in Section 4. A discussion is presented in Section 5, and simple numerical examples are shown to place in evidence the effectiveness of our solutions. This paper concludes in Section 6.

2. BASE EQUATIONS

The Maxwell’s equations for the monochromatic complex fields $\vec{E}$ and $\vec{B}$ varying as $\exp(-i\omega_0 t)$ are [9]

$$\nabla \times \vec{E} = i\omega_0 \vec{B}, \quad (1a)$$

$$\nabla \times \vec{B} = -\frac{i\omega_0}{c} \vec{E}, \quad (1b)$$

$$\nabla \cdot \vec{E} = 0, \quad (1c)$$

$$\nabla \cdot \vec{B} = 0, \quad (1d)$$

where $\omega_0$ is the frequency of a monochromatic pulse, $c = 1/\sqrt{\varepsilon_0 \mu_0}$ is the speed of light in vacuum, $\varepsilon_0$ is the vacuum permittivity, and $\mu_0$ is the vacuum magnetic permeability.

From Eqs. (1a)–(1d), it follows that

$$\nabla \times \nabla \times \vec{E} = (\omega_0/c)^2 \vec{E}, \quad (2a)$$

$$\nabla \cdot \vec{E} = 0. \quad (2b)$$

Assuming that the beam propagates along the $z$ axis and introducing $\vec{F}_\perp$ and $F_z$ as the slowly varying parts of the transverse electric field $\vec{E}_\perp$ and the longitudinal electric field $E_z$, we have $\nabla = \nabla_\perp + \vec{z} \partial_z$ and $\vec{E} = \exp(ikz)(\vec{F}_\perp + \vec{z} F_z)$ [9,19,20]; Eqs. (2a) and (2b) become

$$\nabla_\perp \left( \nabla_\perp \cdot \vec{F}_\perp + \frac{\partial F_z}{\partial z} + ik F_z \right) - \nabla_\perp^2 \vec{F}_\perp - \frac{\partial^2 \vec{F}_\perp}{\partial z^2} - 2ik \frac{\partial \vec{F}_\perp}{\partial z} = 0, \quad (3a)$$

$$\frac{\partial}{\partial z} \left( \nabla_\perp \cdot \vec{F}_\perp + ik \vec{F}_\perp \right) + ik \nabla_\perp \cdot \vec{F}_\perp - \nabla_\perp^2 F_z = k^2 F_z, \quad (3b)$$

where $k = \omega_0/c$ is the wave number. Now we introduce the dimensionless coordinates

$$\vec{r} = r/L, \quad \xi = z/L, \quad \rho = r/Lw_0, \quad \zeta = z/L.$$

where $r$ is the transverse coordinate vector, $\rho$ is the normalized transverse coordinate vector, $L = kw_0^2$ is the diffraction length, and $w_0$ is the beam width, which can be obtained by using the definition of the second-order moment. By substituting Eq. (4) into Eqs. (3a) and (3b), we obtain

$$\nabla_\perp \left( \sigma \nabla_\perp \cdot \vec{F}_\perp + \sigma^2 \frac{\partial F_z}{\partial \xi} + iF_z \right) - \sigma \nabla_\perp^2 \vec{F}_\perp - \sigma^3 \frac{\partial^2 \vec{F}_\perp}{\partial \xi^2} - 2i \sigma \frac{\partial \vec{F}_\perp}{\partial \xi} = 0, \quad (5a)$$

$$\sigma \frac{\partial}{\partial \xi} \left( \nabla_\perp \cdot \vec{F}_\perp \right) + i \sigma \nabla_\perp \cdot \vec{F}_\perp - \sigma^3 \nabla_\perp^2 F_z = F_z, \quad (5b)$$

where $\nabla_\perp = w_0 \nabla_\perp, \quad \sigma = w_0/L = (kw_0)^{-1}$ is a dimensionless perturbation parameter, $\vec{F}_\perp(\vec{r}_\perp, z) \rightarrow F_\perp(\rho, \zeta)$, and $F_z(\vec{r}_\perp, z) \rightarrow F_z(\rho, \zeta)$.

3. MULTISCALE SINGULAR PERTURBATION METHODS

In Eqs. (5a) and (5b), the independent variable $\zeta$ is transformed into several variables by [30,31]

$$\zeta_n = \sigma^n \zeta, \quad n = 0, 1, 2, \ldots, \quad (6)$$

where each $\zeta_n$ is an order of $\sigma$ smaller than the previous one. Thus the derivatives should be replaced by the expansion

$$\partial_\xi = \partial_\zeta + \sigma \partial_{\zeta_1} + \sigma^2 \partial_{\zeta_2} + \cdots. \quad (7)$$

At the same time, $\vec{F}_\perp$ and $F_z$ can be expanded in an asymptotic series

$$\vec{F}_\perp = \vec{F}_{\perp(0)} + \sigma \vec{F}_{\perp(1)} + \sigma^2 \vec{F}_{\perp(2)} + \cdots, \quad (8a)$$

$$F_z = F_{z(0)} + \sigma F_{z(1)} + \sigma^2 F_{z(2)} + \cdots. \quad (8b)$$

From Eqs. (A17) and (A18) in Appendix A, one can obtain the transverse electric field $\vec{E}_{\perp(2N)}$ after considering the $(2N)$th-order correction term and the longitudinal electric field $E_{z(2N+1)}$ after considering the $(2N+1)$th-order correction term,

$$\vec{E}_{\perp(2N)}(\rho, \zeta) = \frac{1}{4\pi^2} \int \vec{A}_0(\xi = 0) \exp(i\varphi_1) d^2 k_\parallel, \quad (9a)$$
where $\tilde{A}_0(\zeta=0)=\tilde{A}_0(\zeta=0,1,2,\ldots,2N=0)$, $\tilde{A}_0(\zeta=0,1,2,\ldots,2N=0)=\tilde{F}_0(0)$ is the Fourier transform of the initial field, $k_t$ is a dimensionless quantity given by the transverse component of the wave vector times $w_0$, $\varphi_1=k_t r + (1-m_1)\zeta/\sigma^2 - \omega_0 t$, $m_1=k_t^2/2+\Sigma_{n=2}^{N+1}(2n-3)!/(2n)!k_{2n}^2\sigma^{2n}$, $m_2=1+k_t^2/2+\Sigma_{n=2}^{N+1}(2n-1)!/(2n)!k_{2n}^2\sigma^{2n}$, $t_0=1$, and $n!=1$. Equations (9a) and (9b) can be changed into

$$E_{(2N+1)}(\tilde{r},\zeta)=\frac{1}{4\pi^2} \int \tilde{A}_0(\zeta=0) \exp(i \varphi_2) d^2k_t,$$

(10a)

$$E_{(2N+1)}(\tilde{r},\zeta)=-\frac{\sigma}{4\pi^2} \int \frac{k_t \cdot \tilde{A}_0(\zeta=0)}{\sqrt{1-k_t^2/\sigma^2}} \exp(i \varphi_2) d^2k_t,$$

(10b)

where $\varphi_2=k_t r + \sqrt{1-k_t^2/\sigma^2}/\sigma^2 - \omega_0 t$. Equations (10a) and (10b) are the exact solutions of Maxwell’s equations. Equation (10a) is consistent with Eq. (3) of Ref. [27], which was obtained by using the vector angular spectrum method.

It is worth mentioning that, for the nonparaxial laser, partial derivation of Eq. (A17) in Appendix A with respect to $\zeta$ gives the expression $\partial \tilde{E}_{(2N)}(\tilde{r},\zeta)/\partial \zeta$; inverse Fourier transforming this expression yields a new equation for the transverse component $\tilde{E}_{(2N)}(\tilde{r},\zeta)$ of the electric vector $\tilde{E}$ for the $(2N)$th-order correction term:

$$\frac{\partial \tilde{E}_{(2N)}(\tilde{r},\zeta)}{\partial \zeta} = \frac{i}{2} \nabla_t^2 \tilde{E}_{(2N)}(\tilde{r},\zeta) - i \sum_{n=2}^{N+1} (-1)^n \frac{(2n-3)!}{(2n)!} \sigma^{2n-2} (\nabla_t^2)^n \tilde{E}_{(2N)}(\tilde{r},\zeta).$$

(11)

By inverse Fourier transforming Eq. (A18), a new equation for the longitudinal component $E_{(2N+1)}(\tilde{r},\zeta)$ of the electric vector $\tilde{E}$ for the $(2N+1)$th-order correction term can be obtained,

$$E_{(2N+1)}(\tilde{r},\zeta)=i \sigma \left[ 1 + \frac{\sigma^2}{2} \nabla_t^2 + \sum_{m=2}^{N} \frac{(2m-1)!}{(2m)!} \right] \nabla_t \cdot \tilde{E}_{(2N)}(\tilde{r},\zeta).$$

(12)

Now let us examine one special case of Eqs. (11) and (12), for the case of $N=1$; i.e., for the lowest-order transverse and longitudinal correction cases, Eqs. (11) and (12) can be simplified to

$$\frac{\partial \tilde{E}_{(2)}(\tilde{r},\zeta)}{\partial \zeta} = \frac{i}{2} \nabla_t^2 \tilde{E}_{(2)}(\tilde{r},\zeta) - i \sigma \frac{\sigma^2}{8} \nabla_t^2 \tilde{E}_{(2)}(\tilde{r},\zeta),$$

(13a)

$$E_{(2)}(\tilde{r},\zeta) = i \sigma \left( 1 + \frac{\sigma^2}{8} \nabla_t^2 \right) \nabla_t \cdot \tilde{E}_{(2)}(\tilde{r},\zeta).$$

(13b)

Equation (13a) is the equation for the transverse component $\tilde{E}_{(2N)}(\tilde{r},\zeta)$ of the electric vector $\tilde{E}$ after considering the second-order correction term. If the electric vector $\tilde{E}$ is linearly polarized, Eq. (13a) is identical to the linear part of Eq. (13) in [33]. Equation (13b) is the equation for the longitudinal component $E_{(2N+1)}(\tilde{r},\zeta)$ of the electric vector $\tilde{E}$ after considering the third-order correction term. In our original scaled variables, Eqs. (11) and (12) yield

$$\frac{\partial \tilde{E}_{(2N)}(\tilde{r},\zeta)}{\partial \zeta} = \frac{i}{2k} \nabla_t^2 \tilde{E}_{(2N)}(\tilde{r},\zeta) - i \sum_{n=2}^{N+1} (-1)^n \frac{(2n-3)!}{(2n)!} \sigma^{2n-2} (\nabla_t^2)^n \tilde{E}_{(2N)}(\tilde{r},\zeta),$$

(14a)

$$E_{(2N+1)}(\tilde{r},\zeta) = \frac{i}{k} \left[ 1 + \frac{1}{2k} \nabla_t^2 + \sum_{m=2}^{N} \frac{(2m-1)!}{(2m)!} \sigma^{2m} (\nabla_t^2)^m \right] \nabla_t \cdot \tilde{E}_{(2N)}(\tilde{r},\zeta).$$

(14b)

Equations (14a) and (14b) suggest that the nonparaxial beam propagations of an arbitrary polarized electromagnetic wave can be expressed in terms of the paraxial wave equation plus an additional series of corrective terms.

4. SOME APPLICATIONS OF THIS MODEL

In this section, we will deal with the nonparaxial propagation of a linearly polarized field in free space by using our solutions, Eqs. (9a) and (9b).

We assume the field is linearly polarized in the $x$ direction so that $\tilde{F}_\perp=\tilde{x} \phi(\tilde{r}_\perp,z)$ and $\tilde{F}_z=\phi(\tilde{r}_\perp,z)$; at the $z=0$ plane, the Dirichlet boundary condition $\tilde{F}_\perp(\tilde{r}_\perp,0)$ is satisfied, where $\phi(\tilde{r}_\perp)$ is an arbitrary complex function. The asymmetric transverse electric and magnetic fields are defined as the electromagnetic field components that lack symmetry in the electric and magnetic fields, since $\tilde{F}_\perp$ is always equal to zero. From Eqs. (9a) and (9b), the asymmetric transverse electric field $F_{(2N+1)}^e(\tilde{r},\zeta)$ for the $(2N+1)$th-order correction term and the asymmetric longitudinal electric field $F_{(2N+1)}^z(\tilde{r},\zeta)$ for the $(2N+1)$th-order correction term can be expressed as

$$F_{z(2N+1)}^e(\tilde{r},\zeta) = \frac{1}{4\pi^2} \int \tilde{F}_0(\tilde{r},\zeta) d\varphi d\kappa,$$

(15a)

$$F_{z(2N+1)}^z(\tilde{r},\zeta) = -\frac{\sigma}{4\pi^2} \int k_\perp \tilde{F}_0 m_2 \exp(i \varphi_1) d\kappa d\eta,$$

(15b)
\( \tilde{\phi}_0 = \int_{-\infty}^{\infty} \phi_0 \exp[-i(k_\xi x + k_\eta y)] d\xi d\eta. \)  

When \( N \) becomes infinity, it is then easy to show that the asymmetric transverse electric field \( E_x^0 \) and the asymmetric longitudinal electric field \( E_z^0 \) can be obtained from Eqs. (15a) and (15b)

\[
E_x^0 = \frac{1}{4\pi^2} \int \int_{-\infty}^{\infty} \phi_0 \exp(i\varphi_2) dk_\xi dk_\eta, \tag{17a}
\]

\[
E_z^0 = -\frac{\sigma}{4\pi^2} \int \int_{-\infty}^{\infty} k_\xi \phi_0 \exp(i\varphi_2) \sqrt{1 - k_\xi^2\sigma^2} dk_\xi dk_\eta, \tag{17b}
\]

Equations (17a) and (17b) are consistent with Eqs. (2) and (3) of [1] and Eqs. (3) and (5) of [2]. Equations (17a) and (17b) can be changed into Eqs. (2) and (3) of [5] if the initial field is chosen as the general flattened Gaussian distribution. The other components of the fields can then be calculated by using the Maxwell equations. The symmetric transverse electric and magnetic fields are defined as the electric and the magnetic field components that are symmetric in the electric and magnetic fields. By imposing a boundary condition identical to \( \tilde{B}(r,z=0) \), repeating the same procedures, and averaging the results [2,5–8,24], the symmetric transverse electric and magnetic fields can be obtained:

\[
E_x(2N) = \frac{1}{8\pi^2} \int \int_{-\infty}^{\infty} \phi_0(2 - m_1 + \sigma^2k_\xi^2m_2)\exp(i\varphi_1) dk_\xi dk_\eta, \tag{18a}
\]

\[
E_y(2N) = -\frac{\sigma^2}{8\pi^2} \int \int_{-\infty}^{\infty} \phi_0 k_\xi k_\eta m_2 \exp(i\varphi_1) dk_\xi dk_\eta, \tag{18b}
\]

\[
E_z(2N+1) = -\frac{\sigma}{8\pi^2} \int \int_{-\infty}^{\infty} \phi_0 k_\xi (1 + m_2) \exp(i\varphi_1) dk_\xi dk_\eta, \tag{18c}
\]

\[
B_x(2N) = \frac{E_y(2N)}{c}, \tag{18d}
\]

\[
B_y(2N) = \frac{1}{8\pi^2c} \int \int_{-\infty}^{\infty} \phi_0(2 - m_1 + \sigma^2k_\xi^2m_2) \exp(i\varphi_1) dk_\xi dk_\eta, \tag{18e}
\]

\[
B_z(2N+1) = -\frac{\sigma}{8\pi^2c} \int \int_{-\infty}^{\infty} \phi_0 k_\xi (1 + m_2) \exp(i\varphi_1) dk_\xi dk_\eta. \tag{18f}
\]

When the parameter \( N \) becomes infinity, the electric and the magnetic field components are then

\[
E_x = \frac{1}{8\pi^2} \int \int_{-\infty}^{\infty} \left(1 + m_3 + \frac{k_\xi^2\sigma^2}{m_3}\right) \phi_0 \exp(i\varphi_2) dk_\xi dk_\eta, \tag{19a}
\]

\[
E_y = -\frac{\sigma^2}{8\pi^2} \int \int_{-\infty}^{\infty} k_\xi \phi_0 \exp(i\varphi_2) dk_\xi dk_\eta, \tag{19b}
\]

\[
E_z = -\frac{\sigma}{8\pi^2} \int \int_{-\infty}^{\infty} k_\xi \left(1 + \frac{m_1}{m_3}\right) \phi_0 \exp(i\varphi_2) dk_\xi dk_\eta, \tag{19c}
\]

\[
B_x = \frac{E_y}{c}, \tag{19d}
\]

\[
B_y = \frac{1}{8\pi^2c^2} \int \int_{-\infty}^{\infty} \left(1 + \frac{k_\xi^2\sigma^2}{m_3}\right) \phi_0 \exp(i\varphi_2) dk_\xi dk_\eta, \tag{19e}
\]

\[
B_z = -\frac{\sigma}{8\pi^2c^2} \int \int_{-\infty}^{\infty} k_\xi \left(1 + \frac{m_1}{m_3}\right) \phi_0 \exp(i\varphi_2) dk_\xi dk_\eta, \tag{19f}
\]

where \( m_3 = \sqrt{1 - k_\xi^2\sigma^2} \). Equations (19a)–(19f) are the exact integral solutions of the six electromagnetic field components. If we take the initial field as a Gaussian profile with a beam waist \( w_0 \), \( E_x(x,y,z=0) = E_0 \exp[-(x^2 + y^2)/w_0^2] \), and we can write its transverse Fourier transform as \( \tilde{\phi}_0 = \pi E_0 \exp[-(k_\xi^2 + k_\eta^2)/4] \); Eqs. (19a)–(19f) can be changed into

\[
E_x = \frac{E_0}{4} \left( I_1 + \frac{\sigma^2}{r_1^2} I_2 - \frac{\sigma^2}{r_1^2} \right), \tag{20a}
\]

\[
E_y = -\frac{\sigma^2 E_0}{4} \frac{\xi}{r_1^3}(r_1 I_3 - 2I_2), \tag{20b}
\]

\[
E_z = -\frac{i\sigma E_0}{4} \frac{\xi}{r_1^4} I_4, \tag{20c}
\]

\[
B_x = \frac{E_y}{c}, \tag{20d}
\]

\[
B_y = \frac{E_0}{4c} \left( I_1 + \frac{\sigma^2}{r_1^2} I_2 + \frac{\sigma^2}{r_1^2} \right), \tag{20e}
\]

\[
B_z = -\frac{i\sigma E_0}{4c} \frac{\eta}{r_1^4} I_4, \tag{20f}
\]

where

\[
I_1 = \int_{0}^{\infty} \left(1 + \sqrt{1 - k_\xi^2\sigma^2}\right) \exp \left(-\frac{k_\xi^2}{4} - i\varphi_3\right) J_0(k_\rho k_\eta) dk_\rho, \tag{21a}
\]

\[
I_2 = \int_{0}^{\infty} \frac{1}{\sqrt{1 - k_\xi^2\sigma^2}} \exp \left(-\frac{k_\xi^2}{4} - i\varphi_3\right) J_1(k_\rho k_\eta) dk_\rho, \tag{21b}
\]
\[ I_3 = \int_0^{1/\sigma^2} \frac{1}{\sqrt{1-\frac{k^2}{n^2}}} \exp\left(-\frac{k^2}{4} - i\varphi_3\right) J_n(kx) k^2 dx, \quad (21c) \]
\[ I_4 = \int_0^{1/\sigma^2} \left(1 + \frac{1}{\sqrt{1-\frac{k^2}{n^2}}} \right) \exp\left(-\frac{k^2}{4} - i\varphi_3\right) J_n(kx) k^2 dx, \quad (21d) \]

\[ \varphi = \varphi_0 - \frac{1}{\sqrt{1-\frac{k^2}{n^2}}} \] and \( J_n(x) \) is the \( n \)-th order Bessel function of the first kind. The real part of Eqs. (20a)–(20f) is consistent with Eqs. (7a)–(7f) of [2], which are obtained by using the angular spectrum representation of plane waves. If the initial field is chosen as a general flattened Gaussian distribution with a beam waist \( w_0 \), i.e., Eq. (1) of [5], the real part of Eqs. (20a)–(20c) can be transformed into Eqs. (7)–(9) of [5]. A similar approach, using Eqs. (9a) and (9b) as a guide, could presumably be used to deal with the nonparaxial propagation of the radially polarized light beams, the circularly polarized light beams, etc., as desired.

5. DISCUSSION

Now, let us provide a discussion in this section. In Eqs. (9) and (15), if we employ the power expansion \( \exp(u) = \sum_{m=0}^{\infty} u^m/m! \), \( u = -i\sum_{n=2}^{\infty} (2n-3)!/(2n)! (k^2 + k^2 y^2) \) and the inverse Fourier transform relation \( \mathcal{F}^{-1}[\mathcal{F}[f]] = 1/(4\pi^2) \rho \, \partial/ \partial r^2 \partial \rho / \partial y^2 \), the general correction field \( E_{x2(2N)} \) and \( E_{y2(2N-1)} \) of Eqs. (9a) and (9b) can be expressed as Eqs. (15) and (17) of [28], and the general correction field \( E_{x2(2N)}^a \) and \( E_{y2(2N-1)}^a \) of Eqs. (15a) and (15b) can be written in a very simple form that is exactly consistent with the result obtained by Cao and Deng [17]. In this case, Eqs. (18a)–(18f) can be reduced to Eqs. (16a)–(16f) of [2] if the initial field is taken as a Gaussian profile. In addition, the lowest-order correction to the paraxial approximation is found to be in good agreement with the result of Lax et al. [9] and Seshadri [18]. Obviously, \( \exp(u) \) can be expressed as the sum of the finite terms only if \( u \) is a small quantity. However, \( u \) is no longer a small quantity as the perturbation parameter \( \sigma \) increases. As a result, the convergence properties of the Lax’s series expansions will become worse and worse when the nonparaxial character of the beam is increased. In this case, the Lax’s method is not effective. Applying a different resummation scheme recently introduced by Weniger [29] and employing Lax’s method [9] preserves the good convergence properties of the correction terms even in case of extremely nonparaxial beams [19]. Recently, Sepke and Umstadter [5] derived the exact analytical solution for the vector electromagnetic field of Gaussian, flattened Gaussian, and annular Gaussian laser modes by using the angular spectrum method. The solutions obtained by Borghi and Santarsiero [19] and by Sepke and Umstadter [5] are suitable for the linearly polarized beams. Here, our solution can be used to deal with linearly polarized beams, radially polarized beams, etc.

It is well known that the group-velocity-dispersion-induced temporal effects [34] for the pulse and the diffraction-induced spatial effects [35] for the beam are a closely analogous pair. Moreover, both of them can be dealt with in Fourier analysis: the temporal frequency of the pulse is the analog of the spatial spectrum of the beam. Using the analogy of the optical beam in the space domain and the optical pulse in the time domain [34,36–38] and comparing Eqs. (14a) and (3.3.1) of [34], we introduce all terms in the last summation term of Eqs. (14a) as the higher-order diffraction terms. The higher the correction term’s order \( N \) is, the higher the diffraction term’s order is. The higher-order diffraction has little effect on the beam propagation in the far-field region because the beam expands in free space, however, the higher-order diffraction has a great effect on the beam propagation in the near-field region, as shown in Figs. 1–4. Equation (14a) shows that the odd-order diffraction terms disappear in the isotropic homogeneous medium. The higher-order diffraction terms can be used to analyze the linear broadening of the beams.

Making a comparison among Lax’s lowest-order solution, Borghi’s lowest-order solution, our lowest-order solution obtained from Eqs. (18a)–(18f), and the exact solution that can be obtained from Eqs. (20a)–(20f), we use the same input beam parameters, i.e., \( \phi(r, \theta) = \exp(-r^2/w_0^2) \), \( r^2 = x^2 + y^2 \) for numerical simulations. The normalized amplitudes of the transverse and the longitudinal electric

![Graph](https://via.placeholder.com/150)
field components are shown as functions of (a) $z/\lambda$; in Figs. 1(a) and 2(a) and as functions of $x/\lambda$ in Figs. 1(b) and 2(b), obtained for the four models for a Gaussian beam with a spot size $w_0/\lambda=1/2$ and the transverse coordinate $y=0$. The behavior of the relative error for the modulus of the transverse and the longitudinal electric fields components, i.e., the quantity $\varepsilon(r,z)=\frac{|\tilde{E}(r,z)|^{\text{exact}} - |\tilde{E}(r,z)|^{\text{correction}}}{|\tilde{E}(r,z)|^{\text{exact}}}$ is plotted in Figs. 3(a) and 4(a) as a function of $z/\lambda$ and as a function of $x/\lambda$ in Figs. 3(b) and 4(b) for a Gaussian beam with a spot size $w_0/\lambda=1/2$ and the transverse coordinate $y=0$. Since the beam expands in the transverse plane with the increase of $z$, the quantitative differences among the four solutions vanish in the far-field region, as presented in Figs. 1(a) and 2(a). It follows from Fig. 3(a) that the maximum value of the relative error of our lowest-order solution is not more than 5.5% if $z > 0.8\lambda$ for the transverse electric field. Figure 3(b) shows that the maximum value of the relative error of our lowest-order solution is not more than 4.0% if $x < 1.2\lambda$ for the transverse electric field. It can be seen from Fig. 4(a) that the maximum value of the relative error of our lowest-order solution is not more than 8.5% if $z > 0.54\lambda$ for the longitudinal electric field. Figure 4(b) shows that the maximum value of the relative error of our lowest-order solution is less than 8.6% if $x < 1.65\lambda$ for the longitudinal electric field. The results shown in Figs. 1–4 indicate that in these cases our lowest-order solution is more accurate than those obtained by Lax’s model and Borghi’s model in dealing with linearly polarized beams.

6. CONCLUSION
In summary, starting from the vector Maxwell’s equations, we have studied nonparaxial beam propagation in free space by applying the singular perturbation methods. Two new equations that yield higher-order corrections have been obtained for transverse and longitudinal electric fields even in the case of tightly focused nonparaxial laser beams. In the near-field region, our correction solutions yield an accurate description of the field for strongly nonparaxial beams that are characterized by large values of the perturbative parameter. In the far-field region, our correction solutions are consistent with all other correction results obtained by others because the beam expands in the transverse plane with the increase of $z$. For weakly nonparaxial beams, our correction solutions can be written in a very simple form that is proved to be exactly consistent with the solutions obtained by Cao and Deng [17]. As well, the lowest-order correction to the paraxial approximation is found to be consistent with the result of Lax et al. [9] and Seshadri [18]. In this case, the six elec-
tromagnetic field components can be reduced to those results obtained by Quesnel and Mora [2] if the initial field is taken as a Gaussian profile. In addition, this treatment allows us to deal with the nonparaxial propagation of circularly polarized light beams, radially polarized light beams, etc., in the presence of arbitrary symmetric boundary conditions.

APPENDIX A

Substituting Eqs. (7) and (8) into Eqs. (5a) and (5b) and equating the coefficients of each power of \( \sigma \), one obtains the following equations of different orders:

\[
F_{(0)}(\rho, \zeta) = 0, \quad (A1)
\]

\[
\nabla^2 \tilde{F}_{(0)} + 2i \frac{\partial \tilde{F}_{(0)}}{\partial \zeta_0} = 0, \quad (A2a)
\]

\[
F_{(1)} = i \nabla \cdot \tilde{F}_{(1)}, \quad (A2b)
\]

\[
\nabla^2 \tilde{F}_{(1)} + 2 \left( \frac{\partial \tilde{F}_{(1)}}{\partial \zeta_0} + \frac{\partial \tilde{F}_{(0)}}{\partial \zeta_1} \right) = 0, \quad (A3a)
\]

\[
F_{(2)} = i \nabla \cdot \tilde{F}_{(2)}, \quad (A3b)
\]

\[
\nabla^2 \tilde{F}_{(2)} + 2i \left( \frac{\partial \tilde{F}_{(2)}}{\partial \zeta_0} + \frac{\partial \tilde{F}_{(1)}}{\partial \zeta_1} + \frac{\partial \tilde{F}_{(0)}}{\partial \zeta_2} \right) = - \frac{\partial^2 \tilde{F}_{(0)}}{\partial \zeta_0^2}, \quad (A4a)
\]

\[
F_{(3)} = i \nabla \cdot \tilde{F}_{(3)} + i \frac{\partial \tilde{F}_{(1)}}{\partial \zeta_0}, \quad (A4b)
\]

\[
\nabla^2 \tilde{F}_{(N)} + 2N \sum_{j=0}^{N-1} \frac{\partial \tilde{F}_{(N-j)}}{\partial \zeta_j} = - \sum_{j=0}^{N-2} \frac{\partial^2 \tilde{F}_{(N-2-j)}}{\partial \zeta_j^2} - 2 \sum_{m=0}^{N-1} \sum_{j=0}^{N-1-j} \left( \frac{\partial^2 \tilde{F}_{(m)}}{\partial \zeta_m \partial \zeta_{N-2-m-j}} \right), \quad (A5a)
\]

\[
F_{(N+1)} = i \nabla \cdot \tilde{F}_{(N)} + i \sum_{j=0}^{N-1} \frac{\partial \tilde{F}_{(N-j)}}{\partial \zeta_{N-1-j}} \quad (N > 2), \quad (A5b)
\]

where square brackets indicate taking the integer. The problem of solving Eqs. (5a) and (5b) for a disturbance \( \tilde{F}(\rho, \zeta) \) becomes solving the set of differential equations (A2a)–(A5a) with the initial conditions

\[
\tilde{F}_{(0)}(\rho, \zeta) = \tilde{F}_{(0)}(\rho, \zeta), \quad \tilde{F}_{(N)}(\rho, \zeta) = 0, \quad N \geq 1, \quad (A6a)
\]

\[
F_{(0)}(\rho, 0) = F_{(N)}(\rho, 0) = 0, \quad N \geq 1. \quad (A6b)
\]

Fourier transforming Eqs. (A2a) and (A2b), it is easy to obtain

\[
\tilde{F}_{(0)}(\rho, \zeta) = \tilde{A}_0(\zeta_0 = 0, \xi_1, \ldots) \exp \left( - \frac{i k_0^2 \zeta_0}{2} \right), \quad (A7a)
\]

\[
\tilde{F}_{(1)}(\rho, \zeta) = - \frac{1}{2} \tilde{F}_{(0)}(\rho, \zeta), \quad (A7b)
\]

where

\[
\tilde{F}_{(0)}(\rho, \zeta) = \int \tilde{F}_{(0)}(\rho, \zeta) \exp(- i \hat{k}_t \cdot \rho) d^2 \rho, \quad (A8a)
\]

\[
\tilde{F}_{(1)}(\rho, \zeta) = \frac{1}{4 \pi^2} \int \tilde{F}_{(0)}(\rho, \zeta) \exp(i \hat{k}_t \cdot \rho) d^2 \rho, \quad (A8b)
\]

\( k_t \) is a dimensionless quantity, and it is given by the transverse component of the wave vector times \( w_0 \cdot \hat{A}_0(\zeta_0 = 0, \xi_1, \ldots) \) is determined by the higher-order perturbed equations, and

\[
\tilde{F}_{(1)}(\rho, \zeta) = \int F_{(1)}(\rho, \zeta) \exp(- i \hat{k}_t \cdot \rho) d^2 \rho, \quad (A9a)
\]

\[
\tilde{F}_{(1)}(\rho, \zeta) = \frac{1}{4 \pi^2} \int \tilde{F}_{(1)}(\rho, \zeta) \exp(i \hat{k}_t \cdot \rho) d^2 \rho, \quad (A9b)
\]

From Eqs. (A3a) and (A3b), one can obtain
When comparing Eq. (A7a) with Eq. (A10a), one can find that \( \sigma \tilde{F}_{(l)}(\tilde{k}, \xi) \) is a small perturbation of \( \tilde{F}_{(0)}(\tilde{k}, \xi) \) only when \( \sigma \xi_0 = \sigma \xi \) is a small perturbation, and \( \xi_0 \partial \tilde{A}_0 / \partial \xi_1 \times \exp(-i k_t^2 \xi_0^2/2) \) must be zero in order to obtain a higher-order effective expansion expression. For example,

\[
\frac{\partial \tilde{A}_0}{\partial \xi_1} = 0. \tag{A11}
\]

Equation (A11) shows that \( \tilde{A}_0 \) is independent of \( \xi_1 \) explicitly. From Eqs. (A10a) and (A10b), it is easy to derive that \( \tilde{F}_{(1)}(\tilde{k}, \xi) = 0, \tilde{F}_{(2)}(\tilde{k}, \xi) = 0. \) Then, repeating the same procedure, the solutions of Eqs. (A4a) and (A4b) are

\[
\tilde{F}_{(3)}(\tilde{k}, \xi) = -\tilde{k}_t \tilde{F}_{(2)}(\tilde{k}, \xi) + i \frac{\partial \tilde{F}_{(1)}(\tilde{k}, \xi)}{\partial \xi_0}. \tag{A12b}
\]

Draw a comparison between Eq. (A7a) and Eq. (A12a): it is easy to find that \( \sigma^2 \tilde{F}_{(2)}(\tilde{k}, \xi) \) is a small perturbation of \( \tilde{F}_{(0)}(\tilde{k}, \xi) \) only when \( \sigma^2 \xi_0 = \sigma^2 \xi \) is a small perturbation, and \( \xi_0 (\partial \tilde{A}_0 / \partial \xi_2 + i k_t^2 \tilde{A}_0 / 8) \exp(-i k_t^2 \xi_0^2/2) \) must be zero in order to obtain a higher-order effective expansion expression. Say,

\[
\frac{\partial \tilde{A}_0}{\partial \xi_2} + \frac{i k_t^4}{8} \tilde{A}_0 = 0. \tag{A13}
\]

From Eq. (A13), it follows that

\[
\tilde{A}_0(\xi_0, \xi_1, \ldots) = \tilde{A}_0(\xi_{0,1,2} = 0, \xi_3, \ldots) \exp \left( -\frac{i k_t^4 \xi_2}{8} \right). \tag{A14}
\]

From Eqs. (A5) and (A7)–(A14), one obtains

\[
\tilde{F}_{(3)}(\tilde{k}, \xi) = 0, \tag{A15a}
\]

\[
\tilde{F}_{(4)}(\tilde{k}, \xi) = 0, \tag{A15b}
\]

\[
\ldots
\]

After a straightforward calculation for the (2N)th-order correction terms, one obtains

\[
\tilde{F}_{(2N)}(\tilde{k}, \xi) = \tilde{A}_0(\xi_{0,1,2N} = 0) \times \exp \left[ -i \sum_{n=2}^{N+1} (2n-3)! k_t^{2n} \xi_0^{2n-2} \right]. \tag{A16}
\]

where \( \bar{A}_0(\xi_{0,1,2N} = 0) = \tilde{F}_{(0)}(\tilde{k}, \xi = 0) \) is the Fourier transform of the initial field. From Eqs. (A7a), (A10a), (A12a), (A15a), and (A16), the Fourier transforming solution \( \tilde{E}_{(2N)}(\tilde{k}, \xi) \) of any transverse component of the electric vector \( \tilde{E} \) after considering the (2N)th-order correction term can be written in the form

\[
\tilde{E}_{(2N)}(\tilde{k}, \xi) = \tilde{A}_0(\xi_{0,1,2N} = 0) \exp \left( -\frac{im \xi_0}{\sigma^2} \right), \tag{A17}
\]

where \( m_1 = k_t^2 \sigma^2 / 2 + \sum_{n=2}^{N+1} (2n-3)! / (2n)! k_t^{2n} \sigma^{2n} \), \( 0! = 1 \) and \( 1! = 1 \); from Eqs. (A7b), (A10b), (A12b), (A15b), and (A17), the Fourier transforming solution \( \tilde{E}_{(2N+1)}(\tilde{k}, \xi) \) of any longitudinal component of the electric vector \( \tilde{E} \) after considering the (2N+1)th-order correction term can be given by

\[
\tilde{E}_{(2N+1)}(\tilde{k}, \xi) = -m_2 \xi_0 \tilde{F}_{(2N)}(\tilde{k}, \xi), \tag{A18}
\]

where \( m_2 = 1 + k_t^2 \sigma^2 / 2 + \sum_{n=2}^{N} (2n-1)! / (2n)! k_t^{2n} \sigma^{2n} \).

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