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Decay of correlations for weakly expansive dynamical systems

Yuan-Ling Ye

Department of Mathematics, South China Normal University, Guangzhou 510631, People's Republic of China

E-mail: ylye@scnu.edu.cn

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Abstract

For a summable variation potential function on a subshift of finite type, Pollicott (2000 *Trans. Am. Math. Soc.* **352** 843–53) gave an estimate of the decay of correlations. It was known that the systems he considered have the bounded distortion property (BDP), and that is a key condition on the systems. In this paper, we study weakly expansive Dini dynamical systems that may not have the BDP. Under some assumptions, our theorem gives an estimate of the decay of correlations.

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1. Introduction

This paper can be regarded as a continuation of our paper [LY], and the notation is adopted from there. The decay of correlation [Bal] or, alternatively, the rate of mixing [Po], is a problem addressed in many fields. It is closely related to the convergence speeds of iterations of the Ruelle operator. They have been extensively studied in dynamical systems [Bal, Bow, Boy, FJ, Li, Ru, Yo, Yu]. For a summable variation potential function on a subshift of finite type, Fan and Pollicott [FP] gave an estimate of the convergence speed of iterations of the Ruelle operator; later on, Pollicott [Po] gave an estimate of the decay of correlations.

In the following we always assume X to be an nonempty compact subset of Euclidean space $(\mathbb{R}^d, |\cdot|)$ with $\overline{X^\circ} = X$, and let T be a self-map on X. We call (X, T) a *weakly expansive dynamical system* if it satisfies the following condition: there exists a finite partition $\{X_j\}_{j=1}^m$ of X such that for each $1 \leqslant j \leqslant m$,

I. X_j is a connected subset of X with piecewise smooth boundary;
II. $T|_{X_j} : X_j \to T(X_j)$ is a homeomorphism with continuous extension to $\overline{X_j}$ and $\overline{T(X_j)} = X$;

III. $T|_{X_j}$ is *weakly expansive*, i.e. for any $t > 0$,

$$\inf_{|x-y|\geqslant t}\{|T(x) - T(y)| : x, y \in X_j\} > t.$$

With such a system (X, T), we associate a potential function $\phi : X \to \mathbb{R}$ such that each $\phi|_{X_j}$ is *Dini continuous* [LY]. The triple (X, T, ϕ) is called a *weakly expansive Dini dynamical system*.

We will see that, under some assumptions, the system (X, T, ϕ) has a unique *equilibrium state* υ [W1]. And then the *correlation function* with respect to continuous function f can be defined by [Po]

$$\Psi_f(n) := \langle \upsilon, (f \circ T^n) f\rangle - (\langle \upsilon, f\rangle)^2.$$

In this paper, we will focus on studying the decay of correlations. For this we define the *expansive ratio function* $r(\cdot)$ of (X, T) by

$$r(y) = \inf_{\substack{x\neq y\\ x,y\in X_j}} \frac{|Tx - Ty|}{|x - y|} \qquad \text{if } y \in X_j.$$

For any multi-index $J = (j_1 j_2 \cdots j_n)$, $1 \leqslant j_i \leqslant m$, we let

$$X_J = X_{j_1} \cap (T^{-1}X_{j_2}) \cap \cdots \cap (T^{-(n-1)}X_{j_n}).$$

We use $|D|$ to denote the diameter of subset $D(\subseteq X)$. Let $\gamma_n = \max_{|J|=n} |X_J|$. By the weakly expansion of T, we will see that $\lim_{n\to\infty} \gamma_n = 0$. For any function p defined on X, we define the *modulus of continuity* of p by $\alpha_p(t) = \sup_{|x-y|\leqslant t} |p(x) - p(y)|$. Let $\alpha(t) = \max_{1\leqslant j\leqslant m} \alpha_{\phi|_{X_j}}(t)$, and define

$$\Phi(t) = \alpha(t) + \int_0^t \frac{\alpha(x)}{x}\,\mathrm{d}x, \qquad 0 \leqslant t \leqslant |X|.$$

Then Φ is well defined because of the Dini continuity of $\phi|_{X_j}$. Our main result is (corollary 4.3).

Theorem. *Suppose that the weakly expansive Dini dynamical system (X, T, ϕ) satisfies the condition*

$$\max_{x\in X} \sum_{y\in T^{-1}x} \exp(\phi(y))(r(y))^{-1} < \min_{x\in X} \sum_{y\in T^{-1}x} \exp(\phi(y)).$$

Then the system has a unique equilibrium state. Moreover, there exist constants $C > 0$, $0 < \lambda < 1$ and $\ell_0 \in \mathbb{N}$ such that for any $f \in C(X)$ and $n \geqslant k\ell$ with $\ell \geqslant \ell_0$, we have

$$|\Psi_f(n)| \leqslant \alpha_f(\gamma_\ell)\|f\| + C\big(\lambda^k + \lambda^\ell + \Phi(\gamma_\ell)\big)\|f\|^2.$$

We remark that the key condition in [FP] and [Po] is that the systems have the bounded distortion property (BDP), which is guaranteed by the summable variation of the potential function. Since a general weakly expansive Dini dynamical system may not have the BDP (see example 4.4), it creates more difficulty (see the remark after definition 3.2). We discuss this in propositions 3.5 and 3.6, that are fundamental propositions of the paper. Our method is to make use of the known results for the IFS in [LY] to define a 'normalized' Ruelle operator and averaging operators [FP], and then get an estimate of decay of correlations.

We organize the paper as follows. In section 2, we will match the weakly expansive Dini dynamical systems with some weakly contractive Dini IFS, and then we convert the Perron–Frobenius properties of the dynamical systems into the IFS. In section 3, we set up the basic propositions. We give an estimate of decay of correlations in section 4.

2. Preliminaries

Without loss of generality, we assume that $|X| = 1$. We let $UC(X^\circ)$ and $C(X)$ denote the space of real-valued uniformly continuous functions on X° and X, respectively, with supremum norm $\|\cdot\|$. Let $M(X)$ denote the set of all regular Borel measures on X. We say that $p \in C(X)$ is *Dini continuous* if $\int_0^1 \alpha_p(t)t^{-1}\,\mathrm{d}t < \infty$. Throughout the paper we always assume (X, T, ϕ) to be a weakly expansive Dini dynamical system defined in the previous section. Then we have the following lemma.

Lemma 2.1. *The Ruelle operator L_ϕ defined on $UC(X^\circ)$ by*

$$L_\phi(f)(x) = \sum_{y\in T^{-1}x} \exp(\phi(y))f(y) \tag{2.1}$$

is a self-map of $UC(X^\circ)$, i.e. $L_\phi(f) \in UC(X^\circ)$ if $f \in UC(X^\circ)$.

Proof. By assumptions I and II on the system (X, T), we have $T(X_j^\circ) = X^\circ$ $\forall 1 \leqslant j \leqslant m$. Then

$$\#\{y : T(y) = x\} = m \qquad \forall x \in X^\circ. \tag{2.2}$$

By the Dini continuity of $\phi|_{X_j}$, we see that the $\phi|_{X_j}$ is uniformly continuous. It, together with (2.2) and assumption II, deduces that $L_\phi(f) \in UC(X^\circ)$ if $f \in UC(X^\circ)$. ■

We remark that the system (X, T) is a special case of *Markov* [Boy] and satisfies condition (2.2). It may not be shared by general Markov systems.

Given an IFS $(X, \{w_j\}_{j=1}^m)$, we let $I^n = \{J = (j_1 j_2 \cdots j_n) : 1 \leqslant j_i \leqslant m\}$. For any multi-index $J \in I^n$, we use $|J|(= n)$ to denote the length of J. Let $w_J(x) = w_{j_1} \circ w_{j_2} \circ \cdots \circ w_{j_n}(x)$, and let $w_J(X) = \{w_J(x) : x \in X\}$. We say that the IFS $(X, \{w_j\}_{j=1}^m)$ is *weakly contractive* if for each $1 \leqslant j \leqslant m$,

$$\alpha_{w_j}(t) := \sup_{|x-y|\leqslant t} |w_j(x) - w_j(y)| < t \qquad \forall t > 0.$$

We call the triple $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ a *weakly contractive Dini IFS* if, moreover, each p_j is Dini continuous on $w_j(X)$. For more details on the IFS, we refer readers to [LY].

Now we are ready to match the weakly expansive Dini dynamical systems with some weakly contractive Dini IFS.

Proposition 2.2. *Suppose that (X, T, ϕ) is a weakly expansive Dini dynamical system with m branches. Then there exists a weakly contractive Dini IFS $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ such that*

(i) $T \circ w_j(x) = x$ $\forall x \in X^\circ, 1 \leqslant j \leqslant m$;
(ii) $\lim_{n\to\infty} \gamma_n = 0$;
(iii) $\sum_{j=1}^m p_j \circ w_j(x) \cdot f \circ w_j(x) = L_\phi f(x)$ $\forall f \in UC(X^\circ), x \in X^\circ$.

Proof. (i) By assumption II, $T|_{X_j}$ has a unique continuous extension to $\overline{X_j}$, and denote it by T_j. By assumption III, we have for any $1 \leqslant j \leqslant m$,

$$|T^{-1}(x) - T^{-1}(y)| \leqslant |x - y| \qquad \forall x, y \in T(X_j).$$

Note that $\overline{T(X_j)} = X$. We follow that $(T|_{X_j})^{-1}$ has a unique continuous extension to X. Denote it by w_j. Hence, we set up an IFS $(X, \{w_j\}_{j=1}^m)$. It is clear that for any $1 \leqslant j \leqslant m$,

$$T_j \circ w_j(x) = x \qquad \forall x \in X. \tag{2.3}$$

In particular, $T \circ w_j(x) = x \qquad \forall x \in X^\circ$.

We will show that the IFS $(X, \{w_j\}_{j=1}^m)$ is weakly contractive. By assumption II, we deduce that for any $t > 0$,

$$R_j(t) := \inf_{|x-y|\geqslant t}\{|T_j(x) - T_j(y)| : x, y \in \overline{X_j}\}$$
$$= \inf_{|x-y|\geqslant t}\{|T_j(x) - T_j(y)| : x, y \in X_j\}.$$

Then by assumption III, we have $R_j(t) > t$, and then T_j is weakly expansive. Let $r_j(t) := 2^{-1}(t + R_j(t))$. Hence for any $t > 0$,

$$r_j^{-1}(t) < t < r_j(t) < R_j(t).$$

We claim that

$$\sup_{|x-y|\leqslant r_j(t)} |w_j(x) - w_j(y)| \leqslant t \qquad \forall t > 0.$$

Otherwise, suppose that there exists some $t_0 > 0$ such that

$$\sup_{|x-y|\leqslant r_j(t_0)} |w_j(x) - w_j(y)| > t_0.$$

Then there exist $x_0, y_0 \in X$ such that

$$|x_0 - y_0| \leqslant r_j(t_0) \qquad \text{and} \qquad |w_j(x_0) - w_j(y_0)| > t_0.$$

This, together with (2.3) and $\overline{X^\circ} = X$, deduces that

$$R_j(t_0) \leqslant |x_0 - y_0| \leqslant r_j(t_0) < R_j(t_0).$$

This contradiction implies the claim. Using this claim, it follows that

$$\alpha_{w_j}(t) = \sup_{|x-y|\leqslant t} |w_j(x) - w_j(y)| \leqslant r_j^{-1}(t) < t.$$

Thus, the IFS $(X, \{w_j\}_{j=1}^m)$ is weakly contractive.

(ii) It is straightforward to check that for any multi-index J,

$$w_J(X^\circ) \subseteq X_J \subseteq w_J(X).$$

Note that $\overline{X^\circ} = X$ and the continuity of w_j. We have $\overline{w_J(X^\circ)} = w_J(X)$. So $|X_J| = |w_J(X)|$. Thus, $\gamma_n = \max_{|J|=n} |w_J(X)|$. Hence by the weak contraction of w_j, we have $0 \leqslant \gamma_{n+1} < \gamma_n$. It follows that $\lim_{n\to\infty} \gamma_n = 0$.

(iii) For each $1 \leqslant j \leqslant m$, we define $q_j : X_j \to \mathbb{R}^+$ by

$$q_j(x) = \exp(\phi(x)) \qquad \text{if } x \in X_j.$$

By the Dini continuity of $\phi|_{X_j}$, it follows that there exists a unique Dini continuous function p_j defined on $\overline{X_j}$ such that $p_j(x) = q_j(x)$ for any $x \in X_j$. Hence, the $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ is a weakly contractive Dini IFS, and

$$\sum_{j=1}^m p_j \circ w_j(x) \cdot f \circ w_j(x) = L_\phi(x) \qquad \forall f \in UC(X^\circ), \quad x \in X^\circ. \qquad \blacksquare$$

Hence, given a weakly expansive Dini dynamical system (X, T, ϕ), we can match it with a weakly contractive Dini IFS. We will call this *weakly contractive Dini IFS induced from the system* (X, T, ϕ). Consequently, corresponding to the system (X, T, ϕ), we can define another *Ruelle operator* P on the induced IFS $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ by

$$P(f)(x) = \sum_{j=1}^m p_j \circ w_j(x) \cdot f \circ w_j(x) \qquad \forall f \in C(X). \tag{2.4}$$

Let $\varrho = \varrho(L_\phi)$ be the spectral radius of L_ϕ, and let L_ϕ^* be the dual operators of L_ϕ on the dual space $(UC(X^\circ))^*$. For any integer n, we let $S_n(x) = \sum_{i=0}^{n-1} \phi(T^i x)$. Then

$$L_\phi^n f(x) = \sum_{y \in T^{-n}x} \exp(S_n(y)) f(y).$$

The PF property (PF stands for Perron–Frobenius) for the Ruelle operator on IFS has been defined in [LY]. We can define a similar property on this dynamical system.

Definition 2.3. *The Ruelle operator L_ϕ on the dynamical system (X, T, ϕ) is said to have the PF property if there exists a unique $0 < h \in UC(X^\circ)$ and a unique probability measure μ defined on X° such that*

$$L_\phi h = \varrho h, \qquad L_\phi^* \mu = \varrho \mu, \qquad \langle \mu, h \rangle = 1$$

and for every $f \in UC(X^\circ)$, $\varrho^{-n} L_\phi^n f$ converges to $\langle \mu, f \rangle h$ in the supremum norm.

Lemma 2.4. *Let L_ϕ and P be defined as (2.1) and (2.4), respectively. Then*

(i) P is a unique continuous extension of L_ϕ;
(ii) L_ϕ has the PF property if and only if P has the PF property.

Proof.

(i) Since $\overline{X^\circ} = X$, then any $f \in UC(X^\circ)$ can be extended uniquely to some $\tilde{f} \in C(X)$, and hence $L_\phi(f)$ has a unique continuous extension $\widetilde{L_\phi(f)}$. By proposition 2.2(iii), it follows $P\tilde{f}(x) = L_\phi f(x)\ \forall x \in X^\circ$. Note that $P(\tilde{f}) \in C(X)$. It follows that $P(\tilde{f}) = \widetilde{L_\phi(f)}$. Hence, P is a unique continuous extension of L_ϕ.

(ii) Let $M(X^\circ)$ denote the set of all regular Borel measures on X°. Define $\pi : UC(X^\circ) \to C(X)$ by $\pi(f) = \tilde{f}$. It is easy to see that π is an isometric isomorphism. Then the adjoint operator π^* is an isometry between $M(X)$ and $M(X^\circ)$. This, together with (i) and $\overline{X^\circ} = X$, deduces that L_ϕ has the PF property if and only if P has the PF property. ■

Hence we can regard L_ϕ and P as coincident. In this paper we always consider L_ϕ as a continuous linear operator on $C(X)$, i.e. $L_\phi = P$. Consequently, we can regard $h \in C(X)$ and $\mu \in M(X)$ in definition 2.3.

Our next theorem gives a sufficient condition for the possession of the PF property. Then discussion on some relevant concepts such as equilibrium state [W1] and mixing [W2] are involved.

Theorem 2.5. *Suppose that the weakly expansive Dini dynamical system (X, T, ϕ) satisfies the condition*

$$\max_{x \in X} \sum_{y \in T^{-1}x} \exp(\phi(y))(r(y))^{-1} < \varrho. \tag{2.5}$$

Then L_ϕ has the PF property. Moreover, if we let $\upsilon = h\mu$, where $0 < h \in C(X)$ and $\mu \in M(X)$ are given by definition 2.3. Then

(i) υ is a probability measure and satisfies the measure separation condition, i.e.

$$\upsilon(\overline{X_I} \bigcap \overline{X_J}) = 0 \qquad \forall I \neq J \quad with \quad |I| = |J|;$$

(ii) υ is the unique equilibrium state of the system (X, T, ϕ);
(iii) υ is mixing for T.

Proof. Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be the weakly contractive Dini IFS induced from the system (X, T, ϕ), and let $r_j = \sup_{x \neq y} |w_j(x) - w_j(y)|/|x - y|$. Note that

$$r(y) = \inf_{\substack{x \neq y \\ x,y \in X_j}} \frac{|Tx - Ty|}{|x - y|}, \qquad \text{if } y \in X_j.$$

From this, together with proposition 2.2(i), it follows that $(r|_{X_j})^{-1} \equiv r_j$ for each $1 \leqslant j \leqslant m$. Hence (2.5) implies that

$$\left\| \sum_{j=1}^{m} p_j \circ w_j(\cdot) r_j \right\| < \varrho.$$

By [LY, theorem 4.4], we see that the operator P defined by (2.4) has the PF property. Combining with lemma 2.4(ii), it follows that L_ϕ has the PF property.

(i) By definition 2.3, we have $\upsilon(X) = \langle \mu, h \rangle = 1$. Again by definition 2.3, we have $\mathrm{supp}(\mu) \subseteq X^\circ$. This, together with $L_\phi^* \mu = \varrho\mu$, deduces that

$$\mathrm{supp}(\mu) \subseteq \bigcup_{|J|=n} X_J^\circ \qquad \forall n \in \mathbb{N}.$$

Then $\mu(\partial X_J) = 0 \ \forall J \in I^n$. Because $\overline{X_J^\circ} = \overline{X_J}$ and $X_I \bigcap X_J = \emptyset$, we have $\overline{X_I} \bigcap \overline{X_J} \subseteq \partial X_I \bigcap \partial X_J$. Hence

$$\mu(\overline{X_I} \bigcap \overline{X_J}) = \mu(\partial X_I \bigcap \partial X_J) = 0.$$

Therefore, $\upsilon = h\mu$ satisfies the measure separation condition.

(ii) The proof can be modified from [W1] on the symbolic space with summable variation potential function. We outline the idea here for completeness. Let $g(x) = \exp(\phi(x))h(x)(\varrho h(Tx))^{-1}$. Then $\sum_{y \in T^{-1}x} g(y) = 1$. Define a 'normalized' operator $L_{\log g} : UC(X^\circ) \to UC(X^\circ)$ by

$$L_{\log g} f(x) = \sum_{y \in T^{-1}x} g(y) f(y).$$

Then $L_{\log g}^* \upsilon = \upsilon$, and for any $n \in \mathbb{N}$,

$$\varrho^{-n} L_\phi^n(fh) = h L_{\log g}^n f.$$

Hence, by making use of theorems 4.14 and 4.18 of [W2], we can prove, similarly to [W1, theorem 2.1], that the system has a unique equilibrium state $\upsilon = h\mu$.

(iii) The argument is standard, and can be modified from [Bow] on the Hölder continuous system. We therefore omit it. ■

Definition 2.6. [Po] *The correlation function* Ψ_f *of* (X, T, ϕ) *with respect to* $f \in C(X)$ *is defined by*

$$\Psi_f(n) = \langle \upsilon, (f \circ T^n) f \rangle - (\langle \upsilon, f \rangle)^2, \qquad n \in \mathbb{N}.$$

We will consider the correlation function in the following sections.

3. Basic propositions

In the following sections we always assume the dynamical system (X, T, ϕ) to be as in theorem 2.5, and let the function g, 'normalized' operator $L_{\log g}$ and equilibrium state υ be as there. Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be the IFS induced from the system (X, T, ϕ). We know from the previous section that the unique equilibrium state υ is mixing. To study the decay of correlation functions, we need consider the convergence speed of iterations $L_{\log g}^n(f)$. For this we set up some basic propositions first.

3.1. The averaging operators

Let $B(X, \upsilon)$ denote the set of all bounded measurable functions on X with essentially supremum norm, i.e. for any $f \in B(X, \upsilon)$,

$$\|f\| = \inf_{\upsilon(E)=0} \sup_{x \in X \setminus E} |f(x)|.$$

We remark that it coincides with the supremum norm if $f \in C(X)$. And for any multi-index J, we can regard that $\mathbf{1}_{X_J} = \mathbf{1}_{w_J(X)}$.

For any $n \geqslant 1$, let $g_n(x) = \prod_{i=0}^{n-1} g(T^i x)$. Then, by induction we have

$$L^n_{\log g} f(x) = \sum_{y \in T^{-n}x} g_n(y) f(y) \qquad \forall f \in B(X, \upsilon).$$

Define the *averaging operator* $P_n : B(X, \upsilon) \to B(X, \upsilon)$ by

$$P_n f(x) = L^n_{\log g} f(T^n x).$$

It is easy to see that

$$\|P_n f\| = \|L^n_{\log g} f\|. \tag{3.1}$$

For any $x \in X$, we denote by $X_n(x)$ the cylinder X_J containing x and with $|J| = n$, and let $\mathcal{B}_n$ be the σ-algebra on X generated by the cylinders of $\{X_J\}_{|J|=n}$. Let $E_n(\cdot) = E(\cdot|\mathcal{B}_n)$ be the conditional expectation with respect to $\mathcal{B}_n$ on the measure space (X, υ). Then for any $f \in B(X, \upsilon)$,

$$E_n(f)(x) = \frac{\int_{X_n(x)} f \, \mathrm{d}\upsilon}{\upsilon(X_n(x))}.$$

Hence,

$$L^n_{\log g} E_n(f)(x) = \sum_{|J|=n} \frac{g_n \circ w_J(x)}{\upsilon(X_J)} \int_{X_J} f \, \mathrm{d}\upsilon \qquad \forall x \in X. \tag{3.2}$$

For any $\ell, n \in \mathbb{N}$ we denote

$$S_n^{(\ell)} = \max_{|x-y| \leqslant \gamma_\ell} \sum_{|J|=n} |g_n \circ w_J(x) - g_n \circ w_J(y)|.$$

Lemma 3.1. *For any $n = k\ell$ $(k, \ell \in \mathbb{N})$ and $f \in C(X)$, we have*

$$\|P_n f\| \leqslant \alpha_f(\gamma_\ell) + \sum_{j=2}^{k-1} S^{(\ell)}_{(j-1)\ell} \cdot \left\| \left(\prod_{i=1}^{j-2} P_{i\ell} E_{i\ell} \right)(f) \right\| + \left\| \left(\prod_{i=1}^{k} P_{i\ell} E_{i\ell} \right)(f) \right\|.$$

Proof. For any $q \in B(X, \upsilon)$, let

$$\mathrm{var}_n(q) = \max_{|J|=n} \sup_{x,y \in X_J} |q(x) - q(y)|.$$

Note that $\|P_n(q)\| \leqslant \|q\|$. We have

$$\|P_n(I - E_n)q\| \leqslant \|(I - E_n)q\| \leqslant \mathrm{var}_n(q).$$

By (3.2) and the $\mathcal{B}_n$-measurability of $E_n(q)$, it follows that

$$\mathrm{var}_{n+\ell}(P_n E_n(q)) \leqslant \mathrm{var}_\ell\big(L^n_{\log g} E_n(q)\big) \leqslant S_n^{(\ell)} \cdot \|q\|.$$

Then for any $j \geqslant 2$,

$$\left\| P_n(I - E_{j\ell}) \left(\prod_{i=1}^{j-1} P_{i\ell} E_{i\ell} \right)(f) \right\| \leqslant S^{(\ell)}_{(j-1)\ell} \left\| \left(\prod_{i=1}^{j-2} P_{i\ell} E_{i\ell} \right)(f) \right\|. \tag{3.3}$$

We know from [FP, theorem 2] that for any $n = k\ell$ ($k, \ell \in \mathbb{N}$),

$$P_n = P_n \left((I - E_\ell) + \sum_{j=2}^{k-1} (I - E_{j\ell}) \prod_{i=1}^{j-1} P_{i\ell} E_{i\ell} + \prod_{i=1}^{k} P_{i\ell} E_{i\ell} \right).$$

Note that

$$\| P_n (I - E_\ell)(f) \| \leqslant \mathrm{var}_\ell (f) \leqslant \alpha_f(\gamma_\ell).$$

This, together with (3.3), deduces the assertion. ■

Hence, we are required to estimate the values of $\| P_n E_n(f) \|$ and $S_{jn}^{(n)}$. Before beginning our estimates, let us recall the following definition.

Definition 3.2. *The dynamical system (X, T, ϕ) is said to have the BDP if there exists $C > 0$ such that for any n,*

$$|S_n \circ w_J(x) - S_n \circ w_J(y)| \leqslant C \qquad \forall J \in I^n \quad \textit{and} \quad x, y \in X.$$

We remark that, under the assumption that the systems have the BDP, Fan and Pollicott [FP, Po] showed easily that there exists $0 < \tau < 1$ such that for any $f \in B(X, \upsilon)$ with $\int_X f \, \mathrm{d}\upsilon = 0$,

$$\| P_n E_n(f) \| \leqslant \tau \| f \|. \tag{3.4}$$

This was the key inequality in their estimates. To our knowledge, (3.4) has not been proved in the absence of the BDP. However, the weakly expansive systems considered may not have the BDP (see example 4.4), hence we are required to make adjustment so that it holds. A similar problem appears when we estimate the value of $S_{jn}^{(n)}$.

3.2. Basic propositions

To check whether the inequality (3.4) holds, we need the following simple inequality.

Lemma 3.3. *Let constant $a > 0$ and $J \in I^n$ satisfy the condition*

$$g_n \circ w_J(x) \leqslant a g_n \circ w_J(y) \qquad \forall x, y \in X.$$

Then

$$a^{-1} \leqslant \frac{g_n \circ w_J(x)}{\upsilon(X_J)} \leqslant a \qquad \forall x \in X.$$

Proof. By theorem 2.5(i) and proposition 2.2(i), we have

$$L^n_{\log g} \mathbf{1}_{X_J}(x) = \sum_{y \in T^{-n}x} g_n(y) \cdot \mathbf{1}_{X_J}(y) = g_n \circ w_J(x) \qquad \text{a.e. } (\upsilon).$$

From the above, together with $L^*_{\log g} \upsilon = \upsilon$, it follows that

$$\upsilon(X_J) = \langle \upsilon, L^n_{\log g} \mathbf{1}_{X_J} \rangle = \langle \upsilon, g_n \circ w_J(\cdot) \rangle. \tag{3.5}$$

We end the proof by noting that $\langle \upsilon, 1 \rangle = 1$. ■

Lemma 3.4. *Suppose that $0 < a^{-1} \leqslant c_J \leqslant a \ \forall J \in I^n$. Then for any $f \in B(X, \upsilon)$ with $\int_X f \, \mathrm{d}\upsilon = 0$, we have*

$$\left| \sum_{|J|=n} c_J \int_{X_J} f \, \mathrm{d}\upsilon \right| \leqslant (1 - a^{-2}) \| f \| \sum_{|J|=n} c_J \cdot \upsilon(X_J).$$

Proof. Since

$$\sum_{|J|=n}\int_{X_J} f\,\mathrm{d}\upsilon = \int_X f\,\mathrm{d}\upsilon = 0,$$

then

$$\begin{aligned}\left|\sum_{|J|=n} c_J \int_{X_J} f\,\mathrm{d}\upsilon\right| &\leqslant \frac{a-a^{-1}}{a+a^{-1}}\sum_{|J|=n} c_J\left|\int_{X_J} f\,\mathrm{d}\upsilon\right| \qquad \text{by ([FP, lemma 1])}\\ &\leqslant (1-a^{-2})\|f\|\sum_{|J|=n} c_J\cdot\upsilon(X_J).\end{aligned}$$

■

Proposition 3.5. *There exists* $0<\tau<1$ *such that for any* $f\in B(X,\upsilon)$ *with* $\int_X f\,\mathrm{d}\upsilon = 0$, *we have*

$$\|P_nE_n(f)\|\leqslant\tau\|f\| \qquad \forall\, n\geqslant 1.$$

Proof. Let $(X,\{w_j\}_{j=1}^m,\{p_j\}_{j=1}^m)$ be the IFS induced from the system (X,T,ϕ). Let $0<h\in C(X)$ and g be as in theorem 2.5. Denote $g_n(x)=\prod_{i=0}^{n-1} g(T^ix)$. Then

$$\sum_{|J|=n} g_n\circ w_J(x) = 1 \qquad \forall n\in\mathbb{N}. \tag{3.6}$$

And for any multi-index $J\in I^n$ and $x,y\in X$,

$$\frac{g_n\circ w_J(x)}{g_n\circ w_J(y)} = \frac{\exp(S_n\circ w_J(x))}{\exp(S_n\circ w_J(y))}\cdot\frac{h\circ w_J(x)}{h\circ w_J(y)}\cdot\frac{h(y)}{h(x)}. \tag{3.7}$$

Let $r_j=\sup_{x\neq y}|w_j(x)-w_j(y)|/|x-y|$. Then $(r(y))^{-1}=r_j$ if $y\in X_j$. Note the fact

$$\varrho\leqslant\max_{x\in X}\sum_{y\in T^{-1}x}\exp(\phi(y)).$$

We deduce from (2.5) that $\min_{1\leqslant j\leqslant m} r_j<1$. Combining with the case $n=1$ of (3.6), it follows that there exists $0<\eta<1$ such that

$$\max_{x\in X}\sum_{j=1}^m g\circ w_j(x)r_j\leqslant\eta.$$

For $J=(j_1j_2\cdots j_n)$, let $r_J=r_{j_1}r_{j_2}\cdots r_{j_n}$. And for $0\leqslant k<\ell\leqslant n$, let $J|_k^\ell=(j_{k+1}j_{k+2}\cdots j_\ell)$. Then by induction we have

$$\max_{x\in X}\sum_{|J|=n} g_n\circ w_J(x)r_J\leqslant\eta^n \qquad \forall n>0. \tag{3.8}$$

Choose θ such that $\eta<\theta<1$ and let

$$\Omega(n,k)=\{J:|J|=n,\ k\text{ smallest with } r_{J|_k^n}\geqslant\theta^{n-k}\}, \qquad 0\leqslant k<n,$$

$$\Omega(n,n)=\{J:|J|=n,\ r_{J|_k^n}<\theta^{n-k}\ \forall\,0\leqslant k<n\}.$$

Then $\Omega(n,k)\bigcap\Omega(n,k')=\emptyset\ \forall k\neq k'$, and $I^n=\bigcup_{k=0}^n\Omega(n,k)$. Let $\delta=\eta\theta^{-1}$. Then $0<\delta<1$. By (3.8), we have for any integer k,

$$\sum_{J\in\Omega(k,0)} g_k\circ w_J(x)\leqslant\delta^k.$$

By $\sum_{j=1}^{m} g \circ w_j(x) = 1$, we deduce that

$$\sum_{J \in \Omega(n,k)} g_n \circ w_J(x) \leqslant \delta^{n-k}. \tag{3.9}$$

Denote

$$\mathcal{A}(n,k) = \bigcup_{i=0}^{n-k} \Omega(n, n-k-i).$$

Let $c = (1-\delta)^{-1}$. Then

$$\sum_{J \in \mathcal{A}(n,k)} g_n \circ w_J(x) \leqslant \sum_{i=0}^{n-k} \delta^{k+i} \leqslant c\delta^k. \tag{3.10}$$

Hence, by (3.5) we have

$$\sum_{J \in \mathcal{A}(n,k)} \upsilon(X_J) = \int_X \left(\sum_{J \in \mathcal{A}(n,k)} g_n \circ w_J(x) \right) \mathrm{d}\upsilon(x) \leqslant c\delta^k. \tag{3.11}$$

Define

$$\alpha(t) = \max_{1 \leqslant j \leqslant m} \sup_{|x-y| \leqslant t} \left| \phi|_{X_j}(x) - \phi|_{X_j}(y) \right|. \tag{3.12}$$

Let $a := \sum_{k=0}^{\infty} \alpha(\theta^k)$. Then a is finite because $\phi|_{X_j}$ is Dini continuous.

For any $J \in \Omega(n, n-k)$, and any $0 \leqslant i < n-k$, since $r_{J|_{n-k}^n} \geqslant \theta^k$ and $r_{J|_i^n} = r_{J|_i^{n-k}} r_{J|_{n-k}^n} < \theta^{n-i}$, we have

$$r_{J|_i^{n-k}} = \frac{r_{J|_i^{n-k}} r_{J|_{n-k}^n}}{r_{J|_{n-k}^n}} \leqslant \theta^{n-k-i}.$$

It deduces that for any $x, y \in X$,

$$\phi(T^i \circ w_{J|_0^{n-k}}(x)) \leqslant \phi(T^i \circ w_{J|_0^{n-k}}(y)) \alpha(\theta^{n-k-i}).$$

Then

$$S_n \circ w_J(x) = \sum_{i=0}^{n-1} \phi(T^i \circ w_J(x)) \leqslant S_n \circ w_J(y) + a + \sum_{i=1}^{k} \alpha(\gamma_i). \tag{3.13}$$

For any $J \notin \mathcal{A}(n,k)$ with $|J| = n$, there exists some $k' < k$ such that $J \in \Omega(n, n-k')$. Then by (3.13), we have

$$S_n \circ w_J(x) \leqslant S_n \circ w_J(y) + a + \sum_{i=1}^{k} \alpha(\gamma_i).$$

Denote

$$a_k := \exp\left(a + 2\alpha_{\log h}(1) + \sum_{i=1}^{k} \alpha(\gamma_i)\right).$$

Hence by (3.7) we have

$$g_n \circ w_J(x) \leqslant a_k g_n \circ w_J(y), \qquad \forall\ J \notin \mathcal{A}(n,k). \tag{3.14}$$

(We use $|X| = 1$ here.)

By proposition 2.2(ii), we have $\lim_{n\to\infty}\alpha(\gamma_n) = 0$. This, together with $0 < \delta < 1$, deduces that there exists integer $k_0 > 0$ such that $a_{k_0}^{-2} - 3c\delta^{k_0} > 0$. Fixing such a k_0, we take

$$\tau_1 := 1 - a_{k_0}^{-2} + 3c\delta^{k_0}.$$

Then $0 < \tau_1 < 1$.

For any $f \in B(X, \upsilon)$ with $\int_X f\,\mathrm{d}\upsilon = 0$, we claim first that

$$|L_{\log g}^n E_n(f)(x)| \leqslant \tau_1 \|f\| \qquad \forall n > k_0. \tag{3.15}$$

Indeed, for any $x \in X$ and $n > k_0$, we let $A_n = \sum_{J\in\mathcal{A}(n,k_0)} \int_{X_J} f\,\mathrm{d}\upsilon$, and let

$$B_n(x) = \sum_{J\notin\mathcal{A}(n,k_0)} \frac{g_n \circ w_J(x)}{\upsilon(X_J)} \int\limits_{X_J} f\,\mathrm{d}\upsilon.$$

By (3.11) we have

$$|A_n| \leqslant \sum_{J\in\mathcal{A}(n,k_0)} \upsilon(X_J)\|f\| \leqslant c\delta^{k_0}\|f\|.$$

By (3.10) we follow that

$$|L_{\log g}^n E_n(f)(x) - B_n(x)| \leqslant \|f\| \sum_{J\in\mathcal{A}(n,k_0)} g_n \circ w_J(x) \leqslant c\delta^{k_0}\|f\|.$$

Define

$$c_J(x) = \begin{cases} 1, & \text{if } J \in \mathcal{A}(n,k_0),\\ \dfrac{g_n \circ w_J(x)}{\upsilon(X_J)}, & \text{if } J \notin \mathcal{A}(n,k_0). \end{cases}$$

By (3.14) and lemma 3.3, we have

$$a_{k_0}^{-1} \leqslant \frac{g_n \circ w_J(x)}{\upsilon(X_J)} \leqslant a_{k_0} \qquad \forall J \notin \mathcal{A}(n,k_0).$$

Then

$$a_{k_0}^{-1} \leqslant c_J(x) \leqslant a_{k_0} \qquad \forall J \in I^n, \quad x \in X.$$

Thus

$$\begin{aligned} |A_n + B_n(x)| &= \left| \sum_{|J|=n} c_J(x) \int_{X_J} f\,\mathrm{d}\upsilon \right| \\ &\leqslant (1 - a_{k_0}^{-2})\|f\| \left(\sum_{J\in\mathcal{A}(n,k_0)} \upsilon(X_J) + \sum_{J\notin\mathcal{A}(n,k_0)} g_n \circ w_J(x) \right) \quad \text{by lemma 3.4} \\ &\leqslant (1 - a_{k_0}^{-2})\|f\|(c\delta^{k_0} + 1) \quad \text{by (3.11) and (3.6)} \\ &\leqslant (1 - a_{k_0}^{-2} + c\delta^{k_0})\|f\|. \end{aligned}$$

Hence for any $n > k_0$,

$$\begin{aligned} |L_{\log g}^n E_n(f)(x)| &\leqslant |L_{\log g}^n E_n(f)(x) - B_n(x)| + |A_n| + |A_n + B_n(x)| \\ &\leqslant c\delta^{k_0}\|f\| + c\delta^{k_0}\|f\| + (1 - a_{k_0}^{-2} + c\delta^{k_0})\|f\| = \tau_1\|f\|. \end{aligned}$$

The claim is thus proved.

For $n \leqslant k_0$, we let

$$b = \exp\left(2\alpha_{\log h}(1) + \sum_{i=1}^{k_0} \alpha(\gamma_i)\right)$$

and let $\tau_2 = 1 - b^{-2}$. Then $0 < \tau_2 < 1$. Note that for any $n \leqslant k_0$,

$$S_n \circ w_J(x) \leqslant \sum_{i=1}^{k_0} \alpha(\gamma_i) + S_n \circ w_J(y) \qquad \forall\, J \in I^n \quad \text{and} \quad x, y \in X.$$

It follows from (3.7) and lemma 3.3 that

$$b^{-1} \leqslant \frac{g_n \circ w_J(x)}{\upsilon(X_J)} \leqslant b \qquad \forall J \in I^n \quad \text{and} \quad x \in X.$$

Then again by (3.2) and lemma 3.4, we have for any $n \leqslant k_0$,

$$|L^n_{\log g} E_n(f)(x)| \leqslant \tau_2 \|f\|.$$

Take $\tau = \max\{\tau_1, \tau_2\}$. Then $0 < \tau < 1$. From this, together with (3.15), it follows that

$$\|L^n_{\log g} E_n(f)\| \leqslant \tau \|f\| \qquad \forall n \geqslant 1.$$

By (3.1), we have $\|P_n E_n(f)\| \leqslant \tau \|f\|$. ■

To estimate the value of $S^{(n)}_{jn}$, we need another function. For this we let α be defined by (3.12), and define

$$\Phi(t) = \alpha(t) + \int_0^t \frac{\alpha(x)}{x}\,\mathrm{d}x, \qquad 0 \leqslant t \leqslant 1.$$

Then Φ is continuous and $\lim_{t\to 0^+} \Phi(t) = \Phi(0) = 0$ because of the Dini continuity of $\phi|_{X_j}$. We remark that for any $0 < \theta < 1$,

$$\sum_{n=0}^{\infty} \alpha(\theta^n t) \leqslant (1-\theta)^{-1}\Phi(t) \qquad \forall\, 0 \leqslant t \leqslant 1. \tag{3.16}$$

Let δ be as in the proof of proposition 3.5. Then we have the following proposition.

Proposition 3.6. *There exist $A > 0$ and $\ell_0 \in \mathbb{N}$ such that for any $\ell \geqslant \ell_0$,*

$$S^{(\ell)}_{j\ell} \leqslant A(\delta^{j\ell} + \Phi(\gamma_\ell)) \qquad \forall j \in \mathbb{N}.$$

Proof. Let $\Omega(n, k)$ be as in the proof of proposition 3.5. For any multi-index $J \in I^n$ and $x, y \in X$, let

$$\alpha_J(x, y) = 2\alpha_{\log h}(|x - y|) + \sum_{k=0}^{n} \alpha(|w_{J|^n_k}(x) - w_{J|^n_k}(y)|).$$

Denote $t = |x - y|$. Note (3.16) and the weak contraction of w_j. It follows that for any $J \in \Omega(n, k)$,

$$\sum_{k=0}^{n} \alpha(|w_{J|^n_k}(x) - w_{J|^n_k}(y)|) \leqslant (1-\theta)^{-1}\Phi(t) + (n-k)\alpha(t).$$

We know from theorems 4.2 and 4.4 of [LY] that

$$\sup_{t>0} \frac{\alpha_{\log h}(t)}{\Phi(t)} < \infty.$$

So there exists $c_1 > 0$ such that for any $J \in \Omega(n, k)$,

$$\alpha_J(x, y) \leqslant c_1\Phi(t) + (n-k)\alpha(t).$$

Thus combining with (3.7), we have

$$\frac{g_n \circ w_J(x)}{g_n \circ w_J(y)} \leqslant \exp(\alpha_J(x, y)) \leqslant \exp(c_1\Phi(t))(\exp(\alpha(t)))^{n-k}.$$

Hence for any $0 \leqslant k \leqslant n$, we have

$$\begin{aligned} E(n,k) &:= \sum_{J\in\Omega(n,k)} |g_n \circ w_J(x) - g_n \circ w_J(y)| \\ &\leqslant (\exp(c_1\Phi(t))(\exp(\alpha(t)))^{n-k} - 1) \sum_{J\in\Omega(n,k)} g_n \circ w_J(x) \\ &\leqslant (\exp(c_1\Phi(t))(\exp(\alpha(t)))^{n-k} - 1)\delta^{n-k} \qquad \text{by (3.9).} \end{aligned}$$

Take $t_0 > 0$ such that $\delta \exp(\alpha(t_0)) < 1$. Then for any $0 \leqslant t \leqslant t_0$,

$$\begin{aligned} \sum_{|J|=n} |g_n \circ w_J(x) - g_n \circ w_J(y)| &= \sum_{k=0}^{n} E(n,k) \\ &\leqslant \exp(c_1\Phi(t)) \sum_{k=0}^{n} (\delta \exp(\alpha(t)))^{n-k} - \sum_{k=0}^{n} \delta^{n-k} \leqslant S + \frac{\delta^{n+1}}{1-\delta} \end{aligned}$$

where

$$S := \frac{\exp(c_1\Phi(t))}{1-\delta\exp(\alpha(t))} - \frac{1}{1-\delta}.$$

Note that $\alpha(0) = \Phi(0) = 0$ and $\alpha \leqslant \Phi$. There exists $c_2 > 0$ such that for any $0 \leqslant t \leqslant t_0$,

$$\exp(c_1\Phi(t)) \leqslant 1 + c_2\Phi(t) \qquad \text{and} \qquad \exp(\alpha(t)) \leqslant 1 + c_2\Phi(t).$$

By using this, it follows that

$$S \leqslant \frac{1 + c_2\Phi(t)}{1-\delta\exp(\alpha(t))} - \frac{1}{1-\delta} \leqslant c_3\Phi(t) \qquad \text{for some } c_3 > 0.$$

Take $A = \max\{c_3, \delta(1-\delta)^{-1}\}$. Then for $0 \leqslant t \leqslant t_0$, we have

$$\sup_{|x-y|\leqslant t} \sum_{|J|=n} |g_n \circ w_J(x) - g_n \circ w_J(y)| \leqslant A(\delta^n + \Phi(t)) \qquad \forall n \in \mathbb{N}.$$

By proposition 2.2(ii), there exists an integer $\ell_0 > 0$ such that $\gamma_\ell \leqslant \gamma_{\ell_0} \leqslant t_0$ for any $\ell \geqslant \ell_0$. Hence in particular

$$S_{j\ell}^{(\ell)} \leqslant A(\delta^{j\ell} + \Phi(\gamma_\ell)) \qquad \forall j \in \mathbb{N}.$$

■

4. Decay of correlations

Having proved propositions 3.5 and 3.6, we are ready to estimate convergence speeds of the iterations $L^n_{\log g}(f)$.

Theorem 4.1. *There exist constants $B > 0$, $0 < \lambda < 1$ and $\ell_0 \in \mathbb{N}$ such that for any $n \geqslant k\ell$ with $\ell \geqslant \ell_0$ and $f \in C(X)$ with $\int f \, d\upsilon = 0$, we have*

$$\|L^n_{\log g} f\| \leqslant \alpha_f(\gamma_\ell) + \lambda^k \|f\| + B(\lambda^\ell + \Phi(\gamma_\ell))\|f\|.$$

Proof. Let $0 < \tau < 1$ be given by proposition 3.5. Let $A > 0$, $0 < \delta < 1$ and $\ell_0 > 0$ be as in proposition 3.6. Take $\lambda := \max\{\tau, \delta\}$. Then $0 < \lambda < 1$. Thus for any $j \geqslant 1$ and $\ell \geqslant \ell_0$,

$$S_{j\ell}^{(\ell)} \leqslant A(\lambda^{j\ell} + \Phi(\gamma_\ell)) \leqslant A(\lambda^\ell + \Phi(\gamma_\ell)).$$

For any $f \in C(X)$ with $\int f \, d\upsilon = 0$, by using repeatedly $\|P_n E_n(f)\| \leqslant \lambda \|f\|$, we deduce that

$$\left\| \left(\prod_{i=1}^{j} P_{i\ell} E_{i\ell} \right)(f) \right\| \leqslant \lambda^j \|f\| \qquad \forall\, j \geqslant 1.$$

Let $B = A\sum_{j=0}^{\infty}\lambda^j$. Hence by lemma 3.1 we have

$$\|P_{k\ell}(f)\| \leqslant \alpha_f(\gamma_\ell) + \sum_{j=2}^{k-1} S^{(\ell)}_{(j-1)\ell}\cdot\lambda^{j-2}\|f\| + \lambda^k\|f\|$$
$$\leqslant \alpha_f(\gamma_\ell) + \lambda^k\|f\| + B(\lambda^\ell + \Phi(\gamma_\ell))\|f\|.$$

Note that (3.1) and $\|L^n_{\log g} f\| \leqslant \|L^{k\ell}_{\log g} f\|\ \forall n \geqslant k\ell$. The result follows. ■

Let the correlation function Ψ_f be given by definition 2.6. We can now estimate the speed of $\Psi_f(n)$ converging to 0.

Theorem 4.2. *For any $f \in C(X)$ and $n \geqslant k\ell$ with $\ell \geqslant \ell_0$, we have*

$$|\Psi_f(n)| \leqslant \alpha_f(\gamma_\ell)\|f\| + 2\lambda^k\|f\|^2 + 2B(\lambda^\ell + \Phi(\gamma_\ell))\|f\|^2.$$

Proof. For any $f \in C(X)$, let $f_0 = f - \langle \upsilon, f\rangle$. Then

$$\langle \upsilon, f_0\rangle = 0,\ \|f_0\| \leqslant 2\|f\| \qquad \text{and} \qquad \alpha_{f_0}(\cdot) = \alpha_f(\cdot).$$

Thus, by theorem 4.1, we have for any $n \geqslant k\ell$ with $\ell \geqslant \ell_0$,

$$\|L^n_{\log g} f_0\| \leqslant \alpha_f(\gamma_\ell) + 2\lambda^k\|f\| + 2B\big(\lambda^\ell + \Phi(\gamma_\ell)\big)\|f\|.$$

It is easy to check that for any integer n,

$$L^n_{\log g}(f\circ T^n\cdot f)(x) = f(x)\cdot L^n_{\log g}(f)(x).$$

Hence

$$\langle \upsilon, (f\circ T^n)f\rangle = \langle \upsilon, L^n_{\log g}(f\circ T^n\cdot f)\rangle = \langle \upsilon, f\cdot L^n_{\log g} f\rangle.$$

Therefore,

$$|\Psi_f(n)| = |\langle \upsilon, f(L^n_{\log g} f - \langle \upsilon, f\rangle)\rangle| = |\langle \upsilon, f\cdot L^n_{\log g} f_0\rangle| \leqslant \|f\|\|L^n_{\log g} f_0\|$$
$$\leqslant \alpha_f(\gamma_\ell)\|f\| + 2\lambda^k\|f\|^2 + 2B(\lambda^\ell + \Phi(\gamma_\ell))\|f\|^2.$$ ■

We remark that the Φ is Hölder continuous if each $\phi|_{X_j}$ is Hölder continuous. It is obvious that if T is strictly expansive, then the condition in the theorem is trivially satisfied. However, the explicit value of ϱ is difficult to find. A simple estimation on the lower bound of ϱ is

$$\min_{x\in X}\sum_{y\in T^{-1}x}\exp(\phi(y)) \leqslant \varrho.$$

By using this we have the following corollary.

Corollary 4.3. *Suppose that the weakly expansive Dini dynamical system (X, T, ϕ) satisfies the condition*

$$\max_{x\in X}\sum_{y\in T^{-1}x}\exp(\phi(y))(r(y))^{-1} < \min_{x\in X}\sum_{y\in T^{-1}x}\exp(\phi(y)).$$

Then there exist constants $C > 0$, $0 < \lambda < 1$ and $\ell_0 \in \mathbb{N}$ such that for any $f \in C(X)$ and $n \geqslant k\ell$ with $\ell \geqslant \ell_0$,

$$|\Psi_f(n)| \leqslant \alpha_f(\gamma_\ell)\|f\| + C\big(\lambda^k + \lambda^\ell + \Phi(\gamma_\ell)\big)\|f\|^2.$$

Example 4.4. *Let* $X = [0, 1]$ *and*

$$T(x) = \begin{cases} \dfrac{x}{1-x}, & \textit{if } x \in \left[0, \frac{1}{2}\right], \\ 2x - 1, & \textit{if } x \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Choose $a > 0$ *such that* $a + (\log 2)^{-2} < 1$. *Define* $\phi : X \to \mathbb{R}$ *by*

$$\phi(x) = \begin{cases} \log(a + (\log T(x))^{-2}), & \textit{if } x \in \left[0, \frac{1}{2}\right], \\ \log(1 - a - (\log T(x))^{-2}), & \textit{if } x \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

It is clear that $\phi|_{[0,1/2]}$ *is not Hölder continuous, and the system* (X, T, ϕ) *satisfies the condition of corollary 4.3. The system, however, does not have the BDP. We can check that* $\gamma_n = (1+n)^{-1}$, *and* $\Phi(t) = \mathrm{O}(-1/\log t)$ *near* 0. *Hence, for any Hölder continuous function* f *on* X, *we have* $|\Psi_f(n^2)| = \mathrm{O}(1/\log n)$.

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References

[Bal] Baladi V 2000 *Positive Transfer Operators and Decay of Correlations* vol 16 (Singapore: World Scientific)

[Bow] Bowen R 1975 *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms* (*Lecture Notes in Mathematics* vol 470) (Berlin: Springer)

[Boy] Boyarsky A 1997 *Laws of Chaos, Invariant Measures and Dynamical Systems in One Dimension* (Basle: Birkhäuser)

[FJ] Fan A H and Jiang Y P 2001 On Ruelle–Perron–Frobenius operators: II. Convergence speeds *Commun. Math. Phys.* **223** 143–59

[FP] Fan A H and Pollicott M 2000 Non-homogeneous equilibrium states and convergence speeds of averaging operators *Math. Proc. Camb. Phil. Soc.* **129** 99–115

[Li] Liverani C 1995 Decay of correlations *Ann. Math.* **142** 239–301

[LY] Lau K S and Ye Y L 2001 Ruelle operator with nonexpansive IFS *Studia Math.* **148** 143–69

[Po] Pollicott M 2000 Rates of mixing for potentials of summable variation *Trans. Am. Math. Soc.* **352** 843–53

[Ru] Ruelle D 1968 Statistical mechanics of a one-dimensional lattice gas *Commun. Math. Phys.* **9** 267–78

[W1] Walters P 1975 Ruelle's operator theorem and g-measure *Trans. Am. Math. Soc.* **214** 375–87

[W2] Walters P 1982 *An Introduction to Ergodic Theory* (Berlin: Springer)

[Yo] Young L S 1999 Recurrence times and rates of mixing *Israel J. Math.* **110** 153–88

[Yu] Yuri M 1996 Decay of correlations for certain multi-dimensional maps *Nonlinearity* **9** 1439–61