Ambiguity aversion and optimal derivative-based pension investment with stochastic income and volatility

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Abstract

This paper provides a derivative-based optimal investment strategy for an ambiguityaverse pension investor who faces not only risks from time-varying income and market return volatility but also an uncertain economic condition over a long time horizon. We derive a robust dynamic derivative strategy and show that the optimal strategy under ambiguity aversion reduces the exposures to the market return risk and the volatility risk, and the investor holds opposite positions in stock and derivative in her optimal portfolio. In the presence of derivative, there are distinct effects of ambiguity on the optimal investment strategy. More importantly, we demonstrate the utility improvement when considering ambiguity and exploiting derivatives and show that ambiguity aversion and derivative trading improve the utility significantly when return volatility increases and that the improvement becomes more significant under ambiguity aversion over a long investment horizon.

JEL classification: C61; G11; G22

Keywords: Robust portfolio choice; DC pension plan; Ambiguity; Derivative; Stochastic volatility; Stochastic salary

1. Introduction

Pension funds hold a significant share of the global market portfolio. Global institutional pension fund assets in 22 major markets are about \$36.4 trillion and increased 4.3% in 2016, and the total pension assets in these countries amount to 62% of their GDP¹. Therefore,

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¹The data are from Willis Towers Watson's Global Pension Assets Study 2017: https://www.willistowers watson.com/en/insights/2017/01/global-pensions-asset-study-2017.

pension investment has become increasingly important. Derivatives enjoy increasing popularity in pension investment and investors are often ambiguity averse. In this paper, we combine these two features and provide a derivative-based optimal investment strategy for an ambiguity-averse pension investor. The investor considers a market with stochastic volatility and faces uncertainties concerning both salary income and economic conditions over a long time horizon. We show that ambiguity-aversion reduces the exposure to market return and volatility risk. In the presence of a derivative, the investor holds opposite positions in stock and derivative. There are distinct effect of the ambiguity over market return risk and stochastic volatility risk on the optimal investment strategy: ambiguity concerning market return risk always reduces both the investment into the stock and the derivative; ambiguity concerning volatility risk reduces the investment into the derivative while increases the investment into the stock. Our analysis further shows that ambiguity aversion and derivative trading improve investors' utility significantly, especially when the return volatility is high or the time horizon is long.

Motivated by recent studies on pension investment, this paper provides an integrated framework for studying an optimal derivative-based pension investment problem. We address various market risks and uncertainties, including market return and stochastic volatility risks and income and economic uncertainties. There are two types of pension funds that are differentiated by their benefit and contribution characteristics: defined benefit (DB) and defined contribution (DC) pension plans. Due to demographic change and development of financial markets, there is an ongoing shift from DB to DC pension plans. Many countries have shifted their pension schemes toward DC plans to ease the pressure on social security programs and transfer investment risk to investors (Poterba et al., 2007). DC pension plans are playing an increasingly important role. As a result, individuals who build their own DC pension funds have been exposed to these risks and uncertainties.

This paper explores various aspects of intertemporal portfolio choices under risk and uncertainty in DC pension plans. In particular, in long pension investment horizon, wealth accumulation depends on investors' contribution which in turn depends on their salary income, and financial market returns. Investors face model instability (structural change of the model economy) and asset return variability. The experimental studies (Bossaerts et al., 2010) demonstrate that investors are averse not only to risk (the known probability distribution) but also to ambiguity (the unknown probability distribution). Also, it is well recognized (Anderson et al., 1999; Merton, 1980) that expected returns are extremely difficult to estimate, thus investors are skeptical of the reliability of standard historical estimates. Therefore, it becomes increasingly important to take ambiguity aversion into account. Moreover, long-term pension investments need to incorporate the risks of salary and the stochastic volatility of stock returns, which are well documented in the empirical literature. On the one hand, salary has significant effects on the optimal long-term portfolio choice of investors. Munk and Sørensen (2010) show that the relation between salary growth and interest rate remains a significant factor determining the optimal investment strategy. On the other hand, as an important improvement of the Black-Scholes model, stochastic volatility has been developed in the literature of option pricing, portfolio selection and related statistics (e.g., Heston, 1993; Kim et al., 1998; Fernndez-Villaverde et al., 2015; Campbell et al., 2016). In this paper, we also take stochastic salary and stochastic volatility into account and shows their effects on the optimal investment decisions.

This paper is also related to the use of derivatives for optimal investment. Theoretically, Liu and Pan (2003) develop an optimal investment strategy using derivatives with stochastic volatility and price jumps. They find that derivatives help to improve investors' utility. In practice, the derivative market is well developed and provides abundant opportunities for pension funds by offering efficient ways to cope with volatility risk. Derivatives are becoming more popular for pension funds in some countries. For example, for the second and third pillars of the UK pension funds, they are invested not only in capital markets such as stocks and bonds, but also in foreign option markets. In this paper, we follow this trend and consider the optimal investment strategy for a DC pension investor who is ambiguity averse and is able to invest in bond, stock, and derivative markets.

This paper is the first, to our knowledge, to explore the joint effect of ambiguity aversion and derivative trading on optimal pension investment and to examine their roles in improving utility. The main contributions of this paper are as follows. First, we derive an optimal investment strategy for the underlying asset and its derivative in a DC pension plan. As noted by Liu and Pan (2003), derivative trading is essential for improving investors' utility. Generally, we investigate two models with and without the derivative. By comparing the two results, we find that trading in derivatives leads to utility improvement by offering additional investment opportunities. Second, after solving the model explicitly, we show that ambiguity aversion affects investor's risk sharing in both the myopic and hedging components. Moreover, the exposures of investor to both market return risk and volatility risk reduce with respect to ambiguity. But for the explicit investment strategies, ambiguity concerning market return risk always reduces both the investment into the stock and the derivative; ambiguity concerning volatility risk reduces the investment into the derivative while increases the investment into the stock. Third, in the DC pension investment, we find that the optimal investment strategy has an additional hedging component that addresses salary risk. In our model, salary risk generates different effects on investor's exposures to market return risk and volatility risk. Finally, we provide an original theoretical proof to show that the optimization problem is well posed, which is ignored in the existing literature. We also present the verification theorems to guarantee the validity of the results.

This paper is related to three strands of the literature. The first strand is on the asset allocation of DC pension funds. Given the widespread use of DC pension plans in practice, there is extensive literature addressing the asset allocation problems of DC pension funds. The existing literature adopts a variety of objectives, such as maximizing the expected utility of terminal wealth (see Blake et al., 2013, 2014; Chen et al., 2017; Deelstra et al., 2004; Emms, 2012; Giacinto et al., 2011) and the mean-variance criterion (see He and Liang, 2013; Sun et al., 2016; Wu and Zeng, 2015).² In the DC pension plan, human capital constitutes an indispensable part of investors' wealth. Therefore, the uncertainty against future salary is considered as a typical background risk. Several scholars have conducted research on portfolio choices with salary risk (e.g., Bodie et al., 1992; Bodie et al., 2004). To explore the effect of stochastic salary on investor's investment behavior, we assume that the salary process follows a general stochastic process and derive an optimal strategy explicitly. We find that, the correlation between the salary and market return/volatility risks results in distinct effects; as salary risk increases, the investor reduces stock investment while short-sells more derivatives.

The second strand of the literature explores certain potentials and roles of derivative trad-

²These papers explore different aspects of factors in the investment of DC pension plans. In the framework of maximizing utility, Deelstra et al. (2004) study an optimal design of guarantees in DC plans. Giacinto et al. (2011) investigate a model of optimal allocation for a DC pension plan with a minimum guarantee. Blake et al. (2013, 2014) use numerical algorithms to solve optimal investment problems under S-shaped utility and Epstein-Zin utility, respectively. Chen et al. (2017) adopt an S-shaped utility to describe the investor's preferences and obtain the optimal investment strategy in closed-form. Under the mean-variance criterion, He and Liang (2013) study a portfolio model for the DC pension plan during the accumulation phase and derive a time-consistent investment strategy within the game theoretic framework. Wu and Zeng (2015) consider the effects of mortality risk on equilibrium strategies. Sun et al. (2016) investigate an optimal investment problem for DC pensions with a jump-diffusion model.

ing in managing stochastic volatility in DC pension plans. There is considerable empirical evidence on time-varying stock return volatility (see Taylor, 1994, for a survey). Following Ílhan et al. (2005) and Liu and Pan $(2003)^3$, Hsuku (2007) studies a dynamic consumption and asset allocation problem with derivative securities under a recursive utility function. Jalal (2013) derives dynamic option-based investment strategies for an investor exhibiting downside loss aversion and provides illustrative results when downside risk is measured by the expected shortfall. Recently, Escobar et al. (2015) consider an optimal investment strategy for an ambiguity-averse investor who can invest in stock and derivative markets. However, there are very limited results on dynamic asset allocation with derivatives in pension investment, despite the increasing popularity of using derivatives in the pension investment market. According to a report by the Singapore Exchange (SGX) from January 6, 2015, the value of securities trading fell 25%, while derivative trading volume rose to a record high in 2014. In the pension market, pension funds have increased their use of derivatives over the past decade. The 2012 NAPF Annual Survey shows that 57% of member schemes include derivatives. Moreover, the Global Pension Assets Study 2016 reports that at the end of 2015, the average global asset allocation of the seven largest markets (Australia, Canada, Japan, the Netherlands, Switzerland, the UK and the US) is 44% equities, 29% bonds, 3% cash and 24% other assets, mainly in derivatives. In this paper, we allow the DC pension investor to invest in a derivative market. By examining cases with and without the derivatives in the portfolios, we show that the use of derivatives always improves investor's utility.

The third strand is on ambiguity in portfolio selection. Ellsberg (1961) first states that most people are ambiguity averse. Then there are numerous theoretical and empirical studies that explore the significance of ambiguity in affecting investor behavior (Bossaerts et al., 2010; Cao et al., 2005; Dimmock et al., 2016, etc). Recent studies consider investment problems with ambiguity and robust decisions. Anderson et al. (2003) develop a constrained worst-case model and derive a robust decision. The model helps the decision maker to assess the fragility of any given decision rule. Maenhout (2004, 2006) also derive the optimal

³Specifically, Liu and Pan (2003) study the optimal investment strategies when an investor has access not only to bond and stock markets but also to a derivative market and provide an example of the role of derivatives in the presence of volatility risk. They find that derivative trading helps to improve investors' utility. Ílhan et al. (2005) investigate an optimal investment problem for an investor who maximizes the expected exponential utility from terminal wealth, combining a static position in derivatives with a traditional dynamic trading strategy in stocks.

investment strategy for an investor who is ambiguity averse with respect to expected market returns. Following Maenhout (2004), some studies address the implications of ambiguity for portfolio choice. For example, Liu (2010) examines an optimal consumption and investment problem for an ambiguity-averse investor with time-varying investment opportunities. Branger and Larsen (2013) consider the optimal portfolio choice under different degrees of ambiguity aversion concerning jump and diffusion risks. Flor and Larsen (2014) obtain an optimal investment strategy for an ambiguity-averse investor in the context of a stochastic interest rate. Munk and Rubtsov (2014) consider a portfolio management problem for an ambiguity-averse investor under stochastic interest risk and inflation risk. Zheng et al. (2016) consider a robust optimal investment-reinsurance problem using a constant elasticity of variance (CEV) model and explicitly solve the case of an exponential utility function. Luo (2016) studies the strategic consumption-portfolio rules with information frictions and salary risk. Our work is related to their works and makes several extensions about ambiguity and portfolio choice.

By considering ambiguity aversion, this paper provides a theoretical explanation of the portfolio choice puzzle of "low portfolio fractions allocated to equity" in the empirical literature (Dimmock et al., 2016). We further explore the distinct effects of different ambiguity attitudes toward market return risk and volatility risk on the risk exposures and investment proportions, respectively. In the presence of a derivative, we show that ambiguity always reduce the derivative investment (in absolute terms), while their effect on stock investment is uncertain. As ambiguity can be regarded as the decrease in the volatility risk premium, derivative investment becomes less attractive. By considering salary risk, our model for DC pension investment is much richer than the classical type of deterministic contribution. A stochastic salary stipulates an exogenous income stream. This makes it difficult to solve the optimization problem. In this paper, we derive a closed-form of the robust investment strategy for DC pension plans (with a stochastic salary). As in Anderson et al. (2003) and Maenhout (2004), the discrepancy between the reference model and the alternative models is defined in terms of relative entropy, which serves as a penalty and quantifies the investor's degree of ambiguity aversion about the reference model. The investor aims to maximize the expected utility from the terminal wealth at retirement. Using the robust control approach. the robust optimal investment strategy is derived in closed-form. In conclusion, our results complement the existing literature.

This paper provides some insights into the efficient investment of DC pension plans.

First, to improve pension funds' investment performance, derivatives can provide an efficient means of diversifying various risk factors. Because the DC pension investment horizon is long, volatility risk has a significant effect on portfolio selection, and derivatives can be very useful to manage such risk. We show that irrespective of ambiguity aversion, utility is always improved by using derivatives. Second, it is desirable to improve the cognitive ability of investors to reduce uncertainty. Recommendations from professional research reports or financial experts would be helpful for investors' investment decisions, especially when facing economic uncertainties. Third, the heterogeneous salary process faced by investors in reality results in different behaviors and has a significant effect on investment strategy. Paying attention to the salary process is necessary for the design of the DC pension plan.

The paper is organized as follows. Section 2 describes the model. Section 3 derives the explicit expressions of the robust optimal risk exposures, investment strategies and the corresponding optimal value function when the derivative is available. Section 4 provides the solutions without derivatives trading. Section 5 presents several numerical examples to illustrate the effects of the model parameters on the robust optimal investment strategy and utility improvements by considering ambiguity aversion and derivative trading. Section 6 concludes the paper.

2. Investment under ambiguity

We study the optimal investment strategy of a DC pension investor who can invest in a financial market consisting of a bond, a stock and a derivative of the stock. The stock price follows a stochastic volatility process. We assume that there are no transaction costs or taxes in the financial market and that trading occurs continuously. In addition to undertaking financial risk, the investor also receives a stochastic salary stream and faces salary risk during her working period. Moreover, she is ambiguity averse regarding both the dynamics of the stock and its stochastic volatility.

2.1. Financial market

The financial market consists of a risk-free bond, a stock and a derivative. The risk-free bond evolves according to

$$dS_0(t) = rS_0(t)dt, \quad S_0(0) = 1,$$
(1)

where r > 0 represents the risk-free interest rate. The stock price follows

$$dS(t) = S(t) \left[(r + \lambda_1 V(t)) dt + \sqrt{V(t)} dW_S(t) \right], \quad S(0) = s_0,$$
(2)

while the stock return variance V(t) is governed by

$$dV(t) = \kappa(\delta - V(t))dt + \sigma_V \sqrt{V(t)} \left(\rho_V dW_S(t) + \sqrt{1 - \rho_V^2} dW_V(t)\right), \quad V(0) = v_0, \quad (3)$$

where $W_S(t)$ and $W_V(t)$ are independent Brownian motions on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P})$ satisfying the usual conditions; T > 0 is a finite constant representing the investment time horizon (retirement date); \mathcal{F}_t denotes the information available until time t; and \mathbb{P} is a reference measure. All stochastic processes throughout this paper are assumed to be well defined and adapted to this probability space. In this model, the instantaneous variance process V(t) is a stochastic process with long-run mean $\delta > 0$, meanreversion rate $\kappa > 0$, and volatility coefficient $\sigma_V > 0$. The price and volatility are correlated, which is captured by the coefficient $\rho_V \in (-1, 1)$ and represents an important feature of the real data. λ_1 is a constant capturing the market price of the risk factor $W_S(t)$.

In addition to investing in the risk-free bond and the stock, the pension investor also has the opportunity to invest in the derivative with the risky asset as the underlying asset. Following Liu and Pan (2003), we consider the derivative with price O(t, S(t), V(t)), (O(t)for short) at time t that depends on the underlying price of the stock S(t) and its volatility V(t), and its payoff structure at the expiration time τ is defined by $O(\tau) = f(S(\tau), V(\tau))$ for some function f. As in the literature, such as Liu and Pan (2003), the derivative includes most traded option types.⁴ To obtain the price of the derivative, we introduce a specific pricing kernel $\{k(t)\}_{t\in[0,T]}$ to price all of the risk factors in the financial market,⁵

$$O(t) = \frac{1}{k(t)} \mathbb{E}_t[k(\tau) f(S(\tau), V(\tau))]$$
(4)

for any $t \leq \tau$. This is consistent with the following parametric pricing kernel and specification of the price dynamics for the derivative (Liu and Pan, 2003)

$$dk(t) = -k(t) \left[r dt + \lambda_1 \sqrt{V(t)} dW_S(t) + \lambda_2 \sqrt{V(t)} dW_V(t) \right], \quad k(0) = 1,$$
(5)

⁴As shown in Liu and Pan (2003), the expiration date τ of the derivative does not need to match the investment horizon T. They present some examples of derivative types. For instance, a derivative with a linear payoff structure $f(S(\tau), V(\tau)) = S(\tau)$ becomes the stock itself. However, for some strike price K > 0, a derivative with the non-linear payoff structure $f(S(\tau), V(\tau)) = (S(\tau), V(\tau)) = (S(\tau) - K)^+$ corresponds to European-style call option, while that with $f(S(\tau), V(\tau)) = (K - S(\tau))^+$ corresponds to European-style put option.

⁵The price of the derivative is defined under measure \mathbb{P} , which is deduced from a risk-neutral measure. Details can be found in Zhang et al. (2012).

and

$$dO(t) = rO(t)dt + (O_sS(t) + \sigma_V\rho_VO_v)\left(\lambda_1V(t)dt + \sqrt{V(t)}dW_S(t)\right) + \sigma_V\sqrt{1 - \rho_V^2}O_v\left(\lambda_2V(t)dt + \sqrt{V(t)}dW_V(t)\right), \quad t \le \tau,$$
(6)
$$O(\tau) = f(S(\tau), V(\tau)),$$

where λ_2 is a constant capturing the market price of stochastic volatility risk $W_V(t)$, O_s and O_v are the partial derivatives of O with respect to (w.r.t.) S(t) and V(t), respectively.

In a DC pension plan, the investor contributes part of her salary to the pension fund before retirement. The salary process is essential when considering a DC pension plan. In this paper, we assume that the dynamics of the investor's salary are described by

$$\begin{cases} dL(t) = L(t) \left[\mu_L dt + \sigma_L \rho_L \left(\lambda_1 V(t) dt + \sqrt{V(t)} dW_S(t) \right) + \sigma_L \sqrt{1 - \rho_L^2} \left(\lambda_2 V(t) dt + \sqrt{V(t)} dW_V(t) \right) \right], \\ L(0) = l_0, \end{cases}$$
(7)

where $\mu_L \ge 0$ is the appreciation rate, $\sigma_L \ge 0$ is the volatility and $\rho_L \in [-1, 1]$ is the coefficient parameter.

Remark 2.1. The salary process plays an important role in pension plans and is analyzed in several studies (Bodie et al., 2004; Chen et al., 2017; Deelstra et al., 2004; Dybvig and Liu, 2010; Guan and Liang, 2014, 2015). Among these contributions, Bodie et al. (2004) and Dybvig and Liu (2010) assume that the salary process is spanned by the stocks in the financial market, which reflects the fact that salary is related to the profitability of the company. Guan and Liang (2014) furthermore assume that the salary process is correlated with the volatility of the stock. In those cases, salary risk is insurable in the stock market. Because the stochastic volatility contains some other risks faced by the investor in our model, we assume the salary to be related to stochastic volatility. It would be interesting and more realistic if the stochastic salary is introduced extra randomness independent of the Brownian motions driving the stock and volatility. However, if does, the part related to l^2 can not be separated in HJB equation, and it becomes difficult for us to derive closed-form solutions to our optimization problems, significantly complicating the analysis of the problems. In this paper, our main focus is to analyze the impacts of ambiguity aversion and derivation trading on the optimal investment strategies and value functions of a DC pension investor. The closed-form solutions is our theoretical results, and with them, we are able to analyze the impacts explicitly.

2.2. Ambiguity

The above-mentioned framework is a traditional portfolio choice model in the DC pension plan, where the investor is assumed to be ambiguity neutral. However, in reality, the investor is usually ambiguity averse and wants to guard herself against worst-case scenarios. To incorporate ambiguity aversion into the investor's investment problem, we assume that the reference model capturing the knowledge of the investor's ambiguity is described by the probability measure \mathbb{P} , but she is skeptical of this reference model and is willing to consider some alternative models, defined by a class of probability measures equivalent to \mathbb{P} as follows (cf. Anderson et al., 2003; Maenhout, 2004):

 $\mathcal{Q} := \{ \mathbb{Q} | \mathbb{Q} \sim \mathbb{P} \}.$

For each $\Phi := \{\phi(t) := (\phi_S(t), \phi_V(t))\}_{t \in [0,T]}$,⁶ define a real-valued process $\{\Lambda^{\Phi}(t) | t \in [0,T]\}$ as

$$\Lambda^{\Phi}(t) = \exp\left\{-\int_{0}^{t} \phi_{S}(s) \mathrm{d}W_{S}(s) - \frac{1}{2}\int_{0}^{t} (\phi_{S}(s))^{2} \mathrm{d}s - \int_{0}^{t} \phi_{V}(s) \mathrm{d}W_{V}(s) - \frac{1}{2}\int_{0}^{t} (\phi_{V}(s))^{2} \mathrm{d}s\right\}$$
(8)

Accordingly, $\Lambda^{\Phi}(t)$ is a \mathbb{P} -martingale. For each Φ , a new alternative measure \mathbb{Q} that is absolutely continuous with \mathbb{P} on \mathcal{F}_T is defined by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_T} = \Lambda^{\Phi}(T).$$

By Girsanov's Theorem, under the alternative measure \mathbb{Q} , we have

$$dW_S^{\Phi}(t) = dW_S(t) + \phi_S(t)dt,$$

$$dW_V^{\Phi}(t) = dW_V(t) + \phi_V(t)dt,$$

where $W_S^{\Phi}(t)$ and $W_V^{\Phi}(t)$ are one-dimensional standard Brownian motions. Furthermore, the price and volatility of the stock, the price of the derivative and the stochastic salary under

⁶Suppose that $\Phi := \{\phi(t) := (\phi_S(t), \phi_V(t))\}_{t \in [0,T]}$ satisfies three conditions: (i) $\phi_S(t)$ and $\phi_V(t)$ are \mathcal{F}_{t} measurable for each $t \in [0,T]$; (ii) $\mathbb{E}\left\{\exp\left\{\frac{1}{2}\int_0^T \left[(\phi_S(t))^2 dt + (\phi_V(t))^2\right] dt\right\}\right\} < \infty$; and (iii) $|\phi(t)|^2 \leq \underline{\kappa}^2 V(t)$ for a.s. $(t,\omega) \in [0,T] \times \Omega$, with constant $\underline{\kappa} \in [\max(\underline{\phi}, \underline{\phi}_3), \kappa/\sigma_V)$, where $\underline{\phi}$, and $\underline{\phi}_3$ are defined in (22) and
(43), respectively. And we will explain $\underline{\phi}$ in footnote 9 and $\underline{\phi}_3$ in footnote 16 below. We denote Θ for the
space of all such processes Φ .

 \mathbb{Q} can be written as

$$dS^{\Phi}(t) = S^{\Phi}(t) \left[\left(r + \lambda_1 V^{\Phi}(t) - \phi_S(t) \sqrt{V^{\Phi}(t)} \right) dt + \sqrt{V^{\Phi}(t)} dW_S^{\Phi}(t) \right],$$

$$dV^{\Phi}(t) = \left[\kappa (\delta - V^{\Phi}(t)) - \sigma_V \sqrt{V^{\Phi}(t)} (\rho_V \phi_S(t) + \sqrt{1 - \rho_V^2} \phi_V(t)) \right] dt$$

$$+ \sigma_V \sqrt{V^{\Phi}(t)} (\rho_V dW_S^{\Phi}(t) + \sqrt{1 - \rho_V^2} dW_V^{\Phi}(t)),$$

$$(10)$$

$$\mathrm{d}O^{\Phi}(t) = rO^{\Phi}(t)\mathrm{d}t + \left(O_{s}S^{\Phi}(t) + \sigma_{V}\rho_{V}O_{v}\right)\left[\lambda_{1}V^{\Phi}(t)\mathrm{d}t - \phi_{S}(t)\sqrt{V^{\Phi}(t)}\mathrm{d}t + \sqrt{V^{\Phi}(t)}\mathrm{d}W^{\Phi}_{S}(t)\right]$$

$$+ \sigma_V \sqrt{1 - \rho_V^2} O_v \left[\lambda_2 V^{\Phi}(t) \mathrm{d}t - \phi_V(t) \sqrt{V^{\Phi}(t)} \mathrm{d}t + \sqrt{V^{\Phi}(t)} \mathrm{d}W_V^{\Phi}(t) \right], \tag{11}$$

$$dL^{\Phi}(t) = L^{\Phi}(t) \left[\mu_L dt + \sigma_L \rho_L(\lambda_1 V^{\Phi}(t) dt - \phi_S(t) \sqrt{V^{\Phi}(t)} dt + \sqrt{V^{\Phi}(t)} dW_S^{\Phi}(t)) + \sigma_L \sqrt{1 - \rho_L^2} (\lambda_2 V^{\Phi}(t) dt - \phi_V(t) \sqrt{V^{\Phi}(t)} dt + \sqrt{V^{\Phi}(t)} dW_V^{\Phi}(t)) \right].$$
(12)

2.3. Wealth process

Let $\mathbf{u} := \{u(t) := (u_S(t), u_O(t))\}_{t \in [0,T]}$ be a trading strategy, and $X^{\mathbf{u}}(t)$ is the wealth process under strategy \mathbf{u} , where $u_S(t), u_O(t), 1 - u_S(t) - u_O(t)$ are the proportions of the wealth invested in the stock, derivative and risk-free bond, respectively. Then, the wealth process $X^{\mathbf{u}}(t)$ under probability measure \mathbb{P} follows

$$\begin{cases} dX^{\mathbf{u}}(t) = X^{\mathbf{u}}(t) \left[(1 - u_S(t) - u_O(t)) \frac{dS_0(t)}{S_0(t)} + u_S(t) \frac{dS(t)}{S(t)} + u_O(t) \frac{dO(t)}{O(t)} \right] + \xi L(t) dt \\ = X^{\mathbf{u}}(t) \left[r dt + \theta_S(t) \left(\lambda_1 V(t) dt + \sqrt{V(t)} dW_S(t) \right) \right. \\ \left. + \theta_V(t) \left(\lambda_2 V(t) + \sqrt{V(t)} dW_V(t) \right) \right] + \xi L(t) dt, \\ X^{\mathbf{u}}(0) = x_0, \end{cases}$$

$$(13)$$

where

$$\theta(t) = \begin{pmatrix} \theta_S(t) \\ \theta_V(t) \end{pmatrix} = \begin{pmatrix} 1 & \frac{O_s S(t) + \sigma_V \rho_V O_v}{O(t)} \\ 0 & \frac{\sigma_V \sqrt{1 - \rho_V^2 O_v}}{O(t)} \end{pmatrix} \begin{pmatrix} u_S(t) \\ u_O(t) \end{pmatrix}$$
(14)

represent the investor's exposures to market return risk $W_S(t)$ and additional volatility risk $W_V(t)$, respectively. Here, we consider the exposures instead of portfolio weights to simplify the analysis.⁷As shown in Liu and Pan (2003), the exposure stems from the dynamics of asset prices and the specific portfolio.

⁷We also provide the non-redundant condition as shown in Eq. (3.3) in Escobar et al. (2015) and Eq. (12) in Liu and Pan (2003). Because we have only one derivative in the model and the relationship between risk exposure and the portfolio weight is shown by Eq. (14), the non-redundant condition becomes $\sqrt{1-\rho_V^2}O_v \neq 0.$

In addition, we assume that the contribution rate of the salary is $\xi \in [0, 1]$. Then under the ambiguity framework, the wealth process $X^{\Phi, \mathbf{u}}(t)$ under probability measure \mathbb{Q} follows

$$dX^{\Phi,\mathbf{u}}(t) = X^{\Phi,\mathbf{u}}(t) \left[rdt + \theta_S(t) \left(\lambda_1 V^{\Phi}(t) dt - \phi_S(t) \sqrt{V^{\Phi}(t)} dt + \sqrt{V^{\Phi}(t)} dW_S^{\Phi}(t) \right) + \theta_V(t) \left(\lambda_2 V^{\Phi}(t) dt - \phi_V(t) \sqrt{V^{\Phi}(t)} dt + \sqrt{V^{\Phi}(t)} dW_V^{\Phi}(t) \right) \right] + \xi L^{\Phi}(t) dt.$$
(15)

Definition 2.2. A strategy $\boldsymbol{u} = \{u(t) := (u_S(t), u_O(t))\}_{t \in [0,T]}$ is said to be admissible if (i) $u_S(t)$ and $u_O(t)$ are \mathcal{F}_t -progressively measurable processes;

(ii) $\forall (t, x, v, l) \in O$, Eq. (15) has a pathwise-unique solution $\{X^{\Phi, u}(t)\}_{t \in [0,T]}$, where $O := [0,T] \times \mathbb{R}^3$;

(*iii*) $\mathrm{E}^{\Phi}_{t,x,v,l}\{\int_{0}^{T}[(u_{S}(t))^{2} + (u_{O}(t))^{2}]\mathrm{d}t\} < \infty$ and $\mathrm{E}^{\Phi}_{t,x,v,l}[U(X^{\Phi,u}(s))] < \infty$, and

$$\mathbf{E}^{\Phi}_{t,x,v,l}[\cdot] = \mathbf{E}^{\Phi}[\cdot|(X^{\Phi,u}(t), V^{\Phi}(t), L^{\Phi}(t)) = (x, v, l)].$$

Denote by Π the set of all admissible strategies.

2.4. Optimization problem

In this paper, the pension investor is assumed to be risk averse with a constant relative risk aversion (CRRA) utility function and seeks to derive an investment strategy during the time interval [0, T] to maximize the expected utility from terminal wealth under the ambiguity framework. Then, the optimization problem for the investor can be written as⁸

$$\sup_{\mathbf{u}\in\Pi} \inf_{\Phi\in\Theta} \mathcal{E}^{\Phi} \left[U(X^{\Phi,\mathbf{u}}(T)) + \int_{0}^{T} \left(\frac{(\phi_{S}(s))^{2}}{2\Psi_{S}(s,x,v,l)} + \frac{(\phi_{V}(s))^{2}}{2\Psi_{V}(s,x,v,l)} \right) ds \right],$$
(16)

where

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma},\tag{17}$$

and γ is the coefficient of relative risk aversion. We assume that $\gamma > 1$ for practical relevance (see Branger and Larsen, 2013; Escobar et al., 2015; Flor and Larsen, 2014). The perturbations $\phi_S(t)$ and $\phi_V(t)$ in the penalty term are scaled by $\Psi_S(t, x, v, l)$ and $\Psi_V(t, x, v, l)$, respectively. $\Psi_S(t, x, v, l)$ and $\Psi_V(t, x, v, l)$ represent the preference parameters for ambiguityaversion, and measure the degree of confidence in the reference model \mathbb{P} at time t; and deviations from the reference measure are penalized by the last integral term in the expectation, which depends on the relative entropy arising from the diffusion risks. According to

⁸Following Anderson et al. (2003) and Maenhout (2004), the alternative models considered by the investor are difficult to distinguish statistically from the reference model. To take this issue into account, the value function includes a penalty term for deviating excessively from the reference model in the sense of relative entropy (the last integral term in the expectation in Eq. (16)), which arises from diffusion risk.

Maenhout (2004), the larger $\Psi_S(t, x, v, l)$ and $\Psi_V(t, x, v, l)$ are, the less the deviations from the reference model are penalized. Furthermore, the pension investor has less faith in the reference model, such that she is more likely to consider alternative models. Hence, the pension investor's ambiguity aversion is increasing w.r.t. $\Psi_S(t, x, v, l)$ and $\Psi_V(t, x, v, l)$.

Define

$$H(t, x, v, l) = \sup_{\mathbf{u} \in \Pi} H^{\Phi^*, \mathbf{u}}(t, x, v, l),$$
(18)

where

$$H^{\Phi^*,\mathbf{u}}(t,x,v,l) = \inf_{\Phi\in\Theta} H^{\Phi,\mathbf{u}}(t,x,v,l) = \inf_{\Phi\in\Theta} \mathcal{E}^{\Phi}_{t,x,v,l} \left[U(X^{\Phi,\mathbf{u}}(T)) + \int_{-t}^{T} \left(\frac{(\phi_S(s))^2}{2\Psi_S(s,x,v,l)} + \frac{(\phi_V(s))^2}{2\Psi_V(s,x,v,l)} \right) \mathrm{d}s \right].$$
(19)

For analytical tractability, we assume that (cf. Pathak, 2002; Branger and Larsen, 2013; Escobar et al., 2015; Flor and Larsen, 2014; Maenhout, 2004)

$$\Psi_S(t, x, v, l) = \frac{\beta_S}{(1 - \gamma)H(t, x, v, l)}, \quad \Psi_V(t, x, v, l) = \frac{\beta_V}{(1 - \gamma)H(t, x, v, l)}, \quad (20)$$

where β_S and β_V are positive constants and called ambiguity aversion parameters; these are used to describe the investor's attitude toward ambiguity. We allow the level of ambiguity concerning the stock price to differ from that concerning the stock's volatility. For convenience, we abuse the notation slightly and interpret β_S as ambiguity aversion regarding market return risk and β_V as ambiguity aversion regarding additional volatility risk.

Proposition 2.3. There exists a unique value function H(t, x, v, l) of the optimal control problem that consists of (18), (19) and (20) subject to (15), (10) and (12).

Proof. See Appendix A.

Based Proposition (2.3), we define H(t, x, v, l) the optimal value function.

3. Optimal investment strategy with a derivative

This section is devoted to deriving the optimal investment strategy for the DC pension investor in the presence of a derivative. We first provide a closed-form solution to the case in which the investor is ambiguity averse in general and then analyze a special case without ambiguity aversion. For convenience, we introduce some notations. Let

$$C^{1,2,2,2}(O) = \{\psi(t,x,v,l) | \psi(t,\cdot,\cdot,\cdot) \text{ is once continuously differentiable on } [0,T]$$

and $\psi(\cdot, x, v, l)$ is twice continuously differentiable on \mathbb{R}^3 .

Let $u = (u_S, u_O)$, $\theta = (\theta_S, \theta_V)$ and $\phi = (\phi_S, \phi_V)$ denote the values that $u(t) = (u_S(t), u_O(t))$, $\theta(t) = (\theta_S(t), \theta_V(t))$ and $\phi(t) = (\phi_S(t), \phi_V(t))$ take, respectively. For any $(t, x, v, l) \in O$ and $\psi(t, x, v, l) \in C^{1,2,2,2}(O)$, we define an infinitesimal generator as

$$\begin{aligned} \mathcal{A}^{\phi,u}\psi(t,x,v,l) &= \psi_t + \left[rx + x\theta_S\lambda_1v + x\theta_V\lambda_2v - x\theta_S\phi_S\sqrt{v} - x\theta_V\phi_V\sqrt{v} + \xi l\right]\psi_x \\ &+ \left[\kappa(\delta-v) - \sigma_V\sqrt{v}\rho_V\phi_S - \sigma_V\sqrt{v}\sqrt{1-\rho_V^2}\phi_V\right]\psi_v \\ &+ \left[\mu_L l + l\sigma_L\lambda_1v\rho_L - l\sigma_L\sqrt{v}\phi_S\rho_L + l\sigma_L\lambda_2v\sqrt{1-\rho_L^2} - l\sigma_L\sqrt{v}\phi_V\sqrt{1-\rho_L^2}\right]\psi_l \\ &+ \frac{1}{2}x^2v(\theta_S^2 + \theta_V^2)\psi_{xx} + \frac{1}{2}\sigma_v^2v\psi_{vv} + \frac{1}{2}l^2\sigma_L^2v\psi_{ll} + l\sigma_Lv\sigma_V\left(\rho_V\rho_L + \sqrt{1-\rho_V^2}\rho_L^2\right)\psi_{lv} \\ &+ (x\sigma_V\theta_Sv\rho_V + x\sigma_V\theta_Vv\sqrt{1-\rho_V^2})\psi_{xv} + (x\theta_S l\sigma_L v\rho_L + x\theta_V l\sigma_L v\sqrt{1-\rho_L^2})\psi_{xl}, \end{aligned}$$

where ψ_t , ψ_x , ψ_v , ψ_l , ψ_{xx} , ψ_{vv} , ψ_{ll} , ψ_{lv} , ψ_{xv} and ψ_{xl} represent the partial derivatives of ψ w.r.t. the corresponding variables.

According to the principle of dynamic programming, the HJB equation with ambiguity aversion can be derived as (see Escobar et al., 2015; Maenhout, 2006; Yi et al., 2013)

$$\sup_{u\in\mathbb{R}^2} \inf_{||\phi||\leq\sqrt{\underline{\kappa}v}} \left\{ \mathcal{A}^{\phi,u} J(t,x,v,l) + \frac{\phi_S^2}{2\Psi_S} + \frac{\phi_V^2}{2\Psi_V} \right\} = 0$$
(21)

with the boundary condition J(T, x, v, l) = U(x).

The following proposition presents the conditions under which the solution of the HJB equation is indeed the value function, and the control is the optimal strategy.

Proposition 3.1. If there exist a function $J(t, x, v, l) \in C^{1,2,2,2}(O)$ and a control $(\mathbf{u}^*, \Phi^*) := \{(u^*(t), \phi^*(t))\}_{t \in [0,T]} \in \Pi \times \Theta$ such that (1) for any $||\phi|| \leq \sqrt{\underline{\kappa}v}$, $\mathcal{A}^{\phi,u^*}J(t, x, v, l) + \frac{\phi_S^2}{2\Psi_S} + \frac{\phi_V^2}{2\Psi_V} \geq 0$; (2) for any $u \in \mathbb{R}^2$, $\mathcal{A}^{\phi^*,u}J(t, x, v, l) + \frac{(\phi_S^*)^2}{2\Psi_S} + \frac{(\phi_V^*)^2}{2\Psi_V} \leq 0$; (3) $\mathcal{A}^{\phi^*,u^*}J(t, x, v, l) + \frac{(\phi_S^*)^2}{2\Psi_S} + \frac{(\phi_V^*)^2}{2\Psi_V} = 0$, with J(T, x, v, l) = U(x); (4) $\{J(\tau, x, v, l)\}_{\tau \in \mathfrak{T}}$ and $\{\frac{(\phi_S^*(\tau))^2}{2\Psi_S(\tau, x, v, l)} + \frac{(\phi_V^*(\tau))^2}{2\Psi_V(\tau, x, v, l)}\}_{\tau \in \mathfrak{T}}$ are uniformly integrable, where \mathfrak{T} denotes the set of stopping times $\tau \leq T$, $u^* = (u_S^*, u_O^*)$ and $\phi^* = (\phi_S^*, \phi_V^*)$ denote the values that $u^*(t) = (u_S^*(t), u_O^*(t))$ and $\phi^*(t) = (\phi_S^*(t), \phi_V^*(t))$ take, respectively. Then J(t, x, v, l) = H(t, x, v, l) and (\mathbf{u}^*, Φ^*) is an optimal control.

Proof. See Appendix B.

According to Proposition 3.1, we know that the optimal investment strategy is \mathbf{u}^* , the optimal risk exposure is

$$\theta^*(t) := \left(\theta^*_S(t), \theta^*_V(t)\right) = \left(\begin{array}{cc} 1 & \frac{O_s S(t) + \sigma_V \rho_V O_v}{O(t)} \\ 0 & \frac{\sigma_V \sqrt{1 - \rho_V^2 O_v}}{O(t)} \end{array}\right) u^*(t),$$

the worst-case measure is Φ^* , and the corresponding optimal value function is J(t, x, v, l) if Novikov's condition is satisfied, which is given below.

Theorem 3.2. For the robust portfolio choice problem (18) with wealth process (15), if the parameters satisfy certain technical conditions 9 , the optimal risk exposure is

$$\theta_{S}^{*}(t) = m(t) \left(1 + \bar{h}(t) \frac{L(t)}{X^{u^{*}}(t)} \right) - \sigma_{L} \rho_{L} \bar{h}(t) \frac{L(t)}{X^{u^{*}}(t)},$$

$$\theta_{V}^{*}(t) = n(t) \left(1 + \bar{h}(t) \frac{L(t)}{X^{u^{*}}(t)} \right) - \sigma_{L} \sqrt{1 - \rho_{L}^{2}} \bar{h}(t) \frac{L(t)}{X^{u^{*}}(t)};$$
(24)

the optimal investment strategy is

$$u_{S}^{*}(t) = \theta_{S}^{*}(t) - \frac{O_{s}S(t) + \sigma_{V}\rho_{V}O_{v}}{O(t)}u_{O}^{*}(t), \qquad u_{O}^{*}(t) = \frac{O(t)\theta_{V}^{*}(t)}{\sigma_{V}\sqrt{1 - \rho_{V}^{2}O_{v}}};$$
(25)

the corresponding optimal value function is

$$J(t, x, v, l) = \frac{(x + \bar{h}(t)l)^{1-\gamma}}{1-\gamma} \exp(\bar{g}(t)v + \hat{g}(t));$$
(26)

and the worst-case measure is given by

$$\phi_{S}^{*}(t) = \frac{\beta_{S}(\lambda_{1}(1-\gamma) + \sigma_{V}\rho_{V}\bar{g}(t))\sqrt{V(t)}}{(1-\gamma)(\beta_{S}+\gamma)}, \quad \phi_{V}^{*}(t) = \frac{\beta_{V}(\lambda_{2}(1-\gamma) + \sigma_{V}\sqrt{1-\rho_{V}^{2}}\bar{g}(t))\sqrt{V(t)}}{(1-\gamma)(\beta_{V}+\gamma)}$$
(27)

 $^9 {\rm The}$ technical conditions are $\underline{\phi} < \kappa^2/\sigma_V^2$ with

$$\underline{\phi} \triangleq \max\left\{\frac{\beta_{S}^{2}\lambda_{1}^{2}}{(\beta_{S}+\gamma)^{2}}, \frac{\beta_{S}^{2}(\lambda_{1}(1-\gamma)+\sigma_{V}\rho_{V}\bar{g}(0))^{2}}{(1-\gamma)^{2}(\beta_{S}+\gamma)^{2}}\right\} + \max\left\{\frac{\beta_{V}^{2}\lambda_{2}^{2}}{(\beta_{V}+\gamma)^{2}}, \frac{\beta_{V}^{2}(\lambda_{2}(1-\gamma)+\sigma_{V}\sqrt{1-\rho_{V}^{2}}\bar{g}(0))^{2}}{(1-\gamma)^{2}(\beta_{V}+\gamma)^{2}}\right\},$$
(22)

and for $\bar{g}(t) \in [\bar{g}(0), 0]$,

$$[64(1-\gamma)^2 - 4(1-\gamma)][(m(t))^2 + (n(t))^2] + 8(1-\gamma)A(t) \le \frac{\kappa^2}{2\sigma_V^2},$$
(23)

which are needed in the verification theorem. According to Dotsis et al. (2007) and Sepp (2008), who give the parameter estimates of the Heston model using the S&P500 index, we know that the value of κ^2/σ_V^2 in the technique conditions is very large (approximately 375.39). Therefore, more parameters can satisfy conditions (22) and (23). where $\{X^*(t)\}_{t\in[0,T]}$ is the wealth process under the corresponding optimal strategy, and

$$m(t) = \frac{\lambda_1 (1 - \gamma) + (1 - (\beta_S + \gamma))\sigma_V \rho_V \bar{g}(t)}{(1 - \gamma)(\beta_S + \gamma)},$$
(28)

$$n(t) = \frac{\lambda_2 (1 - \gamma) + (1 - (\beta_V + \gamma))\sigma_V \sqrt{1 - \rho_V^2} \bar{g}(t)}{(1 - \gamma)(\beta_V + \gamma)},$$
(29)

$$\bar{g}(t) = \frac{\nu_1 \nu_2 - \nu_1 \nu_2 e^{\alpha_2 (\nu_1 - \nu_2)(T-t)}}{\nu_2 - \nu_1 e^{\alpha_2 (\nu_1 - \nu_2)(T-t)}},$$
(30)

$$\hat{g}(t) = \int_{-t}^{T} \left[r(1-\gamma) + \kappa \delta \bar{g}(s) \right] \mathrm{d}s, \tag{31}$$

$$\bar{h}(t) = \frac{\xi}{\mu_L - r} (e^{(\mu_L - r)(T - t)} - 1),$$
(32)

$$\alpha_1 = -\kappa + \frac{\lambda_1 (1 - (\beta_S + \gamma))\sigma_V \rho_V}{\beta_S + \gamma} + \frac{\lambda_2 (1 - (\beta_V + \gamma))\sigma_V \sqrt{1 - \rho_V^2}}{\beta_V + \gamma}, \quad (33)$$

$$\alpha_{2} = \frac{\sigma_{V}^{2}}{2} - \frac{\beta_{S}\sigma_{V}^{2}\rho_{V}^{2} + \beta_{V}\sigma_{V}^{2}(1-\rho_{V}^{2})}{2(1-\gamma)} + \frac{(1-(\beta_{S}+\gamma))^{2}\sigma_{V}^{2}\rho_{V}^{2}}{2(\beta_{S}+\gamma)(1-\gamma)} + \frac{(1-(\beta_{V}+\gamma))^{2}\sigma_{V}^{2}(1-\rho_{V}^{2})}{2(\beta_{V}+\gamma)(1-\gamma)},$$
(34)

$$\alpha_3 = \frac{\lambda_1^2 (1 - \gamma)}{2(\beta_S + \gamma)} + \frac{\lambda_2^2 (1 - \gamma)}{2(\beta_V + \gamma)}, \quad \nu_{1,2} = \frac{\alpha_1 \pm \sqrt{\alpha_1^2 - 4\alpha_2 \alpha_3}}{-2\alpha_2}, \tag{35}$$

$$A(t) = \gamma(m(t))^2 - \frac{\sigma_V \rho_V \bar{g}(t)}{\beta_S + \gamma} m(t) + \gamma(n(t))^2 - \frac{\sigma_V \sqrt{1 - \rho_V^2} \bar{g}(t)}{\beta_V + \gamma} n(t).$$
(36)

Proof. See Appendix C.

Theorem 3.2 presents three features of our results. First, the components m(t) and n(t) in optimal risk exposures $\theta_S^*(t)$ and $\theta_V^*(t)$ consist of traditional components involving the myopic and hedging components. Taking exposure to market return risk $\theta_S^*(t)$ as an example, the myopic component $\frac{\lambda_1}{\beta_S + \gamma}$ is constant and decreases in the ambiguity aversion parameter β_S for stock risk but does not depend on the ambiguity aversion parameter β_V for additional volatility risk. This shows that a myopic investor concentrates solely on ambiguity aversion parameter β_S w.r.t. market return risk. The hedging component $\frac{(1-(\beta_S+\gamma))\sigma_V\rho_V\bar{g}(t)}{(1-\gamma)(\beta_S+\gamma)}$ is time dependent, and for a non-myopic investor, this component depends on β_V , as $\bar{g}(t)$ depends on β_V . That is, the investor is concerned not only with β_S but also with β_V w.r.t. market return risk. The case of exposure to additional volatility risk $\theta_V^*(t)$ is easily analyzed in a similar manner. Second, from the remaining components of optimal risk exposure, we find that the salary process exists in the portfolio and generates a new hedging component w.r.t. salary risk. Due to the assumption that the risk factors $W_S(t)$ and $W_V(t)$ are contained in the salary process, this component is affected by both β_S and β_V . Third, the worst-case

measure is chosen by Eq. (27), which is proportional to volatility $\sqrt{V(t)}$. The case of $\phi_S^*(t)$ is affected by both the ambiguity regarding market return risk β_S and the ambiguity regarding additional volatility risk β_V .

Remark 3.3. In our results, m(t) and n(t) in optimal risk exposure are consistent with the previous studies on ambiguity, such as Branger and Larsen (2013) and Escobar et al. (2015). However, they do not consider the risk of salary, which is very important in a DC pension plan. In this model, the worst-case measure here takes a form similar to that in Escobar et al. (2015).

Theorem 3.4. For problem (18), if there exists a function $J(t, x, v, l) \in C^{1,2,2,2}(O)$, which is a solution to the HJB equation (21) with boundary condition J(T, x, v, l) = U(x) and the parameters satisfy conditions (22) and (23), then the optimal value function is H(t, x, v, l) =J(t, x, v, l), and the optimal strategy is $\mathbf{u}^* = \{(u_S^*(t), u_O^*(t))\}_{t \in [0,T]}$ given in Theorem 3.2.

Proof. See Appendix D.

Remark 3.5. We present several special cases to show the relationships between $\theta_S^*(t)$, $\theta_V^*(t)$ and β_S , β_V and γ . It is obvious that the effects of σ_L on $\theta_S^*(t)$ and $\theta_V^*(t)$ depend on the value of ρ_L . When $\rho_L = 0$, the optimal risk exposure in this case, denoted $\theta_{1S}^*(t)$ and $\theta_{1V}^*(t)$, can be written as $\theta_{1S}^*(t) = m(t) \left(1 + \bar{h}(t) \frac{L(t)}{X^{u^*}(t)}\right)$ and $\theta_{1V}^*(t) = n(t) \left(1 + \bar{h}(t) \frac{L(t)}{X^{u^*}(t)}\right) - \sigma_L \bar{h}(t) \frac{L(t)}{X^{u^*}(t)}$, and the optimal value function in this case, denoted $J_1(t, x, v, l)$, can be written as $J_1(t, x, v, l) = \frac{(x + \bar{h}(t)l)^{1-\gamma}}{1-\gamma} \exp(\bar{g}_1(t)v + \hat{g}_1(t))$.¹⁰ Moreover, as $\bar{h}(t) > 0$, $\bar{g}(t) < 0$ and $\gamma > 1$, following simple calculations, when $\rho_V = 0$, we have $\frac{\partial \theta_{1S}^*(t)}{\partial(\beta_S + \gamma)} < 0$, which implies that the optimal risk exposure decreases w.r.t. the sum of aversion to ambiguity and risk in some cases, which implies that the investor decreases her exposure to market return risk when she

¹⁰The optimal investment strategy when $\rho_L = 0$, denoted $u_{1S}^*(t)$ and $u_{1O}^*(t)$, can be written as $u_{1S}^*(t) = \theta_{1S}^*(t) - \frac{O_s S(t) + \sigma_V \rho_V O_v(t)}{O(t)} u_{1O}^*(t)$ and $u_{1O}^*(t) = \frac{O(t)\theta_{1V}^*(t)}{\sigma_V \sqrt{1 - \rho_V^2 O_v}}$, and the worst-case measure in this case, denoted $\phi_{1S}^*(t)$ and $\phi_{1V}^*(t)$, can be written as $\phi_{1S}^*(t) = \frac{\beta_S(\lambda_1(1-\gamma)+\sigma_V \rho_V \bar{g}_1(t))\sqrt{V(t)}}{(1-\gamma)(\beta_S+\gamma)}$ and $\phi_{1V}^*(t) = \frac{\beta_V(\lambda_2(1-\gamma)+\sigma_V \sqrt{1-\rho_V^2 \bar{g}_1(t)})\sqrt{V(t)}}{(1-\gamma)(\beta_V+\gamma)}$, where

$$\begin{split} \bar{g}_{1}(t) &= \frac{\nu_{11}\nu_{21} - \nu_{11}\nu_{21}e^{\alpha_{21}(\nu_{11} - \nu_{21})(T-t)}}{\nu_{21} - \nu_{11}e^{\alpha_{21}(\nu_{11} - \nu_{21})(T-t)}}, \quad \hat{g}_{1}(t) = \int_{t}^{T} \left[r(1-\gamma) + \kappa \delta \bar{g}_{1}(s) \right] \mathrm{d}s, \\ \alpha_{11} &= -\kappa + \frac{\lambda_{1}(1-(\beta_{S}+\gamma))\sigma_{V}\rho}{\beta_{S}+\gamma} + \frac{\lambda_{2}(1-(\beta_{V}+\gamma))\sigma_{V}\sqrt{1-\rho_{V}^{2}}}{\beta_{V}+\gamma}, \\ \alpha_{21} &= \frac{\sigma_{V}^{2}}{2} - \frac{\beta_{S}^{2}\sigma_{V}^{2}\rho_{V}^{2} + \beta_{V}\sigma_{V}^{2}(1-\rho_{V}^{2})}{2(1-\gamma)} + \frac{(1-(\beta_{S}+\gamma))^{2}\sigma_{V}^{2}\rho_{V}^{2}}{2(\beta_{S}+\gamma)(1-\gamma)} + \frac{(1-(\beta_{V}+\gamma))^{2}\sigma_{V}^{2}(1-\rho_{V}^{2})}{2(\beta_{V}+\gamma)(1-\gamma)}, \\ \alpha_{31} &= \frac{\lambda_{1}^{2}(1-\gamma)}{2(\beta_{S}+\gamma)} + \frac{\lambda_{2}^{2}(1-\gamma)}{2(\beta_{V}+\gamma)}, \quad \nu_{11,21} = \frac{\alpha_{11}\pm\sqrt{\alpha_{11}^{2}-4\alpha_{21}\alpha_{31}}}{-2\alpha_{21}}, \end{split}$$

and $\bar{h}(t)$ is given by Eq. (32). By derivation, we obtain $\alpha_{11}^2 - 4\alpha_{21}\alpha_{31} \ge 0$.

is more ambiguity averse and risk averse.

Remark 3.6. If $\sigma_L = 0$, the salary process is non-stochastic; then the optimal risk exposure in this case, denoted $\theta_{2S}^*(t)$ and $\theta_{2V}^*(t)$, can be written as $\theta_{2S}^*(t) = m(t)(1 + \frac{\hat{h}(t)}{X^{u^*}(t)})$ and $\theta_{2V}^*(t) = n(t)(1 + \frac{\hat{h}(t)}{X^{u^*}(t)})$, and the optimal value function in this case, denoted $J_2(t, x, v)$, can be written as $J_2(t, x, v) = \frac{(x + \hat{h}(t))^{1-\gamma}}{1-\gamma} \exp(\bar{g}(t)v + \hat{g}(t))$, where

$$\hat{h}(t) = \frac{\xi l_0}{\mu_L - r} [\exp(\mu_L T - r(T - t)) - \exp(\mu_L t)],$$
(37)

and m(t), n(t), $\bar{g}(t)$, $\hat{g}(t)$ are given by Eqs. (28), (29), (30) and (31).¹¹ In this case, we find that the optimal risk exposures are proportional to m(t) and n(t).

Furthermore, if there is no salary in our model, i.e., $\xi = 0$ or L(t) = 0, our problem reduces to a portfolio selection problem. The optimal risk exposure in this case, denoted $\theta_{3S}^*(t)$ and $\theta_{3V}^*(t)$, can be written as $\theta_{3S}^*(t) = m(t)$ and $\theta_{3V}^*(t) = n(t)$, and the optimal value function in this case, denoted $J_3(t, x, v)$, can be written as $J_3(t, x, v) = \frac{x^{1-\gamma}}{1-\gamma} \exp(\bar{g}(t)v + \hat{g}(t))$, where m(t), n(t), $\bar{g}(t)$ and $\hat{g}(t)$ are given by Eqs. (28), (29), (30) and (31), respectively.¹² Correspondingly, the optimal risk exposure is independent of wealth x. It is worth noting that the optimal investment strategy obtained in the case without stochastic salary is the same as that given in Escobar et al. (2015) without jumps.

Remark 3.7. If the pension investor is ambiguity neutral, i.e., both ambiguity aversion parameters β_S and β_V equal 0, the optimal risk exposure in this case, denoted $\theta_{4S}^*(t)$ and $\theta_{4V}^*(t)$, can be written as $\theta_{4S}^*(t) = \frac{\lambda_1 + \sigma_V \rho_V \bar{g}_2(t)}{\gamma} \left(1 + \bar{h}(t) \frac{L(t)}{X^{u^*}(t)}\right) - \sigma_L \rho_L \bar{h}(t) \frac{L(t)}{X^{u^*}(t)}$ and $\theta_{4V}^*(t) = \frac{\lambda_2 + \sigma_V \sqrt{1 - \rho_V^2} \bar{g}_2(t)}{\gamma} \left(1 + \bar{h}(t) \frac{L(t)}{X^{u^*}(t)}\right) - \sigma_L \sqrt{1 - \rho_L^2} \bar{h}(t) \frac{L(t)}{X^{u^*}(t)}$, and the optimal value function in this case, denoted $J_2(t, x, v, l)$, can be written as $J_4(t, x, v, l) = \frac{(x + \bar{h}(t)l)^{1-\gamma}}{1-\gamma} \exp(\bar{g}_2(t)v + \hat{g}_2(t))$, where

$$\bar{g}_2(t) = \frac{\nu_{12}\nu_{22} - \nu_{12}\nu_{22}\mathrm{e}^{\alpha_{22}(\nu_{12}-\nu_{22})(T-t)}}{\nu_{22} - \nu_{12}\mathrm{e}^{\alpha_{22}(\nu_{12}-\nu_{22})(T-t)}}, \quad \hat{g}_2(t) = \int_t^T \left[r(1-\gamma) + \kappa\delta\bar{g}_2(s)\right]\mathrm{d}s, \quad (38)$$

¹¹The optimal investment strategy when $\sigma_L = 0$, denoted $u_{2S}^*(t)$ and $u_{2O}^*(t)$, can be written as $u_{2S}^*(t) = \theta_{2S}^*(t) - \frac{O_s S(t) + \sigma_V \rho_V O_v}{O(t)} u_{2O}^*(t)$ and $u_{2O}^*(t) = \frac{O(t)\theta_{2V}^*(t)}{\sigma_V \sqrt{1 - \rho_V^2 O_v}}$, and the worst-case measure in this case, denoted $\phi_{2S}^*(t)$ and $\phi_{2V}^*(t)$, can be written as $\phi_{2S}^*(t) = \frac{\beta_S(\lambda_1(1-\gamma) + \sigma_V \rho_V \bar{g}(t))\sqrt{V(t)}}{(1-\gamma)(\beta_S+\gamma)}$ and $\phi_{2V}^*(t) = \frac{\beta_V(\lambda_2(1-\gamma) + \sigma_V \sqrt{1 - \rho_V^2 \bar{g}(t)})\sqrt{V(t)}}{(1-\gamma)(\beta_S+\gamma)}$.

 $\frac{\beta_V(\lambda_2(1-\gamma)+\sigma_V\sqrt{1-\rho_V^2\bar{g}(t)})\sqrt{V(t)}}{(1-\gamma)(\beta_V+\gamma)}.$ ¹²The optimal investment strategy without stochastic salary, denoted $u_{3S}^*(t)$ and $u_{3O}^*(t)$, can be written as $u_{3S}^*(t) = \theta_{3S}^*(t) - \frac{O_sS(t)+\sigma_V\rho_VO_v}{O(t)}u_{3O}^*(t)$ and $u_{3O}^*(t) = \frac{O(t)\theta_{3V}^*(t)}{\sigma_V\sqrt{1-\rho_V^2O_v}}$, and the worst-case measure in this case, denoted $\phi_{3S}^*(t)$ and $\phi_{3V}^*(t)$, can be written as $\phi_{3S}^*(t) = \frac{\beta_S(\lambda_1(1-\gamma)+\sigma_V\rho_V\bar{g}(t))\sqrt{V(t)}}{(1-\gamma)(\beta_S+\gamma)}$ and $\phi_{3V}^*(t) = \frac{\beta_V(\lambda_2(1-\gamma)+\sigma_V\sqrt{1-\rho_V^2\bar{g}(t)})\sqrt{V(t)}}{(1-\gamma)(\beta_V+\gamma)}.$

and $\bar{h}(t)$ is given by Eq.(32). By derivation, we obtain $\alpha_{12}^2 - 4\alpha_{22}\alpha_{32} \ge 0.^{13}$

Similarly, the following remark provides the optimal investment strategy in the case of no ambiguity and no stochastic salary.

Remark 3.8. If the pension investor is ambiguity neutral and $\sigma_L = 0$, the salary process is non-stochastic, and the optimal risk exposure in this case, denoted $\theta_{5S}^*(t)$ and $\theta_{5V}^*(t)$, can be written as $\theta_{5S}^*(t) = \frac{\lambda_1 + \sigma_V \rho_V \bar{g}_2(t)}{\gamma} (1 + \frac{\hat{h}(t)}{X^{u^*}(t)})$ and $\theta_{5V}^*(t) = \frac{\lambda_2 + \sigma_V \sqrt{1 - \rho_V^2} \bar{g}_2(t)}{\gamma} (1 + \frac{\hat{h}(t)}{X^{u^*}(t)})$, and the optimal value function in this case, denoted $J_5(t, x, v)$, can be written as $J_5(t, x, v) = \frac{(x + \hat{h}(t))^{1-\gamma}}{1-\gamma} \exp(\bar{g}_2(t)v + \hat{g}_2(t))$, where $\hat{h}(t)$, $\bar{g}_2(t)$ and $\hat{g}_2(t)$ are given by Eqs. (37)-(38).¹⁴

Furthermore, if there is no salary and no ambiguity in our model, the optimization problem becomes a portfolio selection problem for an ambiguity-neutral investor; the optimal risk exposure in this case, denoted $\theta_{6S}^*(t)$ and $\theta_{6V}^*(t)$, can be written as $\theta_{6S}^*(t) = \frac{\lambda_1 + \sigma_V \rho_V \bar{g}_2(t)}{\gamma}$ and $\theta_{6V}^*(t) = \frac{\lambda_2 + \sigma_V \sqrt{1 - \rho_V^2} \bar{g}_2(t)}{\gamma}$, and the optimal value function in this case, denoted $J_6(t, x, v)$, can be written as $J_6(t, x, v) = \frac{x^{1-\gamma}}{1-\gamma} \exp(\bar{g}_2(t)v + \hat{g}_2(t))$, where $\bar{g}_2(t)$ and $\hat{g}_2(t)$ are given by Eq. (38).¹⁵ In this case, the result reduces to that of the optimal portfolio problem in the case without jumps in Liu and Pan (2003).

4. Optimal investment strategy without a derivative

In this section, to illustrate the significant role of the derivative, we seek the solution to the case without a derivative and compare it to the result with a derivative.

If there is no derivative security in the financial market, the optimal investment strategy equals the optimal risk exposure to $W_S(t)$, and the surplus process of an ambiguity-averse

¹³The optimal investment strategy without ambiguity, denoted $u_{4S}^*(t)$ and $u_{4O}^*(t)$, can be written as $u_{4S}^*(t) = \theta_{4S}^*(t) - \frac{O_s S(t) + \sigma_V \rho_V O_v}{O(t)} u_{4O}^*(t)$ and $u_{4O}^*(t) = \frac{O(t) \theta_{4V}^*(t)}{\sigma_V \sqrt{1 - \rho_V^2 O_v}}$. In Eq. (38),

$$\begin{split} \alpha_{12} &= -\kappa + \frac{\lambda_1 (1 - \gamma) \sigma_V \rho_V}{\gamma} + \frac{\lambda_2 (1 - \gamma) \sigma_V \sqrt{1 - \rho_V^2}}{\gamma}, \quad \alpha_{22} = \frac{\sigma_V^2}{2\gamma}, \\ \alpha_{32} &= \frac{(\lambda_1^2 + \lambda_2^2) (1 - \gamma)}{2\gamma}, \quad \nu_{12,22} = \frac{\alpha_{12} \pm \sqrt{\alpha_{12}^2 - 4\alpha_{2a_{32}}}}{-2\alpha_{22}}. \end{split}$$

¹⁴The optimal investment strategy when $\sigma_L = 0$ for an ambiguity-neutral pension investor, denoted $u_{5S}^*(t)$ and $u_{5O}^*(t)$, can be written as $u_{5S}^*(t) = \theta_{5S}^*(t) - \frac{O_s S(t) + \sigma_V \rho_V O_v}{O(t)} u_{5O}^*(t)$ and $u_{5O}^*(t) = \frac{O(t) \theta_{5V}^*(t)}{\sigma_V \sqrt{1 - \rho_V^2 O_v}}$. ¹⁵The optimal investment strategy without stochastic salary and ambiguity, denoted $u_{6S}^*(t)$ and $u_{6O}^*(t)$,

¹⁵The optimal investment strategy without stochastic salary and ambiguity, denoted $u_{6S}^*(t)$ and $u_{6O}^*(t)$ can be written as $u_{6S}^*(t) = \theta_{6S}^*(t) - \frac{O_s S(t) + \sigma_V \rho_V O_v}{O(t)} u_{6O}^*(t)$ and $u_{6O}^*(t) = \frac{O(t) \theta_{6V}^*(t)}{\sigma_V \sqrt{1 - \rho_V^2} O_v}$.

pension investor under measure \mathbb{Q} becomes

$$dX^{\tilde{\Phi},\tilde{\mathbf{u}}}(t) = X^{\tilde{\Phi},\tilde{\mathbf{u}}}(t) \left[rdt + \tilde{u}(t) \left(\lambda_1 V^{\tilde{\Phi}}(t) dt - \tilde{\phi}_S(t) \sqrt{V^{\tilde{\Phi}}(t)} dt + \sqrt{V^{\tilde{\Phi}}(t)} dW_S^{\tilde{\Phi}}(t) \right) \right] + \xi L^{\tilde{\Phi}}(t) dt,$$
(39)

where $\tilde{\mathbf{u}} := {\tilde{u}(t)}_{t \in [0,T]}$, $\tilde{\Phi} := {\tilde{\phi}(t) := (\tilde{\phi}_S(t), \tilde{\phi}_V(t))}_{t \in [0,T]}$, and the risk exposure equals the investment strategy, i.e., $\tilde{\theta}_S(t) = \tilde{u}(t)$. The optimization problem becomes

$$\sup_{\tilde{\mathbf{u}}\in\Pi}\inf_{\tilde{\Phi}\in\Theta}\left\{\mathrm{E}_{t,x,v,l}^{\tilde{\Phi}}\left[U(X^{\tilde{\Phi},\tilde{\mathbf{u}}}(T))+\int_{t}^{T}\left(\frac{(\tilde{\phi}_{S}(s))^{2}}{2\tilde{\Psi}_{S}(s,x,v,l)}+\frac{(\tilde{\phi}_{V}(s))^{2}}{2\tilde{\Psi}_{V}(s,x,v,l)}\right)\mathrm{d}s\right]\right\},\qquad(40)$$

and the corresponding HJB equation becomes

$$\sup_{\tilde{u}\in\mathbb{R}}\inf_{||\tilde{\phi}||\leq\sqrt{\underline{\kappa}v}}\left\{\tilde{\mathcal{A}}^{\tilde{\phi},\tilde{u}}\tilde{J}(t,x,v,l)+\frac{\tilde{\phi}_{S}^{2}}{2\tilde{\Psi}_{S}}+\frac{\tilde{\phi}_{V}^{2}}{2\tilde{\Psi}_{V}}\right\}=0,$$
(41)

$$\tilde{\Psi}_S(t,x,v,l) = \frac{\beta_S}{(1-\gamma)\tilde{J}(t,x,v,l)}, \quad \tilde{\Psi}_V(t,x,v,l) = \frac{\beta_V}{(1-\gamma)\tilde{J}(t,x,v,l)}, \quad (42)$$

with the boundary condition $\tilde{J}(T, x, v, l) = U(x)$, where \tilde{u} and $\tilde{\phi} = (\tilde{\phi}_S, \tilde{\phi}_V)$ denote the values that $\tilde{u}(t)$ and $\tilde{\phi}(t) = (\tilde{\phi}_S(t), \tilde{\phi}_V(t))$ take, respectively, and

$$\begin{split} \tilde{\mathcal{A}}^{\tilde{\phi},\tilde{u}}\psi(t,x,v,l) &= \psi_t + [rx + x\tilde{u}\lambda_1v - x\tilde{u}\tilde{\phi}_S\sqrt{v} + \xi l]\psi_x + \frac{1}{2}x^2v\tilde{u}^2\psi_{xx} \\ &+ [\kappa(\delta-v) - \sigma_V\sqrt{v}\rho_V\tilde{\phi}_S - \sigma_V\sqrt{v}\sqrt{1-\rho_V^2}\tilde{\phi}_V]\psi_v + \frac{1}{2}\sigma_v^2v\psi_{vv} \\ &+ \left[\mu_L l + l\sigma_L\lambda_1v\rho_L - l\sigma_L\sqrt{v}\tilde{\phi}_S\rho_L + l\sigma_L\lambda_2v\sqrt{1-\rho_L^2} - l\sigma_L\sqrt{v}\tilde{\phi}_V\sqrt{1-\rho_L^2}\right]\psi_l \\ &+ \frac{1}{2}l^2\sigma_L^2v\psi_{ll} + l\sigma_Lv\sigma_V(\rho_V\rho_L + \sqrt{1-\rho_V^2}\sqrt{1-\rho_L^2})\psi_{lv} + x\sigma_V\tilde{u}v\rho_V\psi_{xv} + x\tilde{u}l\sigma_Lv\rho_L\psi_{xl}. \end{split}$$

The following theorem presents the optimal investment strategy and optimal value function for the DC pension investor without a derivative.

Theorem 4.1. For the robust portfolio choice problem (40) without a derivative, if the parameters satisfy certain technical conditions, ¹⁶ the optimal investment strategy and risk

 $^{16} {\rm The}$ technical conditions are $\underline{\phi}_3 < \kappa/\sigma_V,$ where

$$\underline{\phi}_{3} \triangleq \max\left\{\frac{\beta_{S}^{2}\lambda_{1}^{2}}{(\beta_{S}+\gamma)^{2}}, \frac{\beta_{S}^{2}(\lambda_{1}(1-\gamma)+\sigma_{V}\rho_{V}\bar{g}_{3}(0))^{2}}{(1-\gamma)^{2}(\beta_{S}+\gamma)^{2}}\right\} + \max\left\{\frac{\beta_{V}^{2}\lambda_{2}^{2}}{(\beta_{V}+\gamma)^{2}}, \frac{\beta_{V}^{2}(\lambda_{2}(1-\gamma)+\sigma_{V}\sqrt{1-\rho_{V}^{2}}\bar{g}_{3}(0))^{2}}{(1-\gamma)^{2}(\beta_{V}+\gamma)^{2}}\right\}$$

$$(43)$$

and for $\bar{g}_3(t) \in [\bar{g}_3(0), 0]$,

$$[64(1-\gamma)^2 - 4(1-\gamma)](\tilde{m}(t))^2 + 8(1-\gamma)\gamma(\tilde{m}(t))^2 - 8(1-\gamma)\frac{\sigma_V\rho\bar{g}_3(t)}{\beta_S+\gamma}\tilde{m}(t) \le \frac{\kappa^2}{2\sigma_V^2}.$$
(44)

Similar to conditions $\underline{\phi} < \kappa/\sigma_V$ and (23), conditions $\underline{\phi}_3 < \kappa/\sigma_V$ and (44) are also technical conditions and easily satisfied.

$$\tilde{u}^{*}(t) = \tilde{\theta}^{*}_{S}(t) = \tilde{m}(t) \left(1 + \bar{h}(t) \frac{L(t)}{X^{u^{*}}(t)} \right) - \sigma_{L} \rho_{L} \bar{h}(t) \frac{L(t)}{X^{u^{*}}(t)};$$
(45)

the corresponding optimal value function is

$$\tilde{J}(t, x, v, l) = \frac{(x + \bar{h}(t)l)^{1-\gamma}}{1-\gamma} \exp(\bar{g}_3(t)v + \hat{g}_3(t));$$
(46)

and the worst-case measure is given by

$$\tilde{\phi}_{S}^{*}(t) = \frac{\beta_{S}\sqrt{V(t)}(\lambda_{1}(1-\gamma) + \sigma_{V}\rho_{V}\bar{g}_{3}(t))}{(1-\gamma)(\beta_{S}+\gamma)}, \quad \tilde{\phi}_{V}^{*}(t) = \frac{\beta_{V}\sqrt{V(t)}(\lambda_{2}(1-\gamma) + \sigma_{V}\sqrt{1-\rho_{V}^{2}}\bar{g}_{3}(t))}{(1-\gamma)(\beta_{V}+\gamma)},$$
(47)

where $\{X^*(t)\}_{t\in[0,T]}$ is the wealth process under the corresponding optimal strategy, and

$$\tilde{m}(t) = \frac{\lambda_1(1-\gamma) + (1-(\beta_S+\gamma))\sigma_V \rho_V \bar{g}_3(t)}{(1-\gamma)(\beta_S+\gamma)}, \quad \bar{g}_3(t) = \frac{\tilde{\nu}_1 \tilde{\nu}_2 - \tilde{\nu}_1 \tilde{\nu}_2 e^{\tilde{\alpha}_2(\tilde{\nu}_1 - \tilde{\nu}_2)(T-t)}}{\tilde{\nu}_2 - \tilde{\nu}_1 e^{\tilde{\alpha}_2(\tilde{\nu}_1 - \tilde{\nu}_2)(T-t)}}, \\ \hat{g}_3(t) = \int_t^T \left[r(1-\gamma) + \kappa \delta \bar{g}_3(s) \right] \mathrm{d}s, \quad \tilde{\alpha}_1 = -\kappa + \frac{\lambda_1(1-(\beta_S+\gamma))\sigma_V \rho_V}{\beta_S+\gamma}, \\ \tilde{\alpha}_2 = \frac{\sigma_V^2}{2} - \frac{\beta_S \sigma_V^2 \rho_V^2}{2(1-\gamma)} - \frac{\beta_V \sigma_V^2(1-\rho_V^2)}{2(1-\gamma)} + \frac{(1-(\beta_S+\gamma))^2 \sigma_V^2 \rho_V^2}{2(\beta_S+\gamma)(1-\gamma)}, \\ \tilde{\alpha}_3 = \frac{\lambda_1^2(1-\gamma)}{2(\beta_S+\gamma)}, \quad \tilde{\nu}_{1,2} = \frac{\tilde{\alpha}_1 \pm \sqrt{\tilde{\alpha}_1^2 - 4\tilde{\alpha}_2 \tilde{\alpha}_3}}{-2\tilde{\alpha}_2}, \end{cases}$$
(48)

and $\bar{h}(t)$ is given by Eq.(32). By derivation, we obtain $\tilde{\alpha}_1^2 - 4\tilde{\alpha}_2\tilde{\alpha}_3 \geq 0$.

The proof of Theorem 4.1 is similar to that of Theorem 3.2, and thus, we omit it here.

Theorem 4.2. For problem (40), if there exists a function $\tilde{J}(t, x, v, l) \in C^{1,2,2,2}(O)$ that is a solution to the HJB equation (55) with boundary condition $\tilde{J}(T, x, v, l) = U(x)$, and the parameters satisfy conditions (43) and (44), then the optimal value function is $\tilde{J}(t, x, v, l)$, and the optimal strategy is $\tilde{\boldsymbol{u}}^* = {\tilde{u}^*(t)}_{t \in [0,T]}$ given in Theorem 4.1.

The proof of Theorem 4.2 is similar to that of Theorem 3.4, and thus, we omit it here.

From Theorem 4.1, we find that the optimal investment strategy and risk exposure are both given by Eq. (45). Compared with the former case and optimal exposure to market return risk (24), the difference lies in the form of $\tilde{m}(t)$, particularly, the values of $\nu_{1,2}$ and $\tilde{\nu}_{1,2}$. Here, because the market is incomplete and the investor has only one stock to invest in and obtains one risk premium, the equity premium λ_2 for additional volatility risk is disappearing; as a result, hedging w.r.t. additional volatility risk is less efficient. This quantitative influence depends on the chosen parameters of the model, as illustrated in the following numerical examples. We find that the utility that the pension investor gains is substantially improved when investing in the derivative. Similar results are also found in Escobar et al. (2015). Similar to the case of investment with the derivative, we also provide some special cases if the pension investor has no access to the derivative in Appendix E.

5. Numerical analysis

In this section, we provide several numerical examples to illustrate the effects of model parameters on the robust optimal risk exposures and investment strategies. We also illustrate the utility improvements by considering ambiguity aversion and derivative trading. To examine the empirical properties of our results, we fix a set of base-case parameters for our model (Table 1) using results from existing empirical studies. Details can be referred to Liu and Pan (2003) and Escobar et al. (2015).¹⁷

Table 1: Values of model parameters in the numerical examples.

r	κ	δ	ξ	λ_1	λ_2	μ_L	σ_L	σ_V	γ	β_S	β_V
0.05	5	0.13^{2}	0.2	4	-6	0.08	0.5	0.25	4	3	1
$ ho_V$	$ ho_L$	x	l	v	S	K	au	T	t		
-0.4	0.3	1	1	0.15^{2}	100	100	0.1	5	0		

5.1. Effects of model parameters on risk exposures

Risk exposures θ_S^* and θ_V^* are independent of the types of options; they have more general trends and can describe the exposures to risks W_S and W_V more intuitively. Other corresponding literature also considers the performance of risk exposures; please see Escobar et al. (2015). Therefore, in this subsection, we first consider the effects of model parameters on risk exposures.

Figure 1 shows the effects of the ambiguity aversion parameters β_S and β_V on the optimal market return risk exposure θ_S^* and volatility risk exposure θ_V^* , respectively. We find that the optimal exposure to market return risk θ_S^* decreases in β_S , consistent with Escobar et al. (2015). Another main result is that along with increases in β_V , the optimal exposure to

¹⁷According to Liu and Pan (2003), the empirical properties of the stochastic volatility model are extensively examined using either the time-series data on the S&P 500 index alone (Andersen et al., 2002; Eraker et al., 2003) or the joint time-series data on the S&P 500 index and options (Chernov and Ghysels, 2000; Pan, 2002). Because of different sample periods or empirical approaches in those studies, the exact model estimates may differ from one paper to another. Our chosen model parameters agree with the cases studied by Liu and Pan (2003) and Escobar et al. (2015).

volatility risk θ_V^* is significantly decreasing (in absolute terms). These results show that in an ambiguous environment, the investor becomes less aggressive. We now focus on one specific risk exposure and show how the two ambiguity aversion parameters generate distinct effects on it. Taking market return risk exposure θ_S^* as an example, we find that the stock ambiguity aversion parameter β_S has a relatively larger effect than the volatility ambiguity aversion parameter β_V . This is consistent with the case of volatility risk exposure θ_V^* . Compared to β_V (β_S), β_S (β_V) represents a direct way to affect market return risk exposure (volatility risk exposure).



Figure 1: Effects of β_S and β_V on θ_S^* and θ_V^* .



Figure 2: Effects of κ and σ_V on θ_S^* and θ_V^* .

Figure 2 shows the effects of the mean-reversion rate κ and volatility coefficient σ_V on the optimal market return risk exposure θ_S^* and volatility risk exposure θ_V^* , respectively. In the stock return variance process, a smaller mean-reversion rate κ and larger volatility σ_V usually imply greater additional volatility risk. As a result, optimal volatility risk exposure θ_V^* decreases with κ and increases with σ_V (in absolute terms). The case of market return risk exposure θ_S^* is similar to that of volatility risk exposure, as there is a diversification effect (benefit from risk diversification).



Figure 3: Effects of salary parameters μ_L , σ_L , ξ and l_0 on θ_S^* and θ_V^* .

Figure 3 shows the effects of the salary parameters, appreciation rate μ_L , volatility coefficient σ_L , contribution rate ξ and initial salary l_0 on the optimal market return risk exposure θ_S^* and volatility risk exposure θ_V^* , respectively. We find that both θ_S^* and θ_V^* (in absolute terms) increase with μ_L , ξ and l_0 . When μ_L , ξ and l_0 increase, there will be greater pension fund accumulation. Therefore the investor prefers to undertake more risks to earn more profits. In addition, θ_S^* decreases with σ_L and θ_V^* (in absolute terms) increases with σ_L .



Figure 4: Effects of ρ_V and ρ_L on θ_S^* and θ_V^* .

Figure 4 shows the effects of correlation coefficients ρ_V and ρ_L on the optimal market return risk exposure θ_S^* and volatility risk exposure θ_V^* , respectively. This figure shows that θ_S^* decreases with ρ_V and ρ_L , while θ_V^* (in absolute terms) increases first and then decreases with ρ_V and ρ_L . This behavior stems from the assumption of our model. Eqs. (24), (28) and (29) show that ρ_V and $\sqrt{1-\rho_V^2}$ (ρ_L and $\sqrt{1-\rho_L^2}$) reflect the different properties of a sensitivity analysis for ρ_V (ρ_L). ρ_V (ρ_L) may be negative or non-negative, and $\sqrt{1-\rho_V^2}$ ($\sqrt{1-\rho_L^2}$) is non-negative. Therefore, the risk exposure to W_S decreases with ρ_V and ρ_L , and the risk exposure to W_V (in absolute terms) decreases with $|\rho_V|$ and $|\rho_L|$.

5.2. Effects of model parameters on investment strategies

In this subsection, to further illustrate the role of derivative on the optimal investment strategy, we take the straddle option¹⁸ as an example to show the effects of model parameters on investment strategies. From Figures 5-8, we find that the derivative has an important effect on the investment strategies.

¹⁸The straddle is a portfolio comprising a call option and a put option with the same underlying strike price, time to maturity, and market volatility, and its price is given in Appendix F. We assume that the initial stock price is 100, and the strike price is chosen in a way that makes the straddle "delta-neutral". For details, referred to Liu and Pan (2003) and Cui et al. (2017). We also conduct some numerical examples with other types of options, such as call options and put options. If the readers are interested, we can provide our simulation examples. The analysis is similar. To saving space, we do not include these in our paper.



Figure 5: Effects of β_S and β_V on u_S^* and u_O^* .

Figure 5 shows the effects of the ambiguity aversion parameters β_S and β_V on the optimal proportions invested in stock u_S^* and derivative u_O^* , respectively. We find that both u_S^* and u_O^* (in absolute terms) decrease with β_S . Compared to stock investment, the changes in derivative investment are relatively small. When β_S grows larger, the investor becomes more ambiguity averse to the return of the stock and investment less in the stock. Moreover, u_O^* (in absolute terms) decreases with β_V in a similar way. However, u_S^* increases with β_V . The reason may be as follows. Ambiguity reduces the volatility risk premium and derivative investment becomes less attractive to the ambiguous investor, thereby inducing her to invest more wealth in stock as the result of a substitution effect between the two risky assets.



Figure 6: Effects of κ and σ_V on u_S^* and u_O^* .

Figure 6 shows the effects of mean-reversion rate κ and volatility coefficient σ_V on the optimal proportions invested in stock u_S^* and derivative u_O^* , respectively. As κ increases, both u_S^* and u_O^* (in absolute terms) decrease. As in our model the correlation ρ_V is negative, the uncertainties of the stock price and its volatility change in different ways. Although V(t) will be stable as κ increases, there is an increased probability of a decrease in the stock

price. The decrease affects not only the stock investment but also the derivative investment. Moreover, when $\kappa < 2$, the effect of σ_V on the optimal investment strategies in the stock and the derivative are not monotone; when $\kappa \geq 2$, u_S^* and u_O^* (in absolute terms) decrease as σ_V increases. We attribute to the fact that the larger σ_V is, the more risk the stock has. Therefore the investor will invest less proportion in the stock and hold less opposite position in the derivative.

Figure 7 shows the effects of the salary parameters, appreciation rate μ_L , volatility coefficient σ_L , contribution rate ξ and initial salary l_0 , on the optimal proportions invested in stock u_S^* and derivative u_O^* . We find that both u_S^* and u_O^* (in absolute terms) increase with μ_L , ξ and l_0 : the increasing of μ_L , ξ and l_0 imply that there will be greater pension fund accumulation, and then the investor prefers to undertake more risks to earn more. In addition, u_S^* decreases with σ_L and u_O^* (in absolute terms) increases with σ_L . This is because a larger σ_L implies greater risk of the salary, which involves more risk into the wealth process. Therefore the investor would invest less in the stock to reduce the total risk of the wealth process, while invest more in the derivative to derive an advantage risk-return tradeoff.



Figure 7: Effects of salary parameters μ_L , σ_L , ξ and l_0 on u_S^* and u_O^* .



Figure 8: Effects of ρ_V and ρ_L on u_S^* and u_O^* .

Figure 8 shows the effects of correlation coefficients ρ_V and ρ_L on the optimal proportions invested in stock u_S^* and derivative u_O^* , respectively. On the one hand, both u_S^* and u_O (in absolute terms) increase with ρ_V . When the risks of the financial market become larger, investor goes longs on more stock and shorts more derivative to reduce the portfolio risk. On the other hand, both u_S^* and u_O (in absolute terms) decrease with ρ_L . Because the risk of the salary is difficult to reduce from the portfolio of the stock and the derivative; therefore, the investment in the stock and the derivative will decrease.

5.3. Utility improvement

In this subsection, we study the effects of considering ambiguity aversion and derivative trading on utility improvement. We focus on two cases of utility improvement for the DC pension investor. One is the utility improvement delivered by taking ambiguity aversion into consideration, the other is the utility improvement delivered by allowing the investor to trade in the derivative.

We define the first type of utility improvement. We calculate it by considering ambiguity aversion compared with the case in which the ambiguity-averse investor follows an investment strategy that ignores ambiguity. In particular, we assume that the investor does not adopt the optimal strategy $\mathbf{u}^* = \{(u_S^*(t), u_O^*(t)\}_{t \in [0,T]} \text{ given in Theorem 3.2 but instead makes}$ the decision as if she were ambiguity neutral, i.e., the pension investor follows the strategy $\mathbf{u}_4^* = \{(u_{4S}^*(t), u_{4O}^*(t))\}_{t \in [0,T]}$ given in Remark 3.7. The value function for the pension investor in this case is defined by

$$\bar{J}(t,x,v,l) = \inf_{\bar{\Phi}\in\Theta} \left\{ \mathbf{E}_{t,x,v,l}^{\bar{\Phi}} \left[U(X^{\bar{\Phi},\mathbf{u}_{4}^{*}}(T)) + \int_{t}^{T} \left(\frac{(\bar{\phi}_{S}(s))^{2}}{2\bar{\Psi}_{S}(s,x,v,l)} + \frac{(\bar{\phi}_{V}(s))^{2}}{2\bar{\Psi}_{V}(s,x,v,l)} \right) \mathrm{d}s \right] \right\},$$

where

$$\bar{\Psi}_{S}(t,x,v,l) = \frac{\beta_{S}}{(1-\gamma)\bar{J}(t,x,v,l)}, \quad \bar{\Psi}_{V}(t,x,v,l) = \frac{\beta_{V}}{(1-\gamma)\bar{J}(t,x,v,l)}.$$

Similar to the above derivation, we derive the optimal value function under the suboptimal strategy

$$\bar{J}(t,x,v,l) = \frac{(x+\bar{h}(t)l)^{1-\gamma}}{1-\gamma} \exp(\bar{g}_9(t)v + \hat{g}_9(t)).^{19}$$
(49)

Furthermore, we define the utility improvement by considering the ambiguity aversion given by

$$UI_1(t, x, v, l) := 1 - \frac{J(t, x, v, l)}{\bar{J}(t, x, v, l)} = 1 - \exp((\bar{g}(t) - \bar{g}_9(t))v + \hat{g}(t) - \hat{g}_9(t)),$$
(50)

where J(t, x, v, l) and $\overline{J}(t, x, v, l)$ are given by Eqs. (26) and (49).

We define the second type of utility improvement. We calculate it by considering derivative trading compared with the case in which the pension investor has no access to a derivative. In particular, it is defined by

$$UI_2(t, x, v, l) := 1 - \frac{J(t, x, v, l)}{\tilde{J}(t, x, v, l)} = 1 - \exp((\bar{g}(t) - \bar{g}_3(t))v + \hat{g}(t) - \hat{g}_3(t)),$$
(51)

where J(t, x, v, l) and $\tilde{J}(t, x, v, l)$ are given by Eqs. (26) and (46).

Remark 5.1. From the expressions of $\bar{g}_9(t)$, $\hat{g}_9(t)$, $\bar{g}_3(t)$, utility improvements UI_1 and UI_2 are independent of the salary process.

Remark 5.2. Liu and Pan (2003) state that in a setting with no ambiguity, trading in the derivative can significantly improve the investor's utility. Here, we further show that when the investor is ambiguity averse, there is also utility improvement from having access to the derivatives market. The quantitative improvement is shown in the following numerical examples, which also reveal that the utility improvement delivered by having access to the derivative is large. This implies that the derivative plays a crucial role in providing investment opportunities and improving the efficiency of the market.

¹⁹In Eq. (49),

$$\bar{g}_{9}(t) = \frac{\bar{\nu}_{1}\bar{\nu}_{2}-\bar{\nu}_{1}\bar{\nu}_{2}e^{\bar{\alpha}_{2}(\bar{\nu}_{1}-\bar{\nu}_{2})(T-t)}}{\bar{\nu}_{2}-\bar{\nu}_{1}e^{\bar{\alpha}_{2}(\bar{\nu}_{1}-\bar{\nu}_{2})(T-t)}}, \quad \hat{g}_{9}(t) = \int_{t}^{T} \left[r(1-\gamma) + \kappa\delta\bar{g}_{9}(s)\right] \mathrm{d}s,$$

$$\bar{\alpha}_{1} = -\kappa + \frac{(\lambda_{1}+\sigma_{V}\rho_{V}\bar{g}_{2}(t))(1-(\beta_{S}+\gamma))\sigma_{V}\rho_{V}}{\gamma} + \frac{(\lambda_{2}+\sigma_{V}\sqrt{1-\rho_{V}^{2}}\bar{g}_{2}(t))(1-(\beta_{V}+\gamma))\sigma_{V}\sqrt{1-\rho_{V}^{2}}}{\gamma},$$

$$\bar{\alpha}_{2} = \frac{\sigma_{V}^{2}}{2} - \frac{\beta_{S}\sigma_{V}^{2}\rho_{V}^{2}}{2(1-\gamma)} - \frac{\beta_{V}\sigma_{V}^{2}(1-\rho_{V}^{2})}{2(1-\gamma)}, \quad \bar{\nu}_{1,2} = \frac{\bar{\alpha}_{1}\pm\sqrt{\bar{\alpha}_{1}^{2}-4\bar{\alpha}_{2}\bar{\alpha}_{3}}}{-2\bar{\alpha}_{2}},$$

$$\bar{\alpha}_{3} = -\frac{(\lambda_{1}+\sigma_{V}\rho_{V}\bar{g}_{2}(t))^{2}(1-\gamma)\beta_{S}}{2\gamma^{2}} - \frac{(\lambda_{2}+\sigma_{V}\sqrt{1-\rho_{V}^{2}}\bar{g}_{2}(t))^{2}(1-\gamma)\beta_{V}}{2\gamma^{2}} + \frac{(1-\gamma)(\lambda_{1}^{2}+\lambda_{2}^{2}-\sigma_{V}^{2}(\bar{g}_{2}(t))^{2})}{2\gamma},$$

and $\bar{h}(t)$ and $\bar{g}_2(t)$ are given by Eqs. (32) and (38). After some calculations, we have $\bar{\alpha}_1^2 - 4\bar{\alpha}_2\bar{\alpha}_3 \ge 0$.

Figure 9 shows the effects of the ambiguity aversion parameters β_S and β_V on utility improvements. UI_1 is the utility improvement from considering ambiguity aversion, and we find that it increases with the ambiguity aversion parameters β_S and β_V . Intuitively, when the investor is more uncertain about the reference model, considering ambiguity aversion may deliver greater utility improvements. Furthermore, the ambiguity aversions w.r.t. stock and volatility have different effects on the degree of utility improvement. UI_2 is the utility improvement from trading in the derivative. The effects of β_S and β_V on UI_2 are different from those on UI_1 . This shows that when the investor has no access to the derivative, the effects of β_S and β_V on UI_2 are much less obvious than those on UI_1 , and even if in the absence of ambiguity aversion ($\beta_S = \beta_V = 0$), there is also a high degree of utility improvement for the investor. The reason is as given above, and we reiterate that it is suboptimal to exclude the derivative. The derivative completes the market, provides frequent trading and improves efficiency, which help the investor to pursue good investment performance.



Figure 9: Effects of β_S and β_V on utility improvements.

Figure 10 shows the effects of mean-reversion rate κ and volatility coefficient σ_V on utility improvements. In the stock return variance process, a larger mean-reversion rate κ and smaller volatility σ_V reveal less uncertainty in the variance process. That is, the investor faces low volatility risk. The utility improvement from either considering ambiguity aversion UI_1 or trading in the derivative UI_2 decrease w.r.t. κ and increase w.r.t. σ_V . When the investor has low volatility risk, compared with the case in which the investor ignores ambiguity aversion or has no access to the derivative, the utility improvement is small.²⁰

²⁰This is because there is ambiguity aversion toward the volatility risk and the derivative investment opportunity exists; as a result, when the volatility risk is low, the investor's optimal behavior will lead to less utility improvement than in the case in which volatility risk is high.



Figure 10: Effects of κ and σ_V on utility improvements.

Figure 11 shows the effects of the time horizon T and correlation $\rho_V \in (-1, 1)$ on utility improvements. The figure shows that the utility improvements UI_1 and UI_2 increase w.r.t. the time horizon T. When the investor faces a longer investment horizon, she will gain a greater utility improvement from considering ambiguity aversion or derivative trading. It is therefore necessary to consider ambiguity aversion and derivative trading in a DC pension plan over a long investment period. The case of the correlation ρ_V is interesting. Due to the specific parametrization of the model, the utility improvements (both UI_1 and UI_2) first increase and then decrease in the correlation ρ_V . Note that when $\rho_V \to \pm 1$, two risky assets are almost fully correlated; then, the role of the derivative is weakened once utility improvements are relatively small.



Figure 11: Effects of ρ_V and T on utility improvements.

6. Conclusion

In this paper, we consider a robust optimal investment problem for a DC pension investor facing a stochastic salary. The stock price exhibits stochastic volatility, and the investor has different levels of uncertainty regarding the diffusion component of the stock and its volatility. To cope with volatility risk, she is able to invest her wealth in a derivative. We first solve an optimal investment problem with both ambiguity aversion and a derivative in closed-form and provide verification theorems to guarantee the validity of the solution. Next, we obtain the solutions without the derivative, ambiguity, or salary for some interesting special cases. We also discuss the utility improvements for an investor who considers ambiguity aversion or has access to the derivative. Finally, we explore several detailed conclusions in numerical examples.

We find that three factors play significant roles in the optimal investment strategy in the DC pension plan. The first factor is ambiguity aversion. When an investor experiences uncertainty concerning her reference model, she usually reduces the exposures to market return risk and volatility risk. This is because, in an uncertain environment, it is optimal to adopt a conservative strategy. Moreover, the investor holds opposite positions in stock and derivative and there are distinct effects of ambiguity on the stock and derivative investments. The second factor is the derivative. Derivatives have the convenient properties of providing frequent trading opportunities and improving market efficiency. Investment in derivatives may deliver a large utility improvement. The third factor is salary. In a DC pension plan, the salary and the contribution thereof are essential and generate additional wealth for the investor. More importantly, the salary has an important effect on her investment strategy, and the investor has a new hedge demand in her portfolio to address salary risk. In the numerical examples, we verify the results and find that different model parameters generate distinct properties and that different degrees of ambiguity aversion lead to complicated cases. It is necessary to determine a more accurate relationship between the key factors; this is an interesting problem left for future research.

Acknowledgment

The authors are very grateful to Zhenyu Cui, Xuezhong He, Yanchu Liu, Pei Wang, Pengyu Wei, Xiaole Xue, Jieming Zhou and Yunfan Zhu for their helpful suggestions. This research was supported by the National Natural Science Foundation of China under Grant (Nos. 71771220, 71571195, 11371155, 11326199), For Ying Tung Education Foundation for Young Teachers in the Higher Education Institutions of China (No. 151081), Guangdong Natural Science Foundation for Research Team (No. 2014A030312003), Guangdong Natural Science Funds for Distinguished Young Scholars (No. 2015A030306040) and Natural Science Foundation of Guangdong Province of China (No. 2014A030310195).

Appendix A.

Proof of Proposition 2.3. We will use the contraction mapping principle (see Theorem 5.1 in Gilbarg and Trudinger, 2001) to prove the conclusion. (If the mapping \mathcal{T} from Banach space \mathcal{B} onto itself satisfies that there exists a constant $\theta < 1$ such that $||\mathcal{T}J_1 - \mathcal{T}J_2|| \leq \theta ||J_1 - J_2||$ for all $J_1, J_2 \in \mathcal{B}$, then, there exists a unique solution $J \in \mathcal{B}$ such that $\mathcal{T}J = J$.)

Restrict the initial state (x, v, l) in a compact set $A \subset \mathbb{R}^3$, choose a small enough positive constant δ , defined below, and let $\mathcal{B} = L^{\infty}(B)$ with $B = [T - \delta, T] \times A$, where $L^{\infty}(B)$ is the space of Borel-measurable functions with norm $\operatorname{esssup}\{|J(t, x, v, l)| : (t, x, v, l) \in B\}$. Next, we first consider the optimal control problem on the time interval $[T - \delta, T]$. Fix a function $J \in \mathcal{B}$; then, we denote

$$\Psi_{S}^{J}(s, x, v, l) = \frac{\beta_{S}}{(1 - \gamma)J(s, x, v, l)}, \qquad \Psi_{V}^{J}(s, x, v, l) = \frac{\beta_{V}}{(1 - \gamma)J(s, x, v, l)},$$

and

$$H^{\mathbf{u};J}(t,x,v,l) = \inf_{\Phi \in \Theta} \mathcal{E}^{\Phi}_{t,x,v,l} \left[U(X^{\Phi,\mathbf{u}}(T)) + \int_{t}^{T} \left(\frac{(\phi_{S}(s))^{2}}{2\Psi_{S}^{J}(s,x,v,l)} + \frac{(\phi_{V}(s))^{2}}{2\Psi_{V}^{J}(s,x,v,l)} \right) \, \mathrm{d}s \right]$$

subject to (15), (10) and (12).

Jubjeet to (10), (10) and (12).

Consider the optimal control problem

$$H^{J}(t,x,v,l) = \sup_{\mathbf{u}\in\Pi} H^{\mathbf{u};J}(t,x,v,l), \quad \forall \quad (t,x,v,l) \in B.$$

It is clear that there exists a unique value function $H^J \in \mathcal{B}$ (see Yong and Zhou, 1999) for the above optimal control problem. Thus, we define a mapping $\mathcal{T} : J \to H^J$ from \mathcal{B} onto itself. Suppose that J_1, J_2 are two functions in \mathcal{B} ; then, we compute that for any $\Phi \in \Theta, \mathbf{u} \in \Pi$,

$$\begin{aligned} \|\mathcal{T}(J_{1}) - \mathcal{T}(J_{2})\|_{\mathcal{B}} &= \sup_{(t,x,v,l)\in B} |H^{J_{1}}(t,x,v,l) - H^{J_{2}}(t,x,v,l)| \\ &\leq \sup_{\mathbf{u}\in\Pi,\Phi\in\Theta,(t,x,v,l)\in B} \left| \mathbf{E}_{t,x,v,l}^{\Phi} \left[\int_{t}^{T} \left(\frac{(\phi_{S}(s))^{2}}{2\Psi_{S}^{J_{1}}(s,x,v,l)} + \frac{(\phi_{V}(s))^{2}}{2\Psi_{V}^{J_{1}}(s,x,v,l)} - \frac{(\phi_{S}(s))^{2}}{2\Psi_{V}^{J_{2}}(s,x,v,l)} - \frac{(\phi_{V}(s))^{2}}{2\Psi_{V}^{J_{2}}(s,x,v,l)} \right) \mathrm{d}s \right] \right| \\ &\leq \frac{1-\gamma}{2} \sup_{\Phi\in\Theta,(t,x,v,l)\in B} \mathbf{E}^{\Phi} \left[\int_{T-\delta}^{T} |(J_{1} - J_{2})(s,x,v,l)| \left(\frac{(\phi_{S}(s))^{2}}{\beta_{S}} + \frac{(\phi_{V}(s))^{2}}{\beta_{V}} \right) \mathrm{d}s \right] \\ &\leq \frac{(1-\gamma)||J_{1} - J_{2}||_{B}}{2\min\{\beta_{S},\beta_{V}\}} \sup_{\Phi\in\Theta} \mathbf{E}^{\Phi} \left[\int_{T-\delta}^{T} ||\phi(s)||^{2} \mathrm{d}s \right]. \end{aligned}$$

$$\tag{52}$$

It is not difficult to compute that

$$\sup_{\Phi \in \Theta} \mathbb{E}^{\Phi} \left[\int_{T-\delta}^{T} ||\phi(s)||^{2} ds \right] = \sup_{\Phi \in \Theta} \mathbb{E} \left[\Lambda^{\Phi}(T) \int_{T-\delta}^{T} ||\phi(s)||^{2} ds \right]$$

$$= \sup_{\Phi \in \Theta} \mathbb{E} \left[\left[\Lambda^{\kappa \Phi/\underline{\kappa}\sigma_{V}}(T) \right]^{\underline{\kappa}\sigma_{V}/\kappa} \exp \left\{ \frac{\kappa - \underline{\kappa}\sigma_{V}}{2\underline{\kappa}\sigma_{V}} \int_{0}^{T} ||\phi(s)||^{2} ds \right\} \int_{T-\delta}^{T} ||\phi(s)||^{2} ds \right]$$

$$\leq \sup_{\Phi \in \Theta} \underline{\kappa}^{2} \mathbb{E} \left[\left[\Lambda^{\kappa \Phi/\underline{\kappa}\sigma_{V}}(T) \right]^{\underline{\kappa}\sigma_{V}/\kappa} \exp \left\{ \frac{\underline{\kappa}(\kappa - \underline{\kappa}\sigma_{V})}{2\sigma_{V}} \int_{0}^{T} V(s) ds \right\} \int_{T-\delta}^{T} V(s) ds \right]$$

$$\leq \sup_{\Phi \in \Theta} \underline{\kappa}^{2} \mathbb{E} \left[\Lambda^{\kappa \Phi/\underline{\kappa}\sigma_{V}}(T) \right]^{\underline{\kappa}\sigma_{V}/\kappa} \mathbb{E} \left[\exp \left\{ \frac{\kappa^{2}}{2\sigma_{V}^{2}} \int_{0}^{T} V(s) ds \right\} \right]^{\underline{\kappa}\sigma_{V}(\kappa - \underline{\kappa}\sigma_{V})/\kappa^{2}}$$

$$\mathbb{E} \left[\int_{T-\delta}^{T} (V(s))^{\kappa^{2}/(\kappa - \underline{\kappa}\sigma_{V})^{2}} ds \right]^{(\kappa - \underline{\kappa}\sigma_{V})^{2}/\kappa^{2}} \delta^{1-(\kappa - \underline{\kappa}\sigma_{V})^{2}/\kappa^{2}}, \tag{53}$$

where we used Assumption (iii) in footnote 6 in the first equality and Holder's inequality in the second inequality.

From Theorem 5.1 in Taksar and Zeng (2009), we conclude that

$$\mathbf{E}\left[\exp\left(\frac{\kappa^2}{2\underline{\kappa}^2\sigma_V^2}\int_0^T ||\phi(s)||^2 \mathrm{d}s\right)\right] \le \mathbf{E}\left[\exp\left(\frac{\kappa^2}{2\sigma_V^2}\int_0^T V(s)\mathrm{d}s\right)\right] < \infty,$$

and $\Lambda^{\kappa\Phi/\underline{\kappa}\sigma_V}$ is an exponential martingale. Moreover, the regularity result for SDE implies that

$$\mathbb{E}\left[\int_0^T (V(s))^{\kappa^2/(\kappa-\underline{\kappa}\sigma_V)^2} \,\mathrm{d}s\right] < +\infty.$$

Thus, combining (52) and (53), we can choose a small enough $\delta > 0$ such that

$$\sup_{\Phi \in \Theta} \mathbf{E}^{\Phi} \left[\int_{T-\delta}^{T} ||\phi(s)||^2 \, \mathrm{d}s \right] \leq \frac{\min\{\beta_S, \beta_V\}}{1-\gamma},$$

and

$$\|\mathcal{T}(J_1) - \mathcal{T}(J_2)\|_{\mathcal{B}} \le \frac{1}{2} \|J_1 - J_2\|_{\mathcal{B}}.$$

Hence, the mapping \mathcal{T} is a contraction mapping. According to the contraction mapping principle, the mapping \mathcal{T} has a unique fixed point. This means that there exists a unique value function H(t, x, v, l) of the optimal control problem if $t \in [T - \delta, T]$ and $(x, v, l) \in A$, which consists of (18), (19) and (20) subject to (15), (10) and (12).

Next, we extend the result into the total time interval [0, T]. Suppose that we have proven that there exists a unique value function H(t, x, v, l) of the optimal control problem if $t \in [\widehat{T}, T]$ and $(x, v, l) \in A$.

Then, we choose a small enough positive number δ such that

$$\sup_{\Phi\in\Theta} \mathbf{E}^{\Phi} \left[\int_{\widehat{T}-\delta}^{\widehat{T}} ||\phi(s)||^2 \, \mathrm{d}s \right] = \frac{\min\{\beta_S, \beta_V\}}{1-\gamma}.$$

Moreover, let $\mathcal{B} = L^{\infty}(B)$ with $B = [\widehat{T} - \delta, \widehat{T}] \times A$. Fix a function $J \in \mathcal{B}$; then, we denote

$$\begin{split} \Psi^{J}_{S}(s,x,v,l) &= \frac{\beta_{S}}{(1-\gamma)J(s,x,v,l)I_{\{s\in[\widehat{T}-\delta,\widehat{T}]\}} + (1-\gamma)H(s,x,v,l)I_{\{s\in(\widehat{T},T]\}}}}{\Psi^{J}_{V}(s,x,v,l)} &= \frac{\beta_{V}}{(1-\gamma)J(s,x,v,l)I_{\{s\in[\widehat{T}-\delta,\widehat{T}]\}} + (1-\gamma)H(s,x,v,l)I_{\{s\in(\widehat{T},T]\}}}} \end{split}$$

and

$$H^{\mathbf{u};J}(t,x,v,l) = \inf_{\Phi \in \Theta} \mathcal{E}^{\Phi}_{t,x,v,l} \left[U(X^{\Phi,\mathbf{u}}(T)) + \int_{t}^{T} \left(\frac{(\phi_{S}(s))^{2}}{2\Psi_{S}^{J}(s,x,v,l)} + \frac{(\phi_{V}(s))^{2}}{2\Psi_{V}^{J}(s,x,v,l)} \right) \, \mathrm{d}s \right]$$

subject to (15), (10) and (12).

Consider the optimal control problem

$$H^{J}(t,x,v,l) = \sup_{\mathbf{u}\in\Pi} H^{\mathbf{u};J}(t,x,v,l), \quad \forall \quad (t,x,v,l) \in B.$$

Repeating the same argument as above, we can prove that there exists a unique value function H(t, x, v, l) of the optimal control problem if $(t, x, v, l) \in B$. Repeating the same argument in the domain $[T - 2\delta, T - \delta] \times A$, $[T - 3\delta, T - 2\delta] \times A$, \cdots , we can prove that there exists a unique value function H(t, x, v, l) of the optimal control problem if $(t, x, v, l) \in$ $[0, T] \times A$. Since the set A is arbitrary, and the compatibility in different compact sets is obvious, then we have proven that there exists a unique value function H(t, x, v, l) of the optimal control problem for any $(t, x, v, l) \in [0, T] \times \mathbb{R}^3$.

Appendix B.

Proof of Proposition 3.1. We know that $\Psi_S(t, x, v, l)$, $\Psi_V(t, x, v, l)$ in Proposition 2.3 are $\Psi_S^J(t, x, v, l)$, $\Psi_V^J(t, x, v, l)$, respectively. Consider the optimal control problem

$$H^{J}(t, x, v, l) = \sup_{\mathbf{u}\in\Pi} \inf_{\Phi\in\Theta} \mathcal{E}^{\Phi}_{t, x, v, l} \left[U(X^{\Phi, \mathbf{u}}(T)) + \int_{t}^{T} g(s, x, v, l, \phi_{S}, \phi_{V}) \, ds \right]$$

subject to (15), (10) and (12) for any $(t, x, v, l) \in O$, where

$$g(s, x, v, l, \phi_S, \phi_V) = \frac{\phi_S^2}{2\Psi_S^J(s, x, v, l)} + \frac{\phi_V^2}{2\Psi_V^J(s, x, v, l)}.$$

Note that in this optimal control problem, J in Ψ_S^J and Ψ_V^J is the given function in assumptions rather than the value function. Thus, g is a given function w.r.t. (s, x, v, l, ϕ) , independent of the value function H^J , and the optimal control problem is standard.

Repeating a proof similar to that in Theorem 3.2 in Mataramvura and Øksendal (2008), we deduce that J is the value function of the above optimal control problem. Since the value function and J in Ψ_S^J and Ψ_V^J are the same, J is the value function of the optimal control problem, consisting of (18), (19) and (20) subject to (15), (10) and (12). Thus, by Proposition 2.3, the uniqueness of the value function implies that H(t, x, v, l) = J(t, x, v, l)for any $(t, x, v, l) \in O$, and (\mathbf{u}^*, Φ^*) is an optimal control. \Box

Appendix C.

Proof of Theorem 3.2. According to the first-order optimality conditions, the functions ϕ_S^* and ϕ_V^* , which realize the infimum part of Eq. (21), are given by

$$\phi_S^* = \frac{\beta_S \sqrt{v}}{(1-\gamma)J} \left[x \theta_S J_x + \sigma_V \rho J_v + l \sigma_L \rho_L J_l \right],$$

$$\phi_V^* = \frac{\beta_V \sqrt{v}}{(1-\gamma)J} \left[x \theta_V J_x + \sigma_V \sqrt{1-\rho_L^2} J_v + l \sigma_L \sqrt{1-\rho^2} J_l \right].$$
(54)

Substituting Eq. (54) into Eq. (21), we have

$$J_{t} + (rx + x\theta_{S}\lambda_{1}v + x\theta_{V}\lambda_{2}v + \xi l)J_{x} + \kappa(\delta - v)J_{v} + (\mu_{L}l + l\sigma_{L}\lambda_{1}v\rho_{L} + l\sigma_{L}\lambda_{2}v\sqrt{1 - \rho_{L}^{2}})J_{l} + \frac{1}{2}x^{2}v(\theta_{S}^{2} + \theta_{V}^{2})J_{xx} + \frac{1}{2}\sigma_{V}^{2}vJ_{vv} + \frac{1}{2}l^{2}\sigma_{L}^{2}vJ_{ll} + (x\sigma_{V}\theta_{S}v\rho_{V} + x\sigma_{V}\theta_{V}v\sqrt{1 - \rho_{V}^{2}})J_{xv} + (x\theta_{S}l\sigma_{L}v\rho_{L} + x\theta_{V}l\sigma_{L}v\sqrt{1 - \rho_{L}^{2}})J_{xl} + l\sigma_{L}v\sigma_{V}(\rho_{V}\rho_{L} + \sqrt{1 - \rho_{V}^{2}}\sqrt{1 - \rho_{L}^{2}})J_{lv} - \frac{\beta_{S}v}{2(1 - \gamma)J}[x\theta_{S}J_{x} + \sigma_{V}\rho_{V}J_{v} + l\sigma_{L}\rho_{L}J_{l}]^{2} - \frac{\beta_{V}v}{2(1 - \gamma)J}[x\theta_{V}J_{x} + \sigma_{V}\sqrt{1 - \rho_{V}^{2}}J_{v} + l\sigma_{L}\sqrt{1 - \rho_{L}^{2}}J_{l}]^{2} = 0.$$
(55)

Differentiating Eq. (55) w.r.t. (θ_S, θ_V) implies

$$\theta_{S}^{*} = \frac{\lambda_{1}J_{x} - \frac{\beta_{S}}{(1-\gamma)J}(\sigma_{V}\rho_{V}J_{x}J_{v} + l\sigma_{L}\rho_{L}J_{x}J_{l}) + \sigma_{V}\rho_{V}J_{xv} + l\sigma_{L}\rho_{L}J_{xl}}{x\left[\frac{\beta_{S}}{(1-\gamma)J}J_{x}^{2} - J_{xx}\right]},$$

$$\theta_{V}^{*} = \frac{\lambda_{2}J_{x} - \frac{\beta_{V}}{(1-\gamma)J}(\sigma_{V}\sqrt{1-\rho_{V}^{2}}J_{x}J_{v} + l\sigma_{L}\sqrt{1-\rho_{L}^{2}}J_{x}J_{l}) + \sigma_{V}\sqrt{1-\rho_{V}^{2}}J_{xv} + l\sigma_{L}\sqrt{1-\rho_{L}^{2}}J_{xl}}{x\left[\frac{\beta_{V}}{(1-\gamma)J}J_{x}^{2} - J_{xx}\right]}$$
(56)

Plugging Eq. (56) into Eq. (55) implies

$$\begin{split} J_{t} + (rx + \xi l)J_{x} + \kappa(\delta - v)J_{v} + (\mu_{L}l + l\sigma_{L}\lambda_{1}v\rho_{L} + l\sigma_{L}\lambda_{2}v\sqrt{1 - \rho_{L}^{2}})J_{l} + \frac{1}{2}\sigma_{V}^{2}vJ_{vv} + \frac{1}{2}l^{2}\sigma_{L}^{2}vJ_{ll} \\ + l\sigma_{L}v\sigma_{V}(\rho_{V}\rho_{L} + \sqrt{1 - \rho_{V}^{2}}\sqrt{1 - \rho_{L}^{2}})J_{lv} - \frac{\beta_{S}v}{2(1 - \gamma)J}\left(\sigma_{V}^{2}\rho_{V}^{2}J_{v}^{2} + l^{2}\sigma_{L}^{2}\rho_{L}^{2}J_{l}^{2} + 2\sigma_{V}\sigma_{L}\rho_{V}\rho_{L}lJ_{v}J_{l}\right) \\ - \frac{\beta_{V}v}{2(1 - \gamma)J}\left(\sigma_{V}^{2}(1 - \rho_{V}^{2})J_{v}^{2} + l^{2}\sigma_{L}^{2}(1 - \rho_{L}^{2})J_{l}^{2} + 2\sigma_{V}\sigma_{L}\sqrt{1 - \rho_{V}^{2}}\sqrt{1 - \rho_{L}^{2}}lJ_{v}J_{l}\right) \\ + \frac{v\left[\lambda_{1}J_{x} - \frac{\beta_{S}}{(1 - \gamma)J}(\sigma_{V}\rho_{V}J_{x}J_{v} + l\sigma_{L}\rho_{L}J_{x}J_{l}) + \sigma_{V}\rho_{V}J_{xv} + l\sigma_{L}\rho_{L}J_{xl}\right]^{2}}{2\left[\frac{\beta_{S}}{(1 - \gamma)J}J_{x}^{2} - J_{xx}\right]} \\ + \frac{v[\lambda_{2}J_{x} - \frac{\beta_{V}}{(1 - \gamma)J}(\sigma_{V}\sqrt{1 - \rho_{V}^{2}}J_{x}J_{v} + l\sigma_{L}\sqrt{1 - \rho_{L}^{2}}J_{x}J_{l}) + \sigma_{V}\sqrt{1 - \rho_{V}^{2}}J_{xv} + l\sigma_{L}\sqrt{1 - \rho_{L}^{2}}J_{xl}]^{2}}{2\left[\frac{\beta_{V}}{(1 - \gamma)J}J_{x}^{2} - J_{xx}\right]} = 0. \end{split}$$

To solve Eq. (57), we attempt to conjecture the solution in the following form:

$$J(t, x, v, l) = \frac{(x + h(t, l))^{1 - \gamma}}{1 - \gamma} g(t, v), \quad h(T, l) = 0, \quad g(T, v) = 1,$$
(58)

the partial derivatives of which are

$$J_{t} = g_{t} \frac{(x+h)^{1-\gamma}}{1-\gamma} + g(x+h)^{-\gamma}h_{t}, \quad J_{x} = g(x+h)^{-\gamma}, \quad J_{xx} = -\gamma g(x+h)^{-\gamma-1},$$

$$J_{v} = g_{v} \frac{(x+h)^{1-\gamma}}{1-\gamma}, \quad J_{vv} = g_{vv} \frac{(x+h)^{1-\gamma}}{1-\gamma}, \quad J_{l} = g(x+h)^{-\gamma}h_{l}, \quad J_{lv} = g_{v}(x+h)^{-\gamma}h_{l}$$

$$J_{ll} = -\gamma g(x+h)^{-\gamma-1}h_{l}^{2} + g(x+h)^{-\gamma}h_{ll}, \quad J_{xv} = g_{v}(x+h)^{-\gamma}, \quad J_{xl} = -\gamma g(x+h)^{-\gamma-1}h_{l}.$$

(59)

Substituting Eqs. (58)-(59) into Eq. (57), we have

$$g_{t}\frac{(x+h)^{1-\gamma}}{1-\gamma} + g(x+h)^{-\gamma}h_{t} + rxg(x+h)^{-\gamma} + \xi lg(x+h)^{-\gamma} + \kappa(\delta-v)g_{v}\frac{(x+h)^{1-\gamma}}{1-\gamma} \\ + (\mu_{L}l + l\sigma_{L}\lambda_{1}v\rho_{L} + l\sigma_{L}\lambda_{2}v\sqrt{1-\rho_{L}^{2}})g(x+h)^{-\gamma}h_{l} + \frac{1}{2}\sigma_{V}^{2}vg_{vv}\frac{(x+h)^{1-\gamma}}{1-\gamma} \\ + \frac{1}{2}l^{2}\sigma_{L}^{2}v[-\gamma g(x+h)^{-\gamma-1}h_{l}^{2} + g(x+h)^{-\gamma}h_{ll}] + l\sigma_{L}v\sigma_{V}(\rho_{V}\rho_{L} + \sqrt{1-\rho_{V}^{2}}\sqrt{1-\rho_{L}^{2}})g_{v}(x+h)^{-\gamma}h_{l} \\ - \frac{\beta_{S}v}{2g}\left[\sigma_{V}^{2}\rho_{V}^{2}g_{v}^{2}\frac{(x+h)^{1-\gamma}}{(1-\gamma)^{2}} + l^{2}\sigma_{L}^{2}\rho_{L}^{2}g^{2}(x+h)^{-\gamma-1}h_{l}^{2} + 2\sigma_{L}\sigma_{V}\rho_{V}\rho_{L}lg_{v}\frac{(x+h)^{-\gamma}}{1-\gamma}gh_{l}\right] \\ - \frac{\beta_{V}v}{2g}\left[\sigma_{V}^{2}(1-\rho_{V}^{2})g_{v}^{2}\frac{(x+h)^{1-\gamma}}{(1-\gamma)^{2}} + l^{2}\sigma_{L}^{2}(1-\rho_{L}^{2})g^{2}(x+h)^{-\gamma-1}h_{l}^{2} \\ + 2\sigma_{L}\sigma_{V}\sqrt{1-\rho_{V}^{2}}\sqrt{1-\rho_{L}^{2}}lg_{v}\frac{(x+h)^{-\gamma}}{1-\gamma}gh_{l}\right] \\ + \frac{v\left[\lambda_{1}g(x+h)^{-\gamma} + \frac{1-(\beta_{S}+\gamma)}{1-\gamma}\sigma_{V}\rho_{V}g_{v}(x+h)^{-\gamma} - (\beta_{S}+\gamma)l\sigma_{L}\rho_{L}g(x+h)^{-\gamma-1}h_{l}\right]^{2}}{2(\beta_{S}+\gamma)g(x+h)^{-\gamma-1}} \\ + \frac{v\left[\lambda_{2}g(x+h)^{-\gamma} + \frac{1-(\beta_{V}+\gamma)}{1-\gamma}\sigma_{V}\sqrt{1-\rho_{V}^{2}}g_{v}(x+h)^{-\gamma} - (\beta_{V}+\gamma)l\sigma_{L}\sqrt{1-\rho_{L}^{2}}g(x+h)^{-\gamma-1}h_{l}\right]^{2}}{2(\beta_{V}+\gamma)g(x+h)^{-\gamma-1}}$$

$$(60)$$

Furthermore, let

$$g(t,v) = e^{\bar{g}(t)v + \hat{g}(t)}, \quad \bar{g}(T) = \hat{g}(T) = 0,$$

$$h(t,l) = \bar{h}(t)l + \hat{h}(t), \quad \bar{h}(T) = \hat{h}(T) = 0.$$
(61)

Then,

$$g_t = g(\bar{g}_t v + \hat{g}_t), \quad g_v = g\bar{g}, \quad g_{vv} = g\bar{g}^2, \quad h_t = \bar{h}_t l + \hat{h}_t, \quad h_l = \bar{h}, \quad h_{ll} = 0.$$
 (62)

Inserting Eqs. (61)-(62) into Eq. (60) implies

$$\frac{x+h}{1-\gamma} \left\{ v \left[\bar{g}_t + \left(-\kappa + \frac{\lambda_1 (1-(\beta_S+\gamma))\sigma_V \rho}{\beta_S+\gamma} + \frac{\lambda_2 (1-(\beta_V+\gamma))\sigma_V \sqrt{1-\rho^2}}{\beta_V+\gamma} \right) \bar{g} + \left(\frac{\sigma_V^2}{2} - \frac{\beta_S \sigma_V^2 \rho^2}{2(1-\gamma)} - \frac{\beta_V \sigma_V^2 (1-\rho^2)}{2(1-\gamma)} + \frac{(1-(\beta_S+\gamma))^2 \sigma_V^2 \rho^2}{2(\beta_S+\gamma)(1-\gamma)} + \frac{(1-(\beta_V+\gamma))^2 \sigma_V^2 (1-\rho^2)}{2(\beta_V+\gamma)(1-\gamma)} \right) \bar{g}^2 + \frac{\lambda_1^2 (1-\gamma)}{2(\beta_S+\gamma)} + \frac{\lambda_2^2 (1-\gamma)}{2(\beta_V+\gamma)} \right] + \hat{g}_t + r(1-\gamma) + \kappa \delta \bar{g} \right\} + l \left\{ \bar{h}_t + (\mu_L - r)\bar{h} + \xi \right\} + \hat{h}_t - r\hat{h} = 0.$$
(63)

By separating the variables with and without x, v and l, we can derive the following equations:

$$\begin{split} \bar{g}_t + \left(-\kappa + \frac{\lambda_1 (1 - (\beta_S + \gamma))\sigma_V \rho}{\beta_S + \gamma} + \frac{\lambda_2 (1 - (\beta_V + \gamma))\sigma_V \sqrt{1 - \rho^2}}{\beta_V + \gamma} \right) \bar{g} \\ + \left(\frac{\sigma_V^2}{2} - \frac{\beta_S \sigma_V^2 \rho^2}{2(1 - \gamma)} - \frac{\beta_V \sigma_V^2 (1 - \rho^2)}{2(1 - \gamma)} + \frac{(1 - (\beta_S + \gamma))^2 \sigma_V^2 \rho^2}{2(\beta_S + \gamma)(1 - \gamma)} + \frac{(1 - (\beta_V + \gamma))^2 \sigma_V^2 (1 - \rho^2)}{2(\beta_V + \gamma)(1 - \gamma)} \right) \bar{g}^2 \\ + \frac{\lambda_1^2 (1 - \gamma)}{2(\beta_S + \gamma)} + \frac{\lambda_2^2 (1 - \gamma)}{2(\beta_V + \gamma)} = 0, \\ \hat{g}_t + r(1 - \gamma) + \kappa \delta \bar{g} = 0, \\ \bar{h}_t + (\mu_L - r) \bar{h} + \xi = 0, \quad \hat{h}_t - r \hat{h} = 0. \end{split}$$

Considering the boundary conditions, we have

$$\bar{g}(t) = \frac{\nu_1 \nu_2 - \nu_1 \nu_2 e^{\alpha_2 (\nu_1 - \nu_2)(T-t)}}{\nu_2 - \nu_1 e^{\alpha_2 (\nu_1 - \nu_2)(T-t)}}, \quad \hat{g}(t) = \int_t^T \left[r(1-\gamma) + \kappa \delta g_1(s) \right] \mathrm{d}s,$$

$$\bar{h}(t) = \frac{\xi}{\mu_L - r} (e^{(\mu_L - r)(T-t)} - 1), \quad \hat{h}(t) = 0,$$
(64)

where

$$\begin{split} \alpha_{1} &= -\kappa + \frac{\lambda_{1}(1 - (\beta_{S} + \gamma))\sigma_{V}\rho}{\beta_{S} + \gamma} + \frac{\lambda_{2}(1 - (\beta_{V} + \gamma))\sigma_{V}\sqrt{1 - \rho^{2}}}{\beta_{V} + \gamma}, \\ \alpha_{2} &= \frac{\sigma_{V}^{2}}{2} - \frac{\beta_{S}\sigma_{V}^{2}\rho^{2}}{2(1 - \gamma)} - \frac{\beta_{V}\sigma_{V}^{2}(1 - \rho^{2})}{2(1 - \gamma)} + \frac{(1 - (\beta_{S} + \gamma))^{2}\sigma_{V}^{2}\rho^{2}}{2(\beta_{S} + \gamma)(1 - \gamma)} + \frac{(1 - (\beta_{V} + \gamma))^{2}\sigma_{V}^{2}(1 - \rho^{2})}{2(\beta_{V} + \gamma)(1 - \gamma)}, \\ \alpha_{3} &= \frac{\lambda_{1}^{2}(1 - \gamma)}{2(\beta_{S} + \gamma)} + \frac{\lambda_{2}^{2}(1 - \gamma)}{2(\beta_{V} + \gamma)}, \\ \nu_{1,2} &= \frac{\alpha_{1} \pm \sqrt{\alpha_{1}^{2} - 4\alpha_{2}\alpha_{3}}}{-2\alpha_{2}}. \end{split}$$

Substituting $\bar{g}(t)$, $\hat{g}(t)$, $\bar{h}(t)$ and $\hat{h}(t)$ into Eqs. (54) and (56), we can derive $\theta_S^*(t)$, $\theta_V^*(t)$, $\phi_S^*(t)$ and $\phi_V^*(t)$.

As $\beta_S, \beta_V > 0, \gamma > 1$, we have $\alpha_2 > 0$ and $\alpha_3 < 0$. By calculations, we obtain

$$\begin{split} \alpha_2 &= \frac{\sigma_V^2}{2} - \frac{\beta_S \sigma_V^2 \rho^2}{2(1-\gamma)} - \frac{\beta_V \sigma_V^2 (1-\rho^2)}{2(1-\gamma)} + \frac{(1-(\beta_S+\gamma))^2 \sigma_V^2 \rho^2}{2(\beta_S+\gamma)(1-\gamma)} + \frac{(1-(\beta_V+\gamma))^2 \sigma_V^2 (1-\rho^2)}{2(\beta_V+\gamma)(1-\gamma)}, \\ &= \frac{\sigma_V^2}{2} - \frac{\beta_S \sigma_V^2 \rho^2}{2(1-\gamma)} - \frac{\beta_V \sigma_V^2 (1-\rho^2)}{2(1-\gamma)} + \frac{\sigma_V^2 \rho^2}{2(\beta_S+\gamma)(1-\gamma)} - \frac{\sigma_V^2 \rho^2}{1-\gamma} + \frac{(\beta_S+\gamma) \sigma_V^2 \rho^2}{2(1-\gamma)} \\ &+ \frac{\sigma_V^2 (1-\rho^2)}{2(\beta_V+\gamma)(1-\gamma)} - \frac{\sigma_V^2 (1-\rho^2)}{1-\gamma} + \frac{(\beta_V+\gamma) \sigma_V^2 (1-\rho^2)}{2(1-\gamma)}, \\ &= \frac{\sigma_V^2}{2} + \frac{\sigma_V^2 \rho^2}{2(\beta_S+\gamma)(1-\gamma)} + \frac{\sigma_V^2 (1-\rho^2)}{2(\beta_V+\gamma)(1-\gamma)} - \frac{\sigma_V^2}{1-\gamma} + \frac{\gamma \sigma_V^2}{2(1-\gamma)}. \end{split}$$

As $\gamma > 1$, we have

$$\frac{\sigma_V^2 \rho^2}{2(\beta_S + \gamma)(1 - \gamma)} + \frac{\sigma_V^2 (1 - \rho^2)}{2(\beta_V + \gamma)(1 - \gamma)} > \frac{\sigma_V^2 \rho^2}{2(1 - \gamma)} + \frac{\sigma_V^2 (1 - \rho^2)}{2(1 - \gamma)} = \frac{\sigma_V^2}{2(1 - \gamma)}.$$

Therefore,

$$\alpha_2 > \frac{\sigma_V^2}{2} + \frac{\sigma_V^2}{2(1-\gamma)} - \frac{\sigma_V^2}{1-\gamma} + \frac{\gamma \sigma_V^2}{2(1-\gamma)} = 0.$$

Because $\alpha_3 < 0$, $\alpha_1^2 - 4\alpha_2\alpha_3 > 0$. The proof of Theorem 3.2 is completed.

Appendix D

This appendix mainly provides the proof of Theorem 3.4. Before giving the proof, we present some lemmas, which are used in the proof of Theorem 3.4.

Lemma D.1. $\bar{g}(t)$ given by Eq. (30) is an increasing function of t and $\bar{g}(t) \leq 0, \forall t \in [0, T]$.

Proof. The direct calculation shows that

$$\bar{g}_t(t) = \frac{-\nu_1 \nu_2 (\nu_1 - \nu_2)^2 \alpha_2 e^{\alpha_2 (\nu_1 - \nu_2)(T-t)}}{(\nu_2 - \nu_1 e^{\alpha_2 (\nu_1 - \nu_2)(T-t)})^2}.$$

It is obvious that $\nu_2 > 0 > \nu_1$ and $\alpha_2 > 0$, which implies $\bar{g}_t(t) > 0$, i.e., $\bar{g}(t)$ is an increasing function of t. As $\bar{g}(T) = 0$, then $\bar{g}(t) \le 0$, $\forall t \in [0, T]$.

In Theorem 3.2, we have already derived the optimal risk exposure and the optimal investment strategy. However, we should guarantee that the Radon-Nikodym derivative $\Lambda^*(t)$ of \mathbb{Q} w.r.t. \mathbb{P} corresponding to the optimal worst-case scenario drifts $\phi_S^*(t)$ and $\phi_V^*(t)$, i.e., the expression $\Lambda(t)$ with $\phi_S^*(t)$, $\phi_V^*(t)$ instead of $\phi_S(t)$ and $\phi_V(t)$, is indeed a \mathbb{P} -martingale, which ensures a well-defined \mathbb{Q}^* . The following lemma states sufficient conditions for this scenario based on Novikov's condition and Theorem 5.1 in Taksar and Zeng (2009).

Lemma D.2. Novikov's condition

$$\operatorname{E}\left[\exp\left(\int_{0}^{T}\left(\frac{1}{2}(\phi_{S}^{*}(s))^{2}+\frac{1}{2}(\phi_{V}^{*}(s))^{2}\right)\mathrm{d}s\right)\right]<\infty$$

holds for $\phi_S^*(t)$ and $\phi_V^*(t)$ if the parameters satisfy that for $\forall \bar{g}(t) \in [\bar{g}(0), 0]$,

$$\frac{\beta_S^2 (\lambda_1 (1-\gamma) + \sigma_V \rho_V \bar{g}(t))^2}{(1-\gamma)^2 (\beta_S + \gamma)^2} + \frac{\beta_V^2 (\lambda_2 (1-\gamma) + \sigma_V \sqrt{1-\rho_V^2} \bar{g}(t))^2}{(1-\gamma)^2 (\beta_V + \gamma)^2} < \frac{\kappa^2}{\sigma_V^2}.$$
 (65)

Proof. From Theorem 3.2, we have

$$\phi_{S}^{*}(t) = \frac{\beta_{S}\sqrt{V(t)}(\lambda_{1}(1-\gamma) + \sigma_{V}\rho_{V}\bar{g}(t))}{(1-\gamma)(\beta_{S}+\gamma)}, \quad \phi_{V}^{*}(t) = \frac{\beta_{V}\sqrt{V(t)}(\lambda_{2}(1-\gamma) + \sigma_{V}\sqrt{1-\rho_{V}^{2}}\bar{g}(t))}{(1-\gamma)(\beta_{V}+\gamma)}.$$

Then

$$E\left[\exp\left(\int_{0}^{T} \left(\frac{1}{2}(\phi_{S}^{*}(s))^{2} + \frac{1}{2}(\phi_{V}^{*}(s))^{2}\right) ds\right)\right]$$

$$= E\left[\exp\left(\int_{0}^{T} \left(\frac{\beta_{S}^{2}(\lambda_{1}(1-\gamma) + \sigma_{V}\rho\bar{g}(s))^{2}}{2(1-\gamma)^{2}(\beta_{S}+\gamma)^{2}} + \frac{\beta_{V}^{2}(\lambda_{2}(1-\gamma) + \sigma_{V}\sqrt{1-\rho^{2}}\bar{g}(s))^{2}}{2(1-\gamma)^{2}(\beta_{V}+\gamma)^{2}}\right)V(t)ds\right)\right].$$

With condition (65), we can verify that $\Phi^* := \{\phi^*(t) := (\phi^*_S(t), \phi^*_V(t))\}_{t \in [0,T]}$ satisfies Novikov's condition as follows.

$$\begin{split} & \mathbf{E}\left[\exp\left(\frac{1}{2}\int_{0}^{T}||\phi^{*}(s)||^{2}\mathrm{d}s\right)\right] = \mathbf{E}\left[\exp\left(\int_{0}^{T}\left(\frac{1}{2}(\phi^{*}_{S}(s))^{2} + \frac{1}{2}(\phi^{*}_{V}(s))^{2}\right)\mathrm{d}s\right)\right] \\ & \leq \mathbf{E}\left[\exp\left(\frac{\kappa^{2}}{2\sigma_{V}^{2}}\int_{0}^{T}V(s)\mathrm{d}s\right)\right] < \infty. \end{split}$$

The first estimate follows from condition (65) because of the property of quadratic functions, and the second is from Theorem 5.1 in Taksar and Zeng (2009). \Box

To verify condition (4) in Proposition 3.1, we present another lemma.

Lemma D.3. For problem (18), if J(t, x, v, l) is the solution to the HJB equation (21) and the parameters satisfy that for $\bar{g}(t) \in [\bar{g}(0), 0]$,

$$[64(1-\gamma)^2 - 4(1-\gamma)][(m(t))^2 + (n(t))^2] + 8(1-\gamma)A(t) \le \frac{\kappa^2}{2\sigma_V^2},$$
(66)

we have

$$\mathbf{E}^{\Phi^*}\left[\sup_{t\in[0,T]}|J(t, X^{\Phi^*, \mathbf{u}^*}(t), V(t), L(t))|^4\right] < \infty,$$

and

$$\mathbf{E}^{\Phi^*}\left[\sup_{t\in[0,T]}\left|\frac{(\phi_S^*(t))^2}{2\Psi_S(t,X^{\Phi^*,\mathbf{u}^*}(t),V(t),L(t))} + \frac{(\phi_V^*(t))^2}{2\Psi_V(t,X^{\Phi^*,\mathbf{u}^*}(t),V(t),L(t))}\right|^2\right] < \infty,$$

where

$$A(t) = \gamma(m(t))^2 - \sigma_V \rho_V \bar{g}(t)m(t) + \gamma(n(t))^2 - \sigma_V \sqrt{1 - \rho_V^2} \bar{g}(t)n(t),$$
(67)

and m(t), n(t) are given by Eqs. (28) and (29).

Proof. Step 1. Proof of $E^{\Phi^*} \left[\sup_{t \in [0,T]} |J(t, X^{\Phi^*, \mathbf{u}^*}(t), V(t), L(t))|^4 \right] < \infty.$

Substituting Eqs. (24) and (27) into Eq. (15), we have

$$\frac{\mathrm{d}(X^{\Phi^*,\mathbf{u}^*}(t)+\bar{h}(t)L(t))}{X^{\Phi^*,\mathbf{u}^*}+\bar{h}(t)L(t)} = (r+A(t)V(t))\mathrm{d}t + m(t)\sqrt{V(t)}\mathrm{d}W_S^{\Phi^*}(t) + n(t)\sqrt{V(t)}\mathrm{d}W_V^{\Phi^*}(t),$$
(68)

where m(t), n(t) and A(t) are given by Eqs. (28), (29) and (67). It is easy to obtain that Eq. (68) has a unique positive solution

$$\begin{aligned} X^{\Phi^*,\mathbf{u}^*}(t) + \bar{h}(t)L(t) &= (x_0 + \bar{h}(0)l_0) \exp\left\{\int_0^t r \mathrm{d}s + \int_0^t \underbrace{\left(A(s) - \frac{1}{2}(m(s))^2 - \frac{1}{2}(n(s))^2\right)}_{\bar{A}(s)} V(s) \mathrm{d}s \right. \\ &+ \int_0^t m(s)\sqrt{V(s)} \mathrm{d}W_S^{\Phi^*}(s) + \int_0^t n(s)\sqrt{V(s)} \mathrm{d}W_V^{\Phi^*}(s) \right\}. \end{aligned}$$

Because

$$J(t, X^{\Phi^*, \mathbf{u}^*}(t), V(t), L(t)) = \frac{(X^{\Phi^*, \mathbf{u}^*}(t) + \bar{h}(t)L(t))^{1-\gamma}}{1-\gamma} \exp(\bar{g}(t)V(t) + \hat{g}(t)),$$

 $\bar{g}(t) \in [\bar{g}(0), 0]$, and $\hat{g}(t)$ is bounded, we obtain the following estimate with the appropriate constant $K_1 > 0$,

$$|J(t, X^{\Phi^*, \mathbf{u}^*}(t), V(t), L(t))|^4 = \left| \frac{(X^{\Phi^*, \mathbf{u}^*}(t) + \bar{h}(t)L(t))^{1-\gamma}}{1-\gamma} \exp(\bar{g}(t)V(t) + \hat{g}(t)) \right|^4$$
$$\leq K_1 \left| (X^{\Phi^*, \mathbf{u}^*}(t) + \bar{h}(t)L(t))^{1-\gamma} \right|^4.$$

Next, we focus on $|(X^{\Phi^*,\mathbf{u}^*}(t) + \bar{h}(t)L(t))^{1-\gamma}|^4$.

$$\begin{split} & \left| (X^{\Phi^*,\mathbf{u}^*}(t) + \bar{h}(t)L(t))^{1-\gamma} \right|^4 \\ & \leq K_2 \exp\left\{ \int_0^t 4(1-\gamma)\bar{A}(s)V(s)\mathrm{d}s + \int_0^t 4(1-\gamma)m(s)\sqrt{V(s)}\mathrm{d}W_S^{\Phi^*}(s) \\ & + \int_0^t 4(1-\gamma)n(s)\sqrt{V(s)}\mathrm{d}W_V^{\Phi^*}(s) \right\} \\ & = K_2 \exp\left\{ \int_0^t \left[32(1-\gamma)^2(m(s))^2 + 32(1-\gamma)^2(n(s))^2 + 4(1-\gamma)\bar{A}(s) \right] V(s)\mathrm{d}s \right\} \\ & \cdot \exp\left\{ \int_0^t - 32(1-\gamma)^2(m(s))^2V(s)\mathrm{d}s + \int_0^t 4(1-\gamma)m(s)\sqrt{V(s)}\mathrm{d}W_S^{\Phi^*}(s) \right\} \\ & \cdot \exp\left\{ \int_0^t - 32(1-\gamma)^2(n(s))^2V(s)\mathrm{d}s + \int_0^t 4(1-\gamma)n(s)\sqrt{V(s)}\mathrm{d}W_V^{\Phi^*}(s) \right\} \\ & \cdot \exp\left\{ \int_0^t - 32(1-\gamma)^2(n(s))^2V(s)\mathrm{d}s + \int_0^t 4(1-\gamma)n(s)\sqrt{V(s)}\mathrm{d}W_V^{\Phi^*}(s) \right\}, \end{split}$$

where K_2 is a constant. For the term $F_2(t)$, we can find an estimate as

$$\mathbf{E}^{\Phi^*}[(F_2(t))^4]$$

$$= \mathbf{E}^{\Phi^*} \left[\exp\left(\int_0^t -128(1-\gamma)^2(m(s))^2 V(s) \mathrm{d}s + \int_0^t 16(1-\gamma)m(s)\sqrt{V(s)} \mathrm{d}W_S^{\Phi^*}(s) \right) \right] < \infty.$$

Because $(F_2(t))^4$ is a non-negative local martingale, it is a supermartingale. In fact, $(F_2(t))^4$ is a martingale due to bounded function $16(1 - \gamma)m(t)$ on [0, T] (see Lemma 4.3 in Taksar and Zeng, 2009). Similarly, we have $E^{\Phi^*}[(F_3(t))^4] < \infty$, and $(F_3(t))^4$ is also a martingale.

For the term $F_1(t)$, we estimate $\mathbb{E}^{\Phi^*}[(F_1(t))^2]$ as

$$\mathbf{E}^{\Phi^*}[(F_1(t))^2] = \mathbf{E}^{\Phi^*}\left[\exp\left(\int_0^t \left(64(1-\gamma)^2(m(s))^2 + 64(1-\gamma)^2(n(s))^2 + 8(1-\gamma)\bar{A}(s)\right)V(s)\mathrm{d}s\right)\right].$$

Again applying Theorem 5.1 in Taksar and Zeng (2009), we obtain $E^{\Phi^*}[(F_1(t))^2] < \infty$ if for $\bar{g}(t) \in [\bar{g}(0), 0]$, the following condition holds:

$$64(1-\gamma)^2(m(s))^2 + 64(1-\gamma)^2(n(s))^2 + 8(1-\gamma)\bar{A}(s) \le \frac{\kappa^2}{2\sigma_V^2},$$

i.e.,

$$[64(1-\gamma)^2 - 4(1-\gamma)][(m(s))^2 + (n(s))^2] + 8(1-\gamma)A(s) \le \frac{\kappa^2}{2\sigma_V^2}$$

Applying the Cauchy-Schwartz inequality, we can arrive at

$$\begin{split} & \mathbf{E}^{\Phi^*} |J(t, X^{\Phi^*, \mathbf{u}^*}(t), V(t), L(t))|^4 \le K_3 \mathbf{E}^{\Phi^*} \left[|(X^{\Phi^*, \mathbf{u}^*}(t) + \bar{h}(t)L(t))^{1-\gamma}|^4 \right] \le K_4 \mathbf{E}^{\Phi^*} \left[F_1(t)F_2(t)F_3(t) \right] \\ & \le K_4 \left\{ \mathbf{E}^{\Phi^*} [(F_1(t))^2] \mathbf{E}^{\Phi^*} [(F_2(t)F_3(t))^2] \right\}^{\frac{1}{2}} \\ & \le K_4 \left\{ \mathbf{E}^{\Phi^*} [F_1(t)^2] (\mathbf{E}^{\Phi^*} [(F_2(t))^4] \mathbf{E}^{\Phi^*} [(F_3(t))^4])^{\frac{1}{2}} \right\}^{\frac{1}{2}} < \infty, \end{split}$$

where K_3 and K_4 are appropriate positive constants.

Step 2. Proof of $E^{\Phi^*} \left[\sup_{t \in [0,T]} \left| \frac{(\phi_S^*(t))^2}{2\Psi_S(t, X^{\Phi^*, \mathbf{u}^*}(t), V(t), L(t))} + \frac{(\phi_V^*(t))^2}{2\Psi_V(t, X^{\Phi^*, \mathbf{u}^*}(t), V(t), L(t))} \right|^2 \right] < \infty.$ Inserting Eq. (20) into $E^{\Phi^*} \left[\sup_{t \in [0,T]} \left| \frac{(\phi_S^*(t))^2}{2\Psi_S(t, X^{\Phi^*, \mathbf{u}^*}(t), V(t), L(t))} + \frac{(\phi_V^*(t))^2}{2\Psi_V(t, X^{\Phi^*, \mathbf{u}^*}(t), V(t), L(t))} \right|^2 \right]$ yields

$$\begin{split} & \mathbf{E}^{\Phi^*} \left[\sup_{t \in [0,T]} \left| \frac{(\phi_S^*(t))^2}{2\Psi_S(t, X^{\Phi^*, \mathbf{u}^*}(t), V(t), L(t))} + \frac{(\phi_V^*(t))^2}{2\Psi_V(t, X^{\Phi^*, \mathbf{u}^*}(t), V(t), L(t))} \right|^2 \right] \\ &= \mathbf{E}^{\Phi^*} \left[\sup_{t \in [0,T]} \left| \frac{(1 - \gamma)J(t, X^{\Phi^*, \mathbf{u}^*}(t), V(t), L(t))(\phi_S^*(t))^2}{2\beta_S} \right|^2 \right] \\ &+ \frac{(1 - \gamma)J(t, X^{\Phi^*, \mathbf{u}^*}(t), V(t), L(t))(\phi_V^*(t))^2}{2\beta_S} \right|^2 \right] \\ &\leq \mathbf{E}^{\Phi^*} \left[\sup_{t \in [0,T]} \left| \frac{(1 - \gamma)(\phi_S^*(t))^2}{2\beta_S} + \frac{(1 - \gamma)(\phi_V^*(t))^2}{2\beta_V} \right|^2 \right| J(t, X^{\Phi^*, \mathbf{u}^*}(t), V(t), L(t))|^2 \right] \\ &\leq \mathbf{E}^{\Phi^*} \left[\sup_{t \in [0,T]} \left| \frac{(1 - \gamma)(\phi_S^*(t))^2}{2\beta_S} + \frac{(1 - \gamma)(\phi_V^*(t))^2}{2\beta_V} \right|^4 \right]^{\frac{1}{2}} \mathbf{E}^{\Phi^*} \left[\sup_{t \in [0,T]} |J(t, X^{\Phi^*, \mathbf{u}^*}(t), V(t), L(t))|^4 \right]^{\frac{1}{2}} \\ &< \infty. \end{split}$$

Based on Lemmas D.2 and D.3, we can prove the verification theorem.

Proof of Theorem 3.4. Following the process of solving the HJB equation, conditions (1) and (2) of the admissible strategy hold, and condition (3) of the admissible strategy can be obtained by $E^{\Phi^*} \left[\sup_{t \in [0,T]} |J(t, X^{\Phi^*, \mathbf{u}^*}(t), V(t), L(t))|^4 \right] < \infty$ in Lemma D.3. Thus, \mathbf{u}^* is an admissible strategy. For Lemmas D.2 and D.3, we can simply apply Proposition 3.1 to prove that \mathbf{u}^* is the optimal strategy for problem (18) and J(t, x, v, l) is the corresponding optimal value function.

Appendix E

This appendix provides some special cases when the pension investor has no access to the derivative.

Remark E.1. We present several special cases to show the relationships between $\tilde{u}^*(t)$ and β_S , β_V and γ . It is obvious that the effect of σ_L on $\tilde{u}^*(t)$ depends on the value of ρ_L . When $\rho_L = 0$, the optimal investment strategy in this case, denoted $\tilde{u}_1^*(t)$, can be written as $\tilde{u}_1^*(t) = \tilde{m}_1(t) \left(1 + \bar{h}(t) \frac{L(t)}{X^{u^*}(t)}\right)$, and the optimal value function in this case, denoted $\tilde{J}_1(t, x, v, l)$, can be written as $\tilde{J}_1(t, x, v, l) = \frac{(x + \bar{h}(t)l)^{1-\gamma}}{1-\gamma} \exp(\bar{g}_4(t)v + \hat{g}_4(t))$, where

$$\begin{split} \tilde{m}_{1}(t) &= \frac{\lambda_{1}(1-\gamma) + (1-(\beta_{S}+\gamma))\sigma_{V}\rho_{V}\bar{g}_{3}(t)}{(1-\gamma)(\beta_{S}+\gamma)}, \quad \bar{g}_{4}(t) = \frac{\tilde{\nu}_{11}\tilde{\nu}_{21} - \tilde{\nu}_{11}\tilde{\nu}_{21}e^{\tilde{\alpha}_{21}(\tilde{\nu}_{11}-\tilde{\nu}_{21})(T-t)}}{\tilde{\nu}_{21} - \tilde{\nu}_{11}e^{\tilde{\alpha}_{21}(\tilde{\nu}_{11}-\tilde{\nu}_{21})(T-t)}}, \\ \hat{g}_{4}(t) &= \int_{-t}^{T} \left[r(1-\gamma) + \kappa\delta\bar{g}_{4}(s)\right] \mathrm{d}s, \quad \tilde{\alpha}_{11} = -\kappa + \frac{\lambda_{1}(1-(\beta_{S}+\gamma))\sigma_{V}\rho_{V}}{\beta_{S}+\gamma}, \\ \tilde{\alpha}_{21} &= \frac{\sigma_{V}^{2}}{2} - \frac{\beta_{S}\sigma_{V}^{2}\rho_{V}^{2}}{2(1-\gamma)} - \frac{\beta_{V}\sigma_{V}^{2}(1-\rho_{V}^{2})}{2(1-\gamma)} + \frac{(1-(\beta_{S}+\gamma))^{2}\sigma_{V}^{2}\rho_{V}^{2}}{2(\beta_{S}+\gamma)(1-\gamma)}, \\ \tilde{\alpha}_{31} &= \frac{\lambda_{1}^{2}(1-\gamma)}{2(\beta_{S}+\gamma)}, \quad \tilde{\nu}_{11,21} = \frac{\tilde{\alpha}_{11} \pm \sqrt{\tilde{\alpha}_{11}^{2} - 4\tilde{\alpha}_{21}\tilde{\alpha}_{31}}}{-2\tilde{\alpha}_{21}}, \end{split}$$

and $\bar{h}(t)$ is given by Eq. (32). By derivation, we obtain $\tilde{\alpha}_{11}^2 - 4\tilde{\alpha}_{21}\tilde{\alpha}_{31} \ge 0$. As $\bar{h}(t) > 0$, $\bar{g}_4(t) < 0$, $\rho_V = 0$ and $\gamma > 1$, following simple calculations, we have $\frac{\partial \tilde{u}_1^*(t)}{\partial (\beta_S + \gamma)} < 0$, which implies that the optimal investment strategy decreases w.r.t. aversion to ambiguity and risk in some cases. This result is intuitive and similar to the case involving the derivative. When $\rho_L = 1$, the optimal investment strategy in this case, denoted $\tilde{u}_2^*(t)$, can be written as $\tilde{u}_2^*(t) = \tilde{m}(t) \left(1 + \bar{h}(t) \frac{L(t)}{X^{u^*}(t)}\right) - \sigma_L \bar{h}(t) \frac{L(t)}{X^{u^*}(t)}$, and the optimal value function in this case, denoted $\tilde{J}_2(t, x, v, l)$, can be written as $\tilde{J}_2(t, x, v, l) = \frac{(x + \bar{h}(t)l)^{1-\gamma}}{1-\gamma} \exp(\bar{g}_5(t)v + \hat{g}_5(t))$, where

$$\begin{split} \bar{g}_{5}(t) &= \frac{\tilde{\nu}_{12}\tilde{\nu}_{22} - \tilde{\nu}_{12}\tilde{\nu}_{22}e^{\tilde{\alpha}_{22}(\tilde{\nu}_{12} - \tilde{\nu}_{22})(T-t)}}{\tilde{\nu}_{22} - \tilde{\nu}_{12}e^{\tilde{\alpha}_{22}(\tilde{\nu}_{12} - \tilde{\nu}_{22})(T-t)}}, \\ \hat{g}_{5}(t) &= \int_{-t}^{T} \left[r(1-\gamma) + \kappa\delta\bar{g}_{5}(s) \right] \mathrm{d}s, \\ \tilde{\alpha}_{12} &= -\kappa + \frac{\lambda_{1}(1 - (\beta_{S} + \gamma))\sigma_{V}}{\beta_{S} + \gamma}, \\ \tilde{\alpha}_{22} &= \frac{\sigma_{V}^{2}}{2} - \frac{\beta_{S}\sigma_{V}^{2}}{2(1-\gamma)} + \frac{(1 - (\beta_{S} + \gamma))^{2}\sigma_{V}^{2}}{2(\beta_{S} + \gamma)(1-\gamma)}, \\ \tilde{\alpha}_{32} &= \frac{\lambda_{1}^{2}(1-\gamma)}{2(\beta_{S} + \gamma)}, \quad \tilde{\nu}_{12,22} &= \frac{\tilde{\alpha}_{12} \pm \sqrt{\tilde{\alpha}_{12}^{2} - 4\tilde{\alpha}_{22}\tilde{\alpha}_{32}}}{-2\tilde{\alpha}_{22}} \end{split}$$

and $\bar{h}(t)$ is given by Eq. (32). By derivation, we obtain $\tilde{\alpha}_{12}^2 - 4\tilde{\alpha}_{22}\tilde{\alpha}_{32} \geq 0$. When $\rho_L = -1$, the optimal investment strategy in this case, denoted $\tilde{u}_3^*(t)$, can be written as $\tilde{u}_3^*(t) = \tilde{m}(t) \left(1 + \bar{h}(t) \frac{L(t)}{X^{u^*}(t)}\right) + \sigma_L \bar{h}(t) \frac{L(t)}{X^{u^*}(t)}$, and the optimal value function in this case, denoted $\tilde{J}_3(t, x, v, l)$, can be written as $\tilde{J}_3(t, x, v, l) = \frac{(x + \bar{h}(t)l)^{1-\gamma}}{1-\gamma} \exp(\bar{g}_6(t)v + \hat{g}_6(t))$, where

$$\begin{split} \bar{g}_{6}(t) &= \frac{\tilde{\nu}_{13}\tilde{\nu}_{23} - \tilde{\nu}_{13}\tilde{\nu}_{23}\mathrm{e}^{\tilde{\alpha}_{23}(\tilde{\nu}_{13} - \tilde{\nu}_{23})(T-t)}}{\tilde{\nu}_{23} - \tilde{\nu}_{13}\mathrm{e}^{\tilde{\alpha}_{23}(\tilde{\nu}_{13} - \tilde{\nu}_{23})(T-t)}}, \\ \hat{g}_{6}(t) &= \int_{-t}^{T} \left[r(1-\gamma) + \kappa \delta \bar{g}_{6}(s) \right] \mathrm{d}s, \\ \tilde{\alpha}_{13} &= -\kappa - \frac{\lambda_{1}(1 - (\beta_{S} + \gamma))\sigma_{V}}{\beta_{S} + \gamma}, \\ \tilde{\alpha}_{23} &= \frac{\sigma_{V}^{2}}{2} - \frac{\beta_{S}\sigma_{V}^{2}}{2(1-\gamma)} + \frac{(1 - (\beta_{S} + \gamma))^{2}\sigma_{V}^{2}}{2(\beta_{S} + \gamma)(1-\gamma)}, \\ \tilde{\alpha}_{33} &= \frac{\lambda_{1}^{2}(1-\gamma)}{2(\beta_{S} + \gamma)}, \quad \tilde{\nu}_{13,23} = \frac{\tilde{\alpha}_{13} \pm \sqrt{\tilde{\alpha}_{13}^{2} - 4\tilde{\alpha}_{23}\tilde{\alpha}_{33}}}{-2\tilde{\alpha}_{23}}, \end{split}$$

and $\bar{h}(t)$ is given by Eq. (32). By derivation, we obtain $\tilde{\alpha}_{13}^2 - 4\tilde{\alpha}_{23}\tilde{\alpha}_{33} \ge 0$.

Compared with Remark 3.5, we find that when the investor has no access to the derivative, the non-redundant condition is unnecessary. Therefore, we analyze the case of $\rho = \pm 1$ here and provide related explicit results. From the previous results, we find that the equity premium λ_2 for additional volatility risk is now 0; the investor has no way to cope with the volatility risk. She may increase her wealth invested in the stock (the second part in Eq. (25) is dropped), which causes her to undertake more risk than in the case with the derivative, and decrease her utility at retirement. The following special cases can be studied in a similar way. For a detailed comparison, we list related explicit results below.

Remark E.2. If $\sigma_L = 0$, the salary process is non-stochastic; then, the optimal investment in this case, denoted $\tilde{u}_4^*(t)$, can be written as $\tilde{u}_4^*(t) = \tilde{m}(t)(1 + \frac{\hat{h}(t)}{X^{u^*}(t)})$, and the optimal value function in this case, denoted $\tilde{J}_4(t, x, v)$, can be written as $\tilde{J}_4(t, x, v) = \frac{(x+\hat{h}(t))^{1-\gamma}}{1-\gamma} \exp(\bar{g}_3(t)v + \hat{g}_3(t))$, where $\hat{h}(t)$, $\tilde{m}(t)$, $\bar{g}_3(t)$ and $\hat{g}_3(t)$ are given by Eqs. (37) and (48). In this case, we find that the optimal investment strategy is proportional to $\tilde{m}(t)$. Furthermore, if there is no salary and no derivative, our model reduces to a portfolio selection problem for an ambiguity-averse investor. The optimal investment strategy in this case, denoted $\tilde{u}_5^*(t)$, can be written as $\tilde{u}_5^*(t) = \frac{\lambda_1(1-\gamma)+(1-(\beta_S+\gamma))\sigma_V\rho_V\bar{g}_3(t)}{(1-\gamma)(\beta_S+\gamma)}$, and the optimal value function in this case, denoted $\tilde{J}_5(t, x, v)$, can be written as $\tilde{J}_5(t, x, v) = \frac{x^{1-\gamma}}{1-\gamma} \exp(\bar{g}_3(t)v + \hat{g}_3(t))$, where $\bar{g}_3(t)$ and $\hat{g}_3(t)$ are given by Eq. (48).

Remark E.3. If there is no derivative in the financial market and if the pension investor is ambiguity neutral, then the optimal investment strategy, denoted $\tilde{u}_6^*(t)$, can be written as $\tilde{u}_6^*(t) = \frac{\lambda_1 + \sigma_V \rho_V \bar{g}_7(t)}{\gamma} \left(1 + \bar{h}(t) \frac{L(t)}{X^{u^*}(t)}\right) - \sigma_L \rho_L \bar{h}(t) \frac{L(t)}{X^{u^*}(t)}$, and the optimal value function, denoted $\tilde{J}_6(t, x, v, l)$, can be written as $\tilde{J}_6(t, x, v, l) = \frac{(x + \bar{h}(t)l)^{1-\gamma}}{1-\gamma} \exp(\bar{g}_7(t)v + \hat{g}_7(t))$, where

$$\bar{g}_{7}(t) = \frac{\tilde{\nu}_{14}\tilde{\nu}_{24} - \tilde{\nu}_{14}\tilde{\nu}_{24}e^{\tilde{\alpha}_{24}(\tilde{\nu}_{14} - \tilde{\nu}_{24})(T-t)}}{\tilde{\nu}_{24} - \tilde{\nu}_{14}e^{\tilde{\alpha}_{24}(\tilde{\nu}_{14} - \tilde{\nu}_{24})(T-t)}},$$
(69)

$$\hat{g}_{7}(t) = \int_{t}^{T} \left[r(1-\gamma) + \kappa \delta \bar{g}_{7}(s) \right] \mathrm{d}s,$$

$$\tilde{\alpha}_{14} = -\kappa + \frac{\lambda_{1}(1-\gamma)\sigma_{V}\rho_{V}}{\gamma}, \quad \tilde{\alpha}_{24} = \frac{\sigma_{V}^{2}}{2} + \frac{(1-\gamma)\sigma_{V}^{2}\rho_{V}^{2}}{2\gamma},$$

$$(70)$$

$$\tilde{\alpha}_{34} = \frac{\lambda_1^2 (1 - \gamma)}{2\gamma}, \quad \tilde{\nu}_{14,24} = \frac{\tilde{\alpha}_{14} \pm \sqrt{\tilde{\alpha}_{14}^2 - 4\tilde{\alpha}_{24}\tilde{\alpha}_{34}}}{-2\tilde{\alpha}_{24}},$$

and $\bar{h}(t)$ is given by Eq.(32). By derivation, we obtain $\tilde{\alpha}_{14}^2 - 4\tilde{\alpha}_{24}\tilde{\alpha}_{34} \ge 0$.

Remark E.4. If there is no derivative in the financial market, the pension investor is ambiguity neutral and $\sigma_L = 0$, the salary process is non-stochastic; then, the optimal investment strategy in this case, denoted $\tilde{u}_{7S}^*(t)$, can be written as $\tilde{u}_{7S}^*(t) = \frac{\lambda_1 + \sigma_V \rho_V \bar{g}_7(t)}{\gamma} (1 + \frac{\hat{h}(t)}{X^{u^*}(t)})$, and the optimal value function in this case, denoted $\tilde{J}_7(t, x, v)$, can be written as $\tilde{J}_7(t, x, v) = \frac{(x + \hat{h}(t))^{1-\gamma}}{1-\gamma} \exp(\bar{g}_7(t)v + \hat{g}_7(t))$, where $\hat{h}(t)$, $\bar{g}_7(t)$ and $\hat{g}_7(t)$ are given by Eqs. (37), (69)-(70).

Furthermore, if there is no salary, no ambiguity and no derivative in our model, the optimization problem becomes a portfolio selection problem for an ambiguity-neutral investor; the optimal investment in this case, denoted $\tilde{u}_{8S}^*(t)$, can be written as $\tilde{u}_{8S}^*(t) = \frac{\lambda_1 + \sigma_V \rho_V \bar{g}_7(t)}{\gamma}$, and the optimal value function in this case, denoted $\tilde{J}_8(t, x, v)$, can be written as $\tilde{J}_8(t, x, v) = \frac{x^{1-\gamma}}{1-\gamma} \exp(\bar{g}_7(t)v + \hat{g}_7(t))$, where $\bar{g}_7(t)$ and $\hat{g}_7(t)$ are given by Eqs. (69) and (70).

Remark E.5. If $\sigma_V = 0$, the volatility of the risky asset is non-stochastic, and as noted above, the derivative is indeed redundant. The optimal investment strategy in this special case, denoted $\tilde{u}_9^*(t)$, can be written as $\tilde{u}_9^*(t) = \frac{\lambda_1}{\beta_S + \gamma} (1 + \bar{h}(t) \frac{L(t)}{X^{u^*}(t)}) - \sigma_L \rho_l \bar{h}(t) \frac{L(t)}{X^{u^*}(t)}$, and the optimal value function in this case, denoted $\tilde{J}_9(t, x, l)$, can be written as $\tilde{J}_9(t, x, l) = \frac{(x + \bar{h}(t)l)^{1-\gamma}}{1-\gamma} \exp(\hat{g}_8(t))$, where

$$\hat{g}_8(t) = \left(r(1-\gamma) + \frac{\lambda_1^2(1-\gamma)\delta}{2(\beta_S+\gamma)}\right)(T-t) + \frac{\lambda_1^2(1-\gamma)(v_0-\delta)}{2(\beta_S+\gamma)\kappa} \left(\exp(-\kappa t) - \exp(-\kappa T)\right),$$

and $\bar{h}(t)$ is given by Eq.(32).

Appendix F

This appendix provides the optimal strategy under two special cases, European-style call and put options. Option pricing for the stochastic volatility model adopted here refers to Liu and Pan (2003) and Cui et al. (2017). We derive the prices of European-style call and put options with time τ to expiration and striking at K as follows

$$C(t) = c(t, \tau, S, V; K);$$
 $P(t) = p(t, \tau, S, V; K),$

where S is the spot price and V is the market volatility at time t, and the call and put options' prices are respectively

$$c(t,\tau, S, V; K) = S\mathcal{P}_1(t,\tau, S, V; K) - e^{-r\tau} K\mathcal{P}_2(t,\tau, S, V; K),$$
$$p(t,\tau, S, V; K) = e^{-r\tau} K(1 - \mathcal{P}_2(t,\tau, S, V; K)) - S(1 - \mathcal{P}_1(t,\tau, S, V; K)),$$

where the risk-neutral probabilities \mathcal{P}_1 and \mathcal{P}_2 are recovered from inverting the respective characteristic functions

$$\mathcal{P}_{1}(t,\tau,S,V;K) = \frac{1}{2} - \frac{1}{\pi} \int_{0}^{\infty} \mathbf{Im} \left[\frac{e^{iz(\ln K - \ln S - r\tau)} e^{A(1-iz) + B(1-iz)V}}{z} \right] dz,$$

$$\mathcal{P}_{2}(t,\tau,S,V;K) = \frac{1}{2} - \frac{1}{\pi} \int_{0}^{\infty} \mathbf{Im} \left[\frac{e^{iz(\ln K - \ln S - r\tau)} e^{A(-iz) + B(-iz)V}}{z} \right] dz,$$

where **Im** denotes the imaginary component of a complex number, and A(y), B(y) are given by

$$B(y) = -\frac{a(1 - e^{-q\tau})}{2q - (q + b)(1 - e^{-q\tau})},$$

$$A(y) = -\frac{\kappa^* \delta^*}{\sigma_V^2} \left((q + b)\tau + 2\ln\left(1 - \frac{q + b}{2q}(1 - e^{-q\tau})\right) \right),$$

$$a = y(1 - y), \quad b = \rho_V \sigma_V y - \kappa^*,$$

$$q = \sqrt{b^2 + a\sigma_V^2}, \quad \kappa^* = \kappa + \sigma_V (\rho_V \lambda_1 + \sqrt{1 - \rho_V^2} \lambda_2), \quad \delta^* = \frac{\kappa \delta}{\kappa^*}.$$

The price of the straddle option used in our numerical examples is given by

$$O(t) = c(t, \tau, S, V; K) + p(t, \tau, S, V; K).$$

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