

# Ruelle operator theorem for non-expansive systems

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*Abstract.* The Ruelle operator theorem has been studied extensively both in dynamical systems and iterated function systems. In this paper we study the Ruelle operator theorem for non-expansive systems. Our theorems give some sufficient conditions for the Ruelle operator theorem to be held for a non-expansive system.

## 1. Introduction

Ruelle introduced a convergence theorem to study the equilibrium state of an infinite one-dimensional lattice gas in his famous paper [22]. Bowen [3] further set up the theorem as the convergence of powers of a Ruelle operator on the space of continuous functions on a symbolic space. More precisely, let

$$\Sigma = \{1, \dots, N\}^{\mathbb{N}} = \{\omega = i_0 i_1 \cdots i_{n-1} \cdots \mid i_{n-1} \in \{1, \dots, N\}, n = 1, 2, \dots\}$$

be the one-sided symbolic space and

$$\sigma : \omega = i_0 i_1 \cdots i_{n-1} \cdots \rightarrow \sigma(\omega) = i_1 \cdots i_{n-1} \cdots$$

be the left shift of  $\Sigma$ . Then  $(\Sigma, \sigma)$  is called a symbolic system. Let  $\phi$  be a Hölder continuous function on  $\Sigma$  (a potential). Let  $C(\Sigma)$  be the space of all continuous functions on  $\Sigma$ . The *Ruelle operator* is defined as

$$\mathcal{T}f(x) = \sum_{y \in \sigma^{-1}(x)} e^{\phi(y)} f(y), \quad f \in C(\Sigma). \quad (1.1)$$

It is a positive operator, that is,  $\mathcal{T}f > 0$  whenever  $f > 0$ .

Let  $\varrho$  be the spectral radius of the operator

$$\mathcal{T} : C(\Sigma) \rightarrow C(\Sigma).$$

It is known that  $\varrho$  is the unique positive simple maximal eigenvalue of  $\mathcal{T}$  acting on the space of all Hölder continuous functions on  $\Sigma$  (see, for example, [12]). It was then proved that  $\mathcal{T}$  has a unique positive eigenfunction  $h \in C(\Sigma)$  and a unique probability eigenmeasure  $\mu \in C^*(\Sigma)$  corresponding to the eigenvalue  $\varrho > 0$  (see, for example, [3]). Moreover, for any  $f \in C(\Sigma)$ ,  $\varrho^{-n}\mathcal{T}^n(f)$  converges uniformly to a constant multiple of  $h$ . This is called the Ruelle operator theorem. In this theorem,  $\sigma : \Sigma \rightarrow \Sigma$  is an expanding dynamical system. More general results about the Ruelle operator theorem for expanding dynamical systems and contractive iterated function systems (IFSs) have been also obtained. We give a partial list in the literature [5–8, 25, 26].

Recently a parabolic system has drawn much attention from people who are interested in the Ruelle operator theorem (refer to [1, 16, 17, 21, 24, 27–30]). However, in this case, it is known that the bounded eigenfunction of the spectral radius  $\varrho$  of  $T$  may not exist [14], and even if the eigenfunction exists,  $\varrho$  may not be an isolated point of the spectrum [2]. The results obtained to date are far from satisfactory. The study of such a system remains a challenging problem. Lau and Ye studied the Ruelle operator theorem for a non-expansive system in a recent paper [15]. In the present paper we continue to study the above-mentioned problem for a non-expansive system. In the paper [15], one requirement is that one of the iterations of the IFS must be strictly contractive. It is important to remove this requirement because many examples of IFSs will not satisfy this requirement. In this paper, we remove this requirement, which is a major improvement.

Our IFS  $\{w_j\}_{j=1}^m$  in this paper is weakly contractive as defined by

$$\alpha_{w_j}(t) := \sup_{|x-y| \leq t} |w_j(x) - w_j(y)| < t \quad \text{for all } t > 0, \quad 1 \leq j \leq m$$

or, more generally, non-expansive as defined by

$$|w_j(x) - w_j(y)| \leq |x - y|, \quad 1 \leq j \leq m.$$

For the weakly contractive case, the invariant compact set  $K$  exists as in the contractive case [9]. For the non-expansive case we can take the smallest compact invariant  $K$  (see Proposition 2.2 for the additional assumption). With each  $w_j$ , we associate a positive continuous function  $p_j$  as a weight function (or potential function). We can set up the Ruelle operator as follows on the space  $C(K)$  of continuous functions on  $K$ :

$$T(f)(x) = \sum_{j=1}^m p_j(x) f(w_j(x)), \quad f \in C(K). \quad (1.2)$$

Let  $\varrho$  still be the spectral radius of the operator

$$T : C(K) \rightarrow C(K).$$

*Definition 1.1.* We call  $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$  a non-expansive system, if all maps  $w_j$  are non-expansive and all potentials  $p_j(x)$  are Dini continuous on  $X$ .

The main result in this paper which we are particularly interested in is the following.

**THEOREM 1.2.** (Main theorem) *Let  $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$  be a non-expansive system. Suppose that*

$$\sup_{x \in K} \sum_{j=1}^m p_j(x) \sup_{y \neq x} \frac{|w_j(x) - w_j(y)|}{|x - y|} < \varrho.$$

*Then the Ruelle operator theorem holds for this non-expansive system.*

We prove a more general result (Theorem 4.5) in §4. Actually, the above theorem is a special case of this more general result. The results in this paper extend the results in [15]. However, as we pointed out before, it is a non-trivial generalization: in [15], one of the iterations of the IFS must be strictly contractive and this requirement is removed in this paper. It is an important improvement. Therefore, we provide a Ruelle operator theorem for a system to which each branch contains an indifferent fixed point (see Remark 4.6 and Example 4.7 at the end of this paper).

In practice, it is difficult to calculate the spectral radius  $\varrho$  of  $T$ . However, since  $T$  is a positive operator, we have that  $\|T^n 1\| = \|T^n\|$  and

$$\varrho = \lim_n \|T^n\|^{1/n} = \lim_n \|T^n 1\|^{1/n}.$$

Therefore, from the formula of  $T^n 1$  (see the formula before Proposition 2.3 in §2), a simple but useful lower bound of  $\varrho$  is

$$\min_{x \in K} \sum_{j=1}^m p_j(x) \leq \varrho. \quad (1.3)$$

If we replace the  $\varrho$  by  $\min_{x \in K} \sum_{j=1}^m p_j(x)$  in the above theorem, we can have a simple checkable sufficient condition.

**COROLLARY 1.3.** *Let  $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$  be a non-expansive system. If*

$$\sup_{x \in K} \sum_{j=1}^m p_j(x) \cdot \sup_{y \neq x} \frac{|w_j(x) - w_j(y)|}{|x - y|} < \min_{x \in K} \sum_{j=1}^m p_j(x),$$

*then the Ruelle operator theorem holds for this non-expansive system.*

It is obvious that if  $\{w_j\}_{j=1}^m$  is a contractive IFS, then the conditions in the above theorem and the above corollary and Theorem 4.5 latter are trivially satisfied. The condition of the above theorem is similar to the *average contractive* condition of Barnsley *et al* [2] where they assumed that  $\sum_{j=1}^m p_j(x) = 1$ , hence  $\varrho = 1$ . It is also similar to the condition given by Hennion [10], but he considered the case where each  $p_j$  is a Lipschitz continuous function on  $X$ . Regarding  $T$  as defined on the Lipschitz continuous space, he showed that the essential spectral radius  $\varrho_e(T)$  is strictly less than the spectral radius  $\varrho(T)$ , and then the Ruelle operator theorem holds. Furthermore, a general formula for the essential spectral radius  $\varrho_e(T)$  for a general  $C^\alpha$  IFS or Zygmund IFS can be found in [1]. Using this formula, one can check whether the essential spectral radius  $\varrho_e(T)$  is strictly less than the spectral radius  $\varrho(T)$ , and then check the Ruelle operator theorem. However,

these methods do not work for the weakly contractive (or, more generally, non-expansive) case. The reason is that, in this case,  $\varrho(T)$  is not an isolated point of the spectrum, and  $\varrho(T) = \varrho_e(T)$  (refer to [20, 23]). Note that [13, 19] contain some results showing that  $\varrho(T) = \varrho_e(T)$  is held under some weaker smoothness assumptions (for example, Dini continuity) even in the contractive case. Therefore, the result in this paper provides a new method to check the Ruelle operator theorem for some weakly contractive (or, more generally, non-expansive) IFS.

We note that most people study an IFS on some Euclidean space. This is because the existence of a compact invariant subset  $K$  for a contractive or a weakly contractive IFS needs the structure of an Euclidean space (see [9, 11]). However, arguments in the proofs of this paper only need to assume that  $K$  is a compact Hausdorff metric space, in particular, when we study a dynamical system  $\sigma : K \rightarrow K$  defined on a compact Hausdorff metric space  $K$  satisfying a certain Markov property. More precisely,  $K = \bigcup_{j=1}^m K_j$  is the union of finitely many pairwise disjoint compact subsets  $\{K_j\}_{j=1}^m$  such that each  $\sigma : K_j \rightarrow K$  is a homeomorphism. Then let  $w_j$  be the inverse of  $\sigma : K_j \rightarrow K$  for each  $1 \leq j \leq m$  and define  $(K, \{w_j\}_{j=1}^m)$ . It can be thought of as an IFS as well. Our results in this paper are true for such a non-expansive IFS  $(K, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ .

This paper is organized as follows. In §2, we present some elementary facts about the Ruelle operator and prove Proposition 2.2. We introduce the Ruelle operator theorem in §3 and set up the basic criteria for the assertion of the Ruelle operator theorem. We prove our main result in §4.

## 2. Preliminaries

Consider the system

$$(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m),$$

where  $X \subseteq \mathbb{R}^d$  is a compact subset,  $w_j : X \rightarrow X$ ,  $1 \leq j \leq m$ , are continuous maps and the  $p_j(x)$ ,  $1 \leq j \leq m$ , are positive functions on  $X$  (they are called weights or potentials associated with  $w_j$ ). We say that a map  $w : X \rightarrow X$  is *non-expansive* if

$$|w(x) - w(y)| \leq |x - y| \quad \text{for all } x, y \in X;$$

*weakly contractive* if

$$\alpha_w(t) := \sup_{|x-y| \leq t} |w(x) - w(y)| < t \quad \text{for all } t > 0.$$

It is clear that contractivity implies weak contractivity which also implies non-expansiveness. A simple non-trivial example of a weakly contractive map is  $w(x) = x/(1+x)$  on  $[0, 1]$ . We call

$$(X, \{w_j\}_{j=1}^m)$$

a weakly contractive IFS if all  $w_j$ ,  $1 \leq j \leq m$ , are weakly contractive and a non-expansive IFS if all  $w_j$ ,  $1 \leq j \leq m$ , are non-expansive.

A function  $p(x)$  defined on  $X$  is called Dini continuous if

$$\int_0^1 \frac{\alpha_p(t)}{t} dt < \infty$$

where

$$\alpha_p(t) = \sup_{|x-y| \leq t} |p(x) - p(y)|.$$

For any  $0 < \theta < 1$ , we consider the following summation

$$S_{\theta,p} = \sum_{n=0}^{\infty} \alpha_p(\theta^n a)$$

where  $a$  is the diameter of  $X$ . Then, that  $p(x)$  is Dini continuous is equivalent to saying that  $S_{\theta,p}$  is summable, that is,

$$S_{\theta,p} < \infty.$$

Throughout the paper, we always assume that the potentials  $p_j$  are positive Dini continuous functions on  $X$ . If  $\{w_j\}_{j=1}^m$  is a contractive IFS with the contractive constant  $0 < \tau < 1$ , that is,

$$\sup_{x \neq y \in X} \frac{|w_j(x) - w_j(y)|}{|x - y|} \leq \tau,$$

then the Dini condition on all  $p_j$  can be replaced by the summable condition

$$\max_{1 \leq j \leq m} S_{\tau,p_j} < \infty.$$

However, if  $\{w_j\}_{j=1}^m$  is a non-expansive IFS, we do not have such a constant  $0 < \theta < 1$ . Thus, the Dini condition on potentials is different from the summable condition on potentials. The methods presented before (see e.g. [1, 5, 7, 8, 10, 15–17, 21, 25–27]) do not work for the system considered in this paper. We need to find a sharper method to prove the Ruelle operator theorem under our sufficient conditions.

*Definition 2.1.* Let  $p_j$ ,  $1 \leq j \leq m$ , be positive Dini continuous functions on  $X$ . We call

$$(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m),$$

a non-expansive (or weakly contractive) system, if the IFS  $(X, \{w_j\}_{j=1}^m)$  is non-expansive (or weakly contractive).

Hata studied the invariant sets of the weakly contractive IFS on  $X \subseteq \mathbb{R}^d$  in [9]. By using the existence of fixed points for the weakly contractive maps, he showed the existence of a unique non-empty compact  $K \subseteq X$  invariant under  $\{w_j\}_{j=1}^m$ , i.e.

$$K = \bigcup_{j=1}^m w_j(K).$$

For  $J = (j_1 j_2 \cdots j_n)$ ,  $1 \leq j_i \leq m$ , let

$$w_J(x) = w_{j_1} \circ w_{j_2} \circ \cdots \circ w_{j_n}(x).$$

Then

$$\lim_{|J| \rightarrow \infty} |w_J(K)| = 0$$

and

$$K = \bigcap_{n=1}^{\infty} \bigcup_{|J|=n} w_J(K).$$

However, for a general IFS, an invariant set may not be unique. However, we have the following result.

**PROPOSITION 2.2.** *Suppose that  $\{w_j\}_{j=1}^m$  is a non-expansive IFS on the compact subset  $X$  with at least one  $w_j$  being weakly contractive. Then there exists a unique smallest nonempty compact set  $K$  such that*

$$K = \bigcup_{j=1}^m w_j(K).$$

*Moreover, for any  $x \in K$ , the closure of  $\{w_J(x) \mid |J| = n, n \in \mathbb{N}\}$  is  $K$ , i.e.*

$$\overline{\{w_J(x) \mid |J| = n, n \in \mathbb{N}\}} = K.$$

*Proof.* Let

$$\mathcal{F} = \left\{ F \mid \bigcup_{j=1}^m w_j(F) \subseteq F \right\}.$$

By using the standard Zorn's lemma argument, there exists a minimal compact subset  $K$  such that

$$K = \bigcup_{j=1}^m w_j(K).$$

To show that such  $K$  is unique, we assume without loss of generality that  $w_1$  is weakly contractive. If  $J_n = (1 \cdots 1)$  ( $n$ -times), then  $\lim_{n \rightarrow \infty} |w_{J_n}(X)| = 0$ . Let  $K'$  be another minimal compact invariant set and let  $x \in K$  and  $y \in K'$ . Then

$$\lim_{n \rightarrow \infty} w_{J_n}(x) = \lim_{n \rightarrow \infty} w_{J_n}(y) \in K \cap K'.$$

Hence,

$$K \cap K' \neq \emptyset,$$

and  $w_j(K \cap K') \subseteq K \cap K'$ . From the minimality of  $K$ , we conclude that  $K = K'$ , and deduce the last statement of the proposition.  $\square$

Throughout the paper we consider either weakly contractive IFSs or the IFS in Proposition 2.2. Hence, the set  $K$  is uniquely defined. Furthermore, we can assume without loss of generality that the diameter

$$|K| = \sup\{|x - y| \mid x, y \in K\} = 1.$$

Let  $C(K)$  be the space of all continuous functions on  $K$ . For such a system, we define an operator  $T : C(K) \rightarrow C(K)$  by

$$Tf(x) = \sum_{j=1}^m p_j(x) f(w_j(x)).$$

We call  $T$  the *Ruelle operator* associated with the non-expansive system

$$(K, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m).$$

The dual operator  $T^*$  on the measure space  $M(K)$  is given by

$$T^*\mu(E) = \sum_{j=1}^m \int_{w_j^{-1}(E)} p_j(x) d\mu(x) \quad \text{for any Borel set } E \subseteq K$$

(see e.g. [2]).

For  $J = (j_1 j_2 \cdots j_n)$ ,  $1 \leq j_i \leq m$ , define

$$w_J = w_{j_1} \circ w_{j_2} \circ \cdots \circ w_{j_n}$$

and

$$p_{w_J}(x) = p_{j_1}(w_{j_2} \circ w_{j_3} \circ \cdots \circ w_{j_n}(x)) \cdots p_{j_{n-1}}(w_{j_n}(x)) p_{j_n}(x).$$

Then,

$$T^n f(x) = \sum_{|J|=n} p_{w_J}(x) f(w_J x).$$

Let  $\varrho = \varrho(T)$  be the spectral radius of  $T$ . Since  $T$  is a positive operator, we have that  $\|T^n 1\| = \|T^n\|$  and

$$\varrho = \lim_n \|T^n\|^{1/n} = \lim_n \|T^n 1\|^{1/n}.$$

**PROPOSITION 2.3.** *Let  $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$  be a non-expansive system with at least one weakly contractive  $w_j$ . Let  $T$  be the Ruelle operator on  $C(K)$ . Then:*

- (i)  $\min_{x \in K} \varrho^{-n} T^n 1(x) \leq 1 \leq \max_{x \in K} \varrho^{-n} T^n 1(x)$  for all  $n > 0$ ;
- (ii) if there exist  $\lambda > 0$  and  $0 < h \in C(K)$  such that  $Th = \lambda h$ , then  $\lambda = \varrho$  and there exist  $A, B > 0$  such that

$$A \leq \varrho^{-n} T^n 1(x) \leq B \quad \text{for all } n > 0.$$

*Proof.* We prove the second inequality of (i), the first inequality is similar. Suppose that it is not true, then there exists an integer  $k$  such that  $\|T^k 1\| < \varrho^k$ . Hence,

$$\varrho = (\varrho(T^k))^{1/k} \leq \|T^k\|^{1/k} = \|T^k 1\|^{1/k} < \varrho,$$

which is a contradiction. To prove the second assertion we let  $a_1 = \min_{x \in K} h(x)$ ,  $a_2 = \max_{x \in K} h(x)$ . Then

$$0 < \frac{a_1}{a_2} \leq \frac{h(x)}{a_2} = \frac{\lambda^{-n}}{a_2} T^n h(x) \leq \lambda^{-n} T^n 1(x) = \lambda^{-n} \|T^n\|.$$

Similarly we can show that  $\lambda^{-n} \|T^n\| \leq a_2/a_1$ . Hence,  $\varrho = \lim_{n \rightarrow \infty} \|T^n\|^{1/n} = \lambda$ .  $\square$

We call the operator  $T : C(K) \rightarrow C(K)$  *irreducible* (see [15]) if for any non-trivial, non-negative  $f \in C(K)$  and for any  $x \in K$ , there exists an integer  $n > 0$  such that  $T^n f(x) > 0$ .

PROPOSITION 2.4. *Let  $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$  be a non-expansive system with at least one weakly contractive  $w_j$ . Then the Ruelle operator  $T$  is irreducible and*

$$\dim\{h \in C(K) \mid Th = \varrho h, h \geq 0\} \leq 1.$$

*If  $h \geq 0$  is a  $\varrho$ -eigenfunction of  $T$ , then  $h > 0$ .*

*Proof.* The proof can be found in [15]. We include the details here for the sake of completeness. For any given  $f \in C(K)$  with  $f \geq 0$  and  $f \not\equiv 0$ , let  $V = \{x \in K \mid f(x) > 0\}$ . For any  $x \in K$ , by Proposition 2.3, there exists a multi-index  $J_0$  such that  $w_{J_0}(x) \in V$ . Let  $n_0 = |J_0|$ , then

$$T^{n_0} f(x) = \sum_{|J|=n_0} p_{w_J}(x) f(w_J x) \geq p_{w_{J_0}}(x) f(w_{J_0} x) > 0.$$

This proves that  $T$  is irreducible.

For the dimension of the eigensubspace, we suppose that there exist two independent strictly positive  $\varrho$ -eigenfunctions  $h_1, h_2 \in C(K)$ . Without loss of generality we assume that  $0 < h_1 \leq h_2$  and  $h_1(x_0) = h_2(x_0)$  for some  $x_0 \in K$ . Then  $h = h_2 - h_1$  ( $\geq 0$ ) is a  $\varrho$ -eigenfunction of  $T$  and  $h(x_0) = 0$ . It follows that  $T^n h(x_0) = \varrho^n h(x_0) = 0$ , which contradicts to the irreducibility of  $T$ . Hence, the dimension of the  $\varrho$ -eigensubspace is at most one.

The strict positivity of  $h$  follows directly from the irreducibility of  $T$ .  $\square$

### 3. Ruelle operator theorem

PROPOSITION 3.1. *Let  $\varrho_e$  be the essential spectral radius of  $T$ . Suppose that  $\varrho_e < \varrho$ . Then there exists a  $h \in C(K)$  with  $h > 0$ , a probability measure  $\mu \in M(K)$  and a constant  $0 < b < 1$  such that for any  $f \in C(K)$ ,*

$$\|\varrho^{-n} T^n f - \langle \mu, f \rangle h\|_\infty = O(b^n).$$

*Proof.* Without loss of generality, we assume that

$$\max_{x \in K} \sum_{j=1}^m p_j(x) \leq 1.$$

Then, we can prove, by induction, that

$$\sup_{n>0} \|T^n 1\| = \sup_{n>0} \max_{x \in K} \sum_{|J|=n} p_{w_J}(x) \leq 1.$$

Then, the operators sequence  $n^{-1} T^n$  converges weakly to zero. Note that (see [18] or [1])

$$\varrho_e = \lim_{n \rightarrow \infty} (\inf\{\|T^n - Q\| \mid Q \text{ is compact on } C(K)\})^{1/n}.$$

From this, together with the assumption  $\varrho_e < \varrho$  and [4, Theorem VIII.8.7], it follows that  $T$  is quasi-compact [10]. By making use of Hennion's method [10], we can deduce the assertion.  $\square$



In the following, we are interested in the case where  $\varrho_e = \varrho$ . We first give a basic criterion for the existence of the eigenfunction corresponding to the spectral radius  $\varrho$  in this case.

**PROPOSITION 3.2.** *Let  $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$  be a non-expansive system with at least one weakly contractive  $w_j$ . Suppose that:*

- (i) *there exist  $A, B > 0$  such that  $A \leq \varrho^{-n} T^n 1(x) \leq B$  for any  $x \in K$  and  $n > 0$ ; and*
- (ii) *for any  $f \in C(K)$ ,  $\{\varrho^{-n} T^n f\}_{n=1}^\infty$  is an equicontinuous sequence.*

*Then there exists a unique positive function  $h \in C(K)$  and a unique probability measure  $\mu \in M(K)$  such that*

$$Th = \varrho h, \quad T^* \mu = \varrho \mu, \quad \langle \mu, h \rangle = 1.$$

*Moreover, for every  $f \in C(K)$ ,  $\varrho^{-n} T^n f$  converges to  $\langle \mu, f \rangle h$  in the supremum norm, and for every  $\xi \in M(K)$ ,  $\varrho^{-n} T^{*n} \xi$  converges weakly to  $\langle \xi, h \rangle \mu$ .*

*Proof.* The proof can be found in [15], and we omit it. □

**Definition 3.3.** Let  $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$  be a non-expansive system. We say that the Ruelle operator theorem holds for this system if there exists a unique positive function  $h \in C(K)$  and a unique probability measure  $\mu \in M(K)$  such that

$$Th = \varrho h, \quad T^* \mu = \varrho \mu, \quad \langle \mu, h \rangle = 1,$$

and for every  $f \in C(K)$ ,  $\varrho^{-n} T^n f$  converges to  $\langle \mu, f \rangle h$  in the supremum norm.

In the next section, we study the Ruelle operator theorem for a non-expansive system under the framework in Proposition 3.2.

#### 4. Some sufficient conditions

Throughout this section we consider a non-expansive system  $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ . We assume the non-expansive IFS  $(X, \{w_j\}_{j=1}^m)$  containing at least one weakly contractive  $w_j$ . We prove the Ruelle operator theorem by applying Proposition 3.2.

In the next lemma we see that the Dini condition on all  $p_j$  also implies a similar nature property of the ‘bounded distortion property’. Recall that an equivalent condition for a function  $p(x)$  on  $K$  to be Dini continuous is

$$\sum_{n=0}^{\infty} \alpha_p(\theta^n) < \infty$$

for any  $0 < \theta < 1$ .

**LEMMA 4.1.** *Suppose that  $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$  is a non-expansive system. Let*

$$\alpha(t) = \max_{1 \leq j \leq m} \{\alpha_{\log p_j}(t)\}.$$

*Let  $0 < \theta < 1$  and let*

$$a = \sum_{n=0}^{\infty} \alpha(\theta^n).$$

For any fixed  $x, y \in K$ , if  $J = (j_1 \cdots j_n)$  satisfies the condition:

$$|w_{j_{i+1}} \circ \cdots \circ w_{j_n}(x) - w_{j_{i+1}} \circ \cdots \circ w_{j_n}(y)| \leq \theta^{n-i} \quad \text{for all } 1 \leq i < n.$$

Then

$$p_{w_J}(x) \leq e^a p_{w_J}(y).$$

*Proof.* The inequality follows from the estimate that

$$\begin{aligned} \left| \log \frac{p_{w_J}(x)}{p_{w_J}(y)} \right| &\leq \sum_{i=1}^n |\log p_{j_i}(w_{j_{i+1}} \circ \cdots \circ w_{j_n}(x)) - \log p_{j_i}(w_{j_{i+1}} \circ \cdots \circ w_{j_n}(y))| \\ &\leq \sum_{i=1}^n \alpha(\theta^{n-i}) \leq a. \end{aligned} \quad \square$$

**PROPOSITION 4.2.** Let  $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$  be a non-expansive system. Suppose that:

(i)

$$r := \sup_{x \in K} \min_{1 \leq j \leq m} \sup_{y \neq x} \frac{|w_j(x) - w_j(y)|}{|x - y|} < 1;$$

(ii) there exist constants  $A, B > 0$  such that  $A \leq \varrho^{-n} T^n 1(x) \leq B$  for any  $x \in K$  and  $n > 0$ .

Then the Ruelle operator theorem holds for this IFS.

We would like to point out that condition (i) of Proposition 4.2 is a generalization of condition (i) of [15, Theorem 4.2]. We extend Theorem 4.2 of [15] so that the system considered in this paper satisfies condition (i) of Proposition 4.2.

*Proof.* The proof is the same as that of [15, Theorem 4.2], and we omit it.  $\square$

For any integer  $n$ , we let  $I^n = \{J = (j_1 j_2 \cdots j_n) \mid 1 \leq j_i \leq m\}$ , and let

$$D_n = \{(n_1, n_2, \dots, n_k) \mid 0 < n_i < n_{i+1} \text{ and } n_k \leq n\} \cup \{(0)\}.$$

For any  $J \in I^n$  and any  $0 \leq k < l \leq n$ , we define  $J|_l^k = (j_{n-l+1} j_{n-l+2} \cdots j_{n-k})$ . We let  $J|_l^k = \emptyset$  if  $k = l$ .

For any multi-index  $J$  and  $x \in K$ , we let

$$\gamma_J(x) = \sup_{y \neq x} \frac{|w_J(x) - w_J(y)|}{|x - y|}.$$

For convenience, we let  $\gamma_J(x) = 1$  and  $p_{w_J}(x) = 1$  if  $|J| = 0$ .

**PROPOSITION 4.3.** Let  $\{D(k)\}_{k=1}^\ell$  be a partition of  $I^n$ , and let

$$0 = n_0^{(k)} < n_1^{(k)} < \cdots < n_{t_k}^{(k)} = n \quad \text{for all } 1 \leq k \leq \ell. \quad (4.1)$$

Then, for any  $x \in K$ ,

$$\sum_{k=1}^{\ell} \sum_{J \in D(k)} p_{w_J}(x) \cdot \prod_{t=1}^{t_k} \gamma_{J|_{n_t}^{n_{t-1}^{(k)}}}(w_{J|_{n_{t-1}^{(k)}}}^0 x) \leq a^n,$$

provided that

$$\sup_{x \in K} \sum_{j=1}^m p_j(x) \cdot \gamma_j(x) \leq a. \quad (4.2)$$

*Proof.* Note the fact that

$$p_{w_J}(x) = \prod_{i=0}^{n-1} p_{j_{n-i}}(w_{J|_i^0} x) \quad \text{for all } J = (j_1 j_2 \cdots j_n).$$

From (4.2), we can deduce inductively that for any integer  $n$ ,

$$\sum_{|J|=n} p_{w_J}(x) \cdot \prod_{i=0}^{n-1} \gamma_{J|_{i+1}^i}(w_{J|_i^0} x) \leq a^n. \quad (4.3)$$

For any multi-index  $J = (j_1 j_2 \cdots j_N)$  and  $x \in K$ , we have

$$\frac{|w_J(x) - w_J(y)|}{|x - y|} = \prod_{i=0}^{N-1} \frac{|w_{j_{N-i}}(w_{J|_i^0} x) - w_{j_{N-i}}(w_{J|_i^0} y)|}{|w_{J|_i^0}(x) - w_{J|_i^0}(y)|} \quad \text{for all } y \neq x.$$

This implies that

$$\gamma_J(x) \leq \prod_{i=0}^{N-1} \gamma_{j_{N-i}}(w_{J|_i^0} x). \quad (4.4)$$

From the assumption (4.1), using the same argument as (4.4), we deduce that for any  $J$  with  $|J| = n$ ,

$$\prod_{t=1}^{t_k} \gamma_{J|_{n_{t-1}^{(k)}}}^{n_{t-1}^{(k)}}(w_{J|_{n_{t-1}^{(k)}}} x) \leq \prod_{i=0}^{n-1} \gamma_{J|_{i+1}^i}(w_{J|_i^0} x). \quad (4.5)$$

Note that  $\{D(k)\}_{k=1}^\ell$  is a partition of  $I^n (= \{J \mid |J| = n\})$ . We have

$$\begin{aligned} & \sum_{k=1}^\ell \sum_{J \in D(k)} p_{w_J}(x) \cdot \prod_{t=1}^{t_k} \gamma_{J|_{n_{t-1}^{(k)}}}^{n_{t-1}^{(k)}}(w_{J|_{n_{t-1}^{(k)}}} x) \\ & \leq \sum_{|J|=n} p_{w_J}(x) \cdot \prod_{i=0}^{n-1} \gamma_{J|_{i+1}^i}(w_{J|_i^0} x) \quad (\text{by (4.5)}) \\ & \leq a^n \quad (\text{by (4.3)}). \end{aligned}$$

Thus, the conclusion follows.  $\square$

As a consequence of Proposition 4.2, we have the following.

**PROPOSITION 4.4.** *Let  $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$  be a non-expansive system. Suppose that:*

(i) *there exists  $k$  such that*

$$\sup_{x \in K} \sum_{|J|=k} p_{w_J}(x) \cdot \gamma_J(x) < Q^k;$$

(ii) *there exist constants  $A, B > 0$  such that  $A \leq Q^{-n} T^n 1(x) \leq B$  for any  $x \in K$  and  $n > 0$ .*

*Then the Ruelle operator theorem holds.*

*Proof.* By (i) there exists a  $0 < \eta < 1$  such that

$$\sup_{x \in K} \sum_{|J|=k} p_{w_J}(x) \cdot \gamma_J(x) \leq \eta \varrho^k.$$

This, together with Proposition 4.3, implies that for any  $x \in K$  and  $\ell \in \mathbb{N}$ ,

$$\sum_{|J|=\ell k} p_{w_J}(x) \cdot \prod_{t=1}^{\ell} \gamma_{J|_{tk}}^{(t-1)k}(w_{J|_{(t-1)k}}^0 x) \leq \eta^\ell \varrho^{\ell k}.$$

By using the argument similar to (4.4), we can prove that for any multi-index  $J$  with  $|J| = \ell k$ ,

$$\gamma_J(x) \leq \prod_{t=1}^{\ell} \gamma_{J|_{tk}}^{(t-1)k}(w_{J|_{(t-1)k}}^0 x).$$

It follows that

$$\sum_{|J|=\ell k} p_{w_J}(x) \cdot \gamma_J(x) \leq \eta^\ell \varrho^{\ell k}. \quad (4.6)$$

We claim that

$$\sup_{x \in K} \inf_{\ell \in \mathbb{N}} \min_{|J|=\ell k} \gamma_J(x) = 0.$$

Otherwise, we suppose that

$$\sup_{x \in K} \inf_{\ell \in \mathbb{N}} \min_{|J|=\ell k} \gamma_J(x) > 0.$$

Then, there exists a  $b_0 > 0$  and a  $x_0 \in K$  such that

$$\inf_{\ell \in \mathbb{N}} \min_{|J|=\ell k} \gamma_J(x_0) \geq b_0.$$

This, combined with (4.6) and (ii), implies that for any  $\ell \in \mathbb{N}$ ,

$$\begin{aligned} \eta^\ell &\geq \varrho^{-\ell k} \sum_{|J|=\ell k} p_{w_J}(x_0) \cdot \gamma_J(x_0) \geq b_0 \cdot \varrho^{-\ell k} \sum_{|J|=\ell k} p_{w_J}(x_0) \\ &= b_0 \cdot \varrho^{-\ell k} T^{\ell k} 1(x_0) \geq b_0 A. \quad (\text{by (ii)}) \end{aligned}$$

This contradicts the choice of  $0 < \eta < 1$ . Then, the claim follows. Thus, there exists a  $\ell_0 \in \mathbb{N}$  and a  $J_0$  with  $|J_0| = \ell_0 k$  such that  $\sup_{x \in K} \gamma_{J_0}(x) < 1$ . Hence, by Proposition 4.2, the Ruelle operator theorem for  $T^{\ell_0 k}$  holds. This implies that the Ruelle operator theorem for  $T$  holds.  $\square$

**THEOREM 4.5.** *Suppose that  $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$  is a non-expansive system. If there exists  $k$  such that*

$$\sup_{x \in K} \sum_{|J|=k} p_{w_J}(x) \cdot \gamma_J(x) < \varrho^k, \quad (4.7)$$

*then the Ruelle operator theorem holds.*

*Proof.* Since the Ruelle operator theorem for  $T^k$  implies the Ruelle operator theorem for  $T$ , we may assume  $k = 1$  in the hypothesis, so that (4.7) is reduced to

$$\sup_{x \in K} \sum_{j=1}^m p_j(x) \cdot \gamma_j(x) < \varrho. \quad (4.8)$$

This means that condition (i) of Proposition 4.4 is satisfied. Hence, we need only to show that condition (ii) of Proposition 4.4 is also satisfied, i.e. there exist  $A, B > 0$  such that

$$A \leq \varrho^{-n} \sum_{|J|=n} p_{w_J}(x) \leq B \quad \text{for all } n.$$

By (4.8) we can find  $0 < \eta < 1$  such that

$$\sup_{x \in K} \sum_{j=1}^m p_j(x) \cdot \gamma_j(x) \leq \eta \varrho. \quad (4.9)$$

For any fixed  $x \in K$ , choose  $\theta$  such that  $0 < \eta < \theta < 1$ . For any integer  $n$  and  $J \in I^n$ , let  $n_1$  be the largest integer such that

$$\gamma_{J|_{n_1}}^0(x) \geq \theta^{n_1},$$

and let  $n_2(>n_1)$  be the largest integer such that

$$\gamma_{J|_{n_2}}^{n_1}(w_{J|_{n_1}}^0 x) \geq \theta^{n_2-n_1},$$

and so on. Then, we find a sequence  $\{n_i\}_{i=1}^{t_J}$  such that

$$\gamma_{J|_{n_{i+1}}}^{n_i}(w_{J|_{n_i}}^0 x) \geq \theta^{n_{i+1}-n_i} \quad \text{for all } 1 \leq i \leq n_{t_J} - 1,$$

and

$$\gamma_{J|_{n_i}}^{n_{t_J}}(w_{J|_{n_{t_J}}}^0(x)) < \theta^{i-n_{t_J}} \quad \text{for all } n_{t_J} < i \leq n. \quad (4.10)$$

Define  $\sigma : I^n \rightarrow D_n$  by

$$\sigma(J) = (n_1, n_2, \dots, n_{t_J}).$$

Then  $\#\sigma(I^n) < \infty$ . Denote  $\sigma(I^n) = \{A_k\}_{k=1}^\ell$ , where  $A_k \in D_n$ . Let

$$D(k) = \{J \mid \sigma(J) = A_k\} \quad \text{for all } 1 \leq k \leq \ell.$$

It is clear that

$$D(i) \cap D(j) = \emptyset \quad \text{for all } i \neq j.$$

Hence,  $\{D(k)\}_{k=1}^\ell$  is a partition of  $I^n$ .

For any  $1 \leq k \leq \ell$ , let  $A_k = (n_1^{(k)}, n_2^{(k)}, \dots, n_{t_k}^{(k)})$ . For convenience, we let  $n_0^{(k)} = 0$  and let  $n_{t_k}^{(k)} = n$ . By making use of (4.9), it follows from Proposition 4.3 that

$$S_0 := \sum_{k=1}^\ell \sum_{J \in D(k)} p_{w_J}(x) \cdot \prod_{t=1}^{t_k} \gamma_{J|_{n_t}^{n_{t-1}^{(k)}}}(w_{J|_{n_{t-1}^{(k)}}}^0 x) \leq (\eta \varrho)^n. \quad (4.11)$$

Let

$$\begin{aligned}\Omega(n, k) &= \{J \mid |J| = n \text{ and } n_{t_J} = k\}, \quad 1 \leq k \leq n, \\ \Omega(n, 0) &= \{J \mid |J| = n \text{ and } n_{t_J} = 0\}.\end{aligned}$$

Then

$$I^n = \bigcup_{k=0}^n \Omega(n, k).$$

Without loss of generality, we assume that  $\Omega(n, n) = \{D(k)\}_{k=1}^{\ell_0}$ , where  $\ell_0 \leq \ell$ . We also let

$$S_1 := \sum_{k=1}^{\ell_0} \sum_{J \in D(k)} p_{w_J}(x) \cdot \prod_{t=1}^{t_k} \gamma_{J|_{n_t}^{n_{t-1}^{(k)}}} (w_{J|_{n_t}^0} x).$$

For any  $1 \leq k \leq \ell_0$  and any  $J \in D(k)$ , we have  $n_{t_{k-1}}^{(k)} = n_{t_J} = n$ , and this implies that

$$\prod_{t=1}^{t_k} \gamma_{J|_{n_t}^{n_{t-1}^{(k)}}} (w_{J|_{n_t}^0} x) \geq \prod_{t=1}^{t_k} \theta^{n_t^{(k)} - n_{t-1}^{(k)}} = \theta^n.$$

From this, we conclude that

$$S_1 \geq \sum_{k=1}^{\ell_0} \sum_{J \in D(k)} p_{w_J}(x) \cdot \theta^n = \sum_{J \in \Omega(n, n)} p_{w_J}(x) \cdot \theta^n.$$

This, combined with (4.11), implies that

$$\sum_{J \in \Omega(n, n)} p_{w_J}(x) \cdot \theta^n \leq S_1 \leq S_0 \leq (\eta \varrho)^n.$$

Thus, it follows that

$$\varrho^{-n} \sum_{J \in \Omega(n, n)} p_{w_J}(x) \leq \left(\frac{\eta}{\theta}\right)^n. \quad (4.12)$$

Remember that

$$\alpha(t) = \max_{1 \leq j \leq m} \alpha_{\log p_j}(t)$$

and

$$a = \sum_{k=0}^{\infty} \alpha(\theta^k).$$

Then  $a$  is finite because all of the  $p_i$  are Dini continuous functions on  $X$ . For any  $n > 0$ , we can make use of Proposition 2.3(i) to find  $x_n \in K$  such that

$$\varrho^{-n} \sum_{|J|=n} p_{w_J}(x_n) \leq 1. \quad (4.13)$$

For any  $J = (j_1 j_2 \cdots j_n) \in \Omega(n, k)$ , we have  $J|_k^0 \in \Omega(k, k)$ . By using (4.10), we can deduce from Lemma 4.1 that

$$p_{w_{J|_n^k}}(w_{J|_k^0} x) \leq e^a p_{w_{J|_n^k}}(y) \quad \text{for all } y \in K.$$

(We use  $|K| = 1$  here.) Hence,

$$p_{w_J}(x) = p_{w_{J|_n^k}}(w_{J|_k^0}x)p_{w_{J|_k^0}}(x) \leq e^a p_{w_{J|_n^k}}(x_{n-k})p_{w_{J|_k^0}}(x). \quad (4.14)$$

It follows that

$$\begin{aligned} \varrho^{-n} \sum_{|J|=n} p_{w_J}(x) &= \varrho^{-n} \sum_{k=0}^n \sum_{J \in \Omega(n,k)} p_{w_J}(x) \\ &\leq \varrho^{-n} \sum_{k=0}^n \sum_{J \in \Omega(n,k)} e^a p_{w_{J|_n^k}}(x_{n-k})p_{w_{J|_k^0}}(x) \quad (\text{by (4.14)}) \\ &\leq e^a \sum_{k=0}^n \left( \varrho^{-n+k} \sum_{|J'|=n-k} p_{w_{J'}}(x_{n-k}) \right) \left( \varrho^{-k} \sum_{J'' \in \Omega(k,k)} p_{w_{J''}}(x) \right) \\ &\leq e^a \sum_{k=0}^n 1 \cdot \left( \frac{\eta}{\theta} \right)^k \quad (\text{by (4.12), (4.13)}). \end{aligned} \quad (4.15)$$

The last term is bounded by  $e^a \sum_{k=0}^\infty (\eta/\theta)^k := B_1$ . This concludes the upper bound estimate.

For the lower bound estimation, we note that Proposition 2.3(i) and (4.15) implies that for any  $n > 0$ , there exists  $y_n \in K$  such that

$$1 \leq C_n := \varrho^{-n} \sum_{|J|=n} p_{w_J}(y_n) \leq B_1.$$

For any fixed  $x \in K$ , we let

$$\alpha_J = \sum_{i=0}^{n-1} \alpha(|w_{J|_i^0}(x) - w_{J|_i^0}(y_n)|).$$

Then, we have

$$p_{w_J}(y_n) \leq p_{w_J}(x)e^{\alpha_J}.$$

By (4.10), we have for any  $J \in \Omega(n, k)$ ,

$$|w_{J|_i^0}(x) - w_{J|_i^0}(y_n)| < \theta^{i-k} \quad \text{for all } k < i \leq n.$$

(We use  $|K| = 1$  here.) It follows that

$$\alpha_J \leq a + k\alpha(1) \quad \text{for all } J \in \Omega(n, k).$$

Using the same argument as (4.15), we can deduce that

$$\varrho^{-n} \sum_{J \in \Omega(n,k)} p_{w_J}(y_n) \leq e^a \left( \frac{\eta}{\theta} \right)^k.$$

Then, we have

$$\begin{aligned} \varrho^{-n} \sum_{|J|=n} p_{w_J}(y_n)\alpha_J &= \varrho^{-n} \sum_{k=0}^n \sum_{J \in \Omega(n,k)} p_{w_J}(y_n)\alpha_J \\ &\leq \varrho^{-n} \sum_{k=0}^n (a + k\alpha(1)) \sum_{J \in \Omega(n,k)} p_{w_J}(y_n) \\ &\leq e^a \sum_{k=0}^n (a + k\alpha(1)) \left( \frac{\eta}{\theta} \right)^k \leq B_2, \end{aligned}$$

where  $B_2 := e^a \sum_{k=0}^{\infty} (a + k\alpha(1))(\eta/\theta)^k$ . By the convexity of function  $e^x$ , we have

$$\begin{aligned} \varrho^{-n} \sum_{|J|=n} p_{w_J}(x) &\geq \varrho^{-n} \sum_{|J|=n} p_{w_J}(y_n) e^{-\alpha_J} \geq \frac{\varrho^{-n}}{C_n} \sum_{|J|=n} p_{w_J}(y_n) e^{-\alpha_J} \\ &\geq e^{-\frac{1}{C_n} \varrho^{-n} \sum_{|J|=n} p_{w_J}(y_n) \alpha_J} \geq e^{-B_2}. \end{aligned}$$

This completes the proof.  $\square$

*Remark 4.6.* We note that for any multi-index  $J$  and  $x \in K$ ,

$$\gamma_J(x) \leq \sup_{y \neq z} \frac{|w_J(z) - w_J(y)|}{|z - y|}.$$

Then, for any integer  $n$ , we have

$$\sum_{|J|=n} p_{w_J}(x) \cdot \gamma_J(x) \leq \sum_{|J|=n} p_{w_J}(x) \cdot \sup_{y \neq z} \frac{|w_J(z) - w_J(y)|}{|z - y|}.$$

Hence, Theorem 4.5 in this paper is a generalization of [15, Theorem 4.4]. However, the following example indicates that this generalization is non-trivial.

*Example 4.7.* Let  $X = [0, 1]$ , and let  $w_1(x) = x - x^2/2$ ,  $w_2(x) = 1/2 + x^2/2$ . Then  $w'_1(\cdot) \geq 0$ ,  $w'_2(\cdot) \geq 0$ . So  $w_1(0) = 0$ ,  $w_2(1) = 1$ ,  $w'_1(0) = w'_2(1) = 1$  and  $w'_1(1) = w'_2(0) = 0$ .

In this example, both  $w_1$  and  $w_2$  are not strictly contractive. In fact, 0 is the indifferent fixed point of  $w_1$  and 1 is the indifferent fixed point of  $w_2$ . It is easy to see that the IFS  $(X, \{w_j\}_{j=1}^2)$  is weakly contractive.

Let  $p_1$  be any positive Dini function (not a Lipschitz function) on  $X$  with the inequalities  $0 < p_1(\cdot) < 1$ . Let

$$\delta = \frac{1}{5} \cdot \min_{x \in X} \{p_1(x), 1 - p_1(x)\} > 0,$$

and let

$$g(x) = \begin{cases} \delta - 2^{-1} + x & \text{if } 2^{-1} - \delta < x \leq 2^{-1}, \\ \delta + 2^{-1} - x & \text{if } 2^{-1} < x < 2^{-1} + \delta, \\ 0 & \text{otherwise.} \end{cases}$$

Define a Dini function  $p_2$  on  $X$  by

$$p_2(x) = 1 - p_1(x) + g(x) \quad \text{for all } x \in X.$$

Then

$$1 \leq \sum_{j=1}^2 p_j(x) = 1 + g(x) \leq 1 + \delta.$$

So, for any  $x \in X$ ,

$$g(x) - \frac{1}{4}p_1(x) < 0 \quad \text{and} \quad g(x) - \frac{1}{4}p_2(x) < 0. \quad (4.16)$$



Let  $K$  be the invariant set of the IFS  $\{w_j\}_{j=1}^2$ . Define

$$Tf(x) = \sum_{j=1}^2 p_j(x) f \circ w_j(x) \quad \text{for all } f \in C(K).$$

Let  $\varrho$  be the spectral radius of the operator  $T$ . Then, we have

$$1 \leq \varrho \leq 1 + \delta. \quad (4.17)$$

Note that

$$\begin{aligned} \gamma_1(x) &= \sup_{y \neq x} \frac{|w_1(y) - w_1(x)|}{|y - x|} = \sup_{y \neq x} \frac{|y - x - 2^{-1}y^2 + 2^{-1}x^2|}{|y - x|} \\ &= \sup_{y \neq x} \left( 1 - \frac{1}{2}(x + y) \right) = 1 - \frac{x}{2}; \\ \gamma_2(x) &= \sup_{y \neq x} \frac{|w_2(y) - w_2(x)|}{|y - x|} = \sup_{y \neq x} \frac{|2^{-1}y^2 - 2^{-1}x^2|}{|y - x|} = \frac{1}{2}(1 + x). \end{aligned}$$

We have

$$\begin{aligned} \sum_{j=1}^2 p_j(x) \cdot \gamma_j(x) &= p_1(x) \cdot \left( 1 - \frac{x}{2} \right) + p_2(x) \cdot \frac{1 + x}{2} \\ &\leq \begin{cases} p_1(x) + \frac{3}{4}p_2(x) & \text{if } 0 \leq x \leq \frac{1}{2}, \\ \frac{3}{4}p_1(x) + p_2(x) & \text{if } \frac{1}{2} < x \leq 1, \end{cases} \\ &\leq \begin{cases} 1 + (g(x) - \frac{1}{4}p_2(x)) & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 1 + (g(x) - \frac{1}{4}p_1(x)) & \text{if } \frac{1}{2} < x \leq 1, \end{cases} \\ &< 1 \quad (\text{by (4.16)}). \end{aligned}$$

This, together with (4.17), implies that

$$\sup_{x \in X} \sum_{j=1}^2 p_j(x) \cdot \gamma_j(x) < \varrho.$$

Then, Theorem 4.5 implies that the Ruelle operator theorem holds for this weakly contractive system.

Owing to the equalities

$$\sup_{y \neq z} \frac{|w_j(y) - w_j(z)|}{|y - z|} = 1 \quad \text{for all } j = 1, 2,$$

and

$$\sup_{x \in X} \sum_{j=1}^2 p_j(x) = \sup_{x \in X} (1 + g(x)) = 1 + \delta,$$

by noting that (4.17), the following inequality:

$$\sup_{x \in X} \sum_{j=1}^2 p_j(x) \cdot 1 < \varrho$$

does not hold. Hence, for this system, the condition of [15, Theorem 1.2] is not satisfied.

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