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Optimal Consumption and Portfolio Selection with Early Retirement Option

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Abstract. In this paper we propose an approach to investigate a model of consumption and investment with a mandatory retirement date and early retirement option; we analyze properties of the optimal strategy and thereby contribute to understanding the interaction between retirement, consumption, and portfolio decisions in the presence of both the important features of retirement. In particular, we provide a characterization of the threshold of wealth as a function of time, and we show that it is strictly decreasing near the mandatory retirement date. The threshold is similar to the early exercise boundary of an American option in the sense that if the agent's wealth is above or equal to the threshold level, then the agent immediately retires. We also provide comparative static analysis.

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Keywords: mandatory retirement • early retirement option • consumption • portfolio selection • variational inequality

1. Introduction

An increasing number of people are approaching retirement, as population aging is progressing rapidly in both the developed and developing worlds. The decisions of the people close to retirement are affected significantly by the outcomes of financial markets. For example, the big stock market boom between 1995 and 2000 led to a dramatic increase in the number of people who chose voluntary early retirement (Fahri and Panageas [9]). In reverse causality, these people's decisions on retirement, consumption, and savings are expected to have a significant effect on aggregate consumption and investments, and to have a large influence on the world's financial markets and economy.[1]

In this paper we study a model of consumption and investment with a mandatory retirement date and early retirement option.[2] More specifically, we study the optimal consumption and portfolio choice of an agent/wage earner who faces a prespecified mandatory retirement date but has an option to retire earlier than that date. We investigate the properties of the optimal choice of retirement time, consumption, and portfolio of assets. There have been studies of models with one feature of retirement: either only with voluntary retirement (Choi and Shim [3], Choi et al. [4], Dybvig and Liu [6], Fahri and Panageas [9]) or only with mandatory retirement (Dybvig and Liu [6]). There, however, has not been a serious theoretical study devoted to the model where both features—the mandatory retirement date and early retirement option—are present. For example, Dybvig and Liu [6] study two different models, each with only one feature of retirement; the authors state that "an alternative model of mandatory retirement that allows for early retirement is more complicated because of the extra time dimension, but can be solved using the randomization method employed by Liu and Loewenstein . . ." (p. 886). The method of Liu and Loewenstein [23], mentioned in this quote, is a model of transaction costs, and not that of retirement. It provides an approximation to the final horizon by a random time and thus is an approximation method, and it is not a method to study the true optimal solution. Fahri and Panageas [9] provide an approximate solution to the problem with both features of retirement. Their investigation, however, is an addendum to that of an infinite horizon problem, and they have not conducted a thorough analysis of the true solution, as we do here.

Consideration of both the mandatory retirement requirement and voluntary early retirement option makes it necessary to study a model with a finite horizon. Thus, in our model an agent chooses consumption and the portfolio of assets, faces a mandatory retirement date T, and has an early retirement option (i.e., he or she can choose retirement time $\tau \leq T$). Thus, the model provides a challenge in the following two senses. First, it is a problem where the optimal choice of consumption and investment is coupled with the optimal stopping problem of choosing the retirement date, where the two decisions interact with each other. Second, the embedded optimal stopping problem has a finite horizon, and so similar to an American option, which does not admit a simple closed-form solution.

In this paper we propose an approach to investigate the model with a finite horizon and analyze properties of the optimal strategy, and thereby contribute to understanding the interaction between retirement, consumption, and portfolio decisions in the presence of both the important features of retirement. In particular, we provide a characterization of the threshold of wealth as a function of time and show that it is strictly decreasing near the mandatory retirement date. The threshold is similar to the early exercise boundary of an American option in the sense that if the agent's wealth is above or equal to the threshold level, then the agent immediately retires. We also provide useful comparative static analysis. For example, we show that the threshold tends to go higher if the wage rate increases, and the threshold tends to go down if the utility cost of labor increases, for every $t \in [0, T)$, extending the comparative static analysis by Choi and Shim [3], originally derived in an infinite horizon without a mandatory retirement requirement.

In the literature there exist three common methods to analyze the properties of the value function and optimal strategy of an optimal control problem without a choice of a stopping time: first, the martingale method with a dual transformation (see, e.g., Cox and Huang [5], Karatzas and Shreve [16], Karatzas et al. [18]), second, the transformation of the associated Hamilton-Jacobi-Bellman (HJB) equation into a linear partial differential equation (PDE) through the Legendre transformation (see, e.g., Fleming and Soner [10], Karatzas et al. [19]), and third, the stochastic maximum principle (see, e.g., Yong and Zhou [32]). But all of the methods cannot be directly applied to our problem because the state equations before and after the retirement date are not the same, and the dual transformations and Legendre transformations in the two stages are different. Moreover, the stopping time is unknown and interacts with the optimal control. For an optimal stopping problem, there are also three commonly used methods: the martingale method (see, e.g., Karatzas and Shreve [16], Karatzas and Wang [17]), the probabilistic method (see, e.g., Peskir and Shiryaev [25]), and the PDE method via its associated variational inequality (see, e.g., Bensoussan and Frideman [2], Friedman [11]). But it is difficult to discover the properties of the value function and the optimal strategy only by the martingale method or by the probabilistic method, because the optimal stopping time discovered with these methods is relatively abstract. The PDE method attempts to identify the optimal stopping boundary for a Markovian optimal stopping problem, but it cannot be directly applied to our problem, because the associated HJB equation is a variational inequality with a fully nonlinear parabolic differential operator. Thus, it is very difficult to discover the properties of the solution. If the horizon is infinite, the associated HJB equation is an ordinary differential equation, and an explicit solution can be found and properties can be discovered by investigating the explicit form (e.g., Choi and Shim [3], Dybvig and Liu [6, 7]). But it is impossible to obtain an explicit solution of a finite horizon problem, even for a simple problem without optimal control, such as pricing a standard American call/put option.

In this paper we propose methods to get over the difficulty and discover important properties of the value function and the optimal strategy of the agent's choice problem with only a minimal assumption on the agent's utility function. Because of the aforementioned difficulty, the results a researcher can expect, under the assumption of a general utility function, are usually a characterization of the value function as a unique solution to an associated HJB equation, and there is a vast literature with such limited results (see, e.g., Hamadene et al. [14], Touzi and Vieille [26]). For example, Koo et al. [20] have considered a finite horizon problem similar to ours and have shown only the existence and uniqueness of the value function; they have not been able to derive the properties of the optimal strategy. We overcome the difficulty in the following ways. First, we conduct successive transformations (see Section 3): first transforming the original problem into its dual problem (similar to Karatzas and Wang [17]), then transforming the dual problem into a variational inequality. Second, we use the PDE method to analyze the properties of the optimal strategy in the dual coordinate system. Finally, we come back to the original problem and discuss the properties in the original coordinate system.

The rest of this paper is organized as follows. In Section 2 we explain our model. In Section 3 we transform the original problem into a variational inequality and provide the verification theorem. In Section 4 we provide a solution to the variational inequality and explain its properties. In Section 5 we study the optimal retirement threshold. In Section 6 we conduct comparative static analysis, and in Section 7 we conclude.

2. A Model of Retirement, Investment, and Consumption Choice

2.1. Objective of an Agent Facing Retirement

We consider an economic agent who receives a constant stream of labor income at a rate equal to $\varrho > 0$. The agent's objective is to maximize the following utility function by choosing the consumption, portfolio of assets, and time of retirement:

$$U \equiv \mathbb{E}\left[\int_0^{T^1} e^{-\beta t}(U_1(t,c_t) - lI_{\{t\le\tau\}})\,dt + e^{-\beta T^1}U_2(T^1, W_{T^1})\right], \tag{1}$$

where c_t is the agent's rate of consumption at time t, W_{T^1} is his or her wealth at time T^1, $\beta > 0$ is the subjective discount rate, U_1 is the felicity function of consumption, U_2 is the bequest function, $l > 0$ is the utility cost of labor, τ is the time of retirement, I_A is the characteristic function of set A, and $\mathbb{E}$ denotes expectation. Time T^1 is the final time of the agent's horizon—that is, the time when the agent's consumption plan ends and he or she makes a bequest. We assume that the subjective discount rate β, the utility cost of labor l, and the final time T^1 are fixed constants. Our model, however, can be extended to accommodate a random final time—for example, a random time of the agent's death, with the assumption that there is no bequest motive (i.e., $U_2 = 0$). Consider the case where $U_2 = 0$, and T^1 is a random variable taking value in $[T, T^2]$, where T is the mandatory retirement date explained below and T^2 is a constant (an upper bound of the agent's life span).[3] Suppose that the probability density function and the cumulative distribution function of T^1 are $f(\xi)$ and $F(\xi)$, respectively. Then the agent's objective function can be written as

$$\begin{aligned} U &= \mathbb{E}\left[\int_0^{T} e^{-\beta t}(U_1(t,c_t) - lI_{\{t\le\tau\}})\,dt + \int_T^{T^2}\left(\int_T^{\xi} e^{-\beta t}U_1(t,c_t)\,dt\right)f(\xi)\,d\xi\right] \\ &= \mathbb{E}\left[\int_0^{T} e^{-\beta t}(U_1(t,c_t) - lI_{\{t\le\tau\}})\,dt + \int_T^{T^2}\left(\int_t^{T^2} f(\xi)\,d\xi\right)e^{-\beta t}U_1(t,c_t)\,dt\right] \\ &= \mathbb{E}\left[\int_0^{T^2} e^{-\beta t}((1-F(t))U_1(t,c_t) - lI_{\{t\le\tau\}})\,dt\right]. \end{aligned} \tag{2}$$

The above objective function belongs to the class defined in (1).

There is a mandatory retirement date $T \le T^1$ when the agent is forced to retire from work. We assume that T is a constant. Choi and Shim [3] have employed an objective function similar to (1) in their infinite horizon model without a mandatory retirement date. After retirement, the agent does not have an option to go back to work, and thus retirement is an irreversible decision. The agent, however, is allowed to retire earlier than the mandatory retirement date and hence able to choose retirement time $\tau \le T$. We assume that the agent does not receive income after retirement.[4] The agent thus faces a choice between earning higher income by delaying retirement and saving the utility cost of labor by retiring immediately.

For later use, we define $U_i(t,0) \triangleq \lim_{c\to 0+} U_i(t,c)$.[5] We make the following assumptions with regard to the felicity function U_1 and the bequest function U_2.

Assumption 1. *The utility functions $U_i(t,c) \in C^\infty([0,T^1]\times(0,+\infty))$, $i = 1,2$, take values in $\mathbb{R}$,[6] are strictly concave with respect to c, and satisfy the following conditions*:

$$\lim_{c\to 0^+} \partial_c U_i(t,c) = +\infty, \qquad \lim_{c\to +\infty} \partial_c U_i(t,c) = 0, \qquad \limsup_{c\to+\infty} \max_{t\in[0,T^1]} \partial_c U_i(t,c)c^k \le C$$

for any $t \in [0,T^1]$, where C and k are positive constants.

Remark 1. The first two conditions in Assumption 1 are called *Inada conditions*, which are commonly employed in models of economic growth and consumption/savings (see, e.g., Inada [15], Uzawa [28]). Indeed, we can dispense with the first condition, $\lim_{c\to 0^+} \partial_c U_i(t,c) = +\infty$, and consider the case $\lim_{c\to 0^+} \partial_c U_i(t,c) < +\infty$ for $i = 1,2$. Treatment of the latter case, however, turns out to be very similar to analysis of the problem under the first Inada condition, and we will mainly focus our analysis on the case where the Inada condition is satisfied.[7] From Assumption 1, we can deduce that (see Karatzas and Shreve [16])

$$\lim_{c\to 0^+} \min_{t\in[0,T^1]} \partial_c U_i(t,c) = +\infty, \qquad \lim_{c\to+\infty} \max_{t\in[0,T^1]} \partial_c U_i(t,c) = 0.$$

Moreover, since utility functions U_i are concave, the inequality in Assumption 1 implies that

$$\mathcal{J}_{U_i}(t,x) \le C_1(1 + x^{-1/k}), \quad \forall (t,x) \in [0,T^1]\times(0,+\infty), \ \ i = 1,2, \tag{3}$$

where $\mathcal{J}_{U_i}(t,\cdot)$ are the inverse functions of $\partial_c U_i(t,\cdot)$, and C_1 is a positive constant.

The last inequality in Assumption 1 is a technical assumption, which is satisfied by many commonly used utility functions. For example, it is satisfied by the following CRRA utility function and constant absolute risk aversion utility function; that is,

$$U_i(t,c) = \frac{c^{1-\gamma}}{1-\gamma}(0<\gamma\neq 1) \qquad \text{or} \qquad U_i(t,c) = \ln c \qquad \text{or} \qquad U_i(t,c) = 1-e^{-\alpha c}(\alpha>0). \tag{4}$$

There exist examples of a time-inhomogeneous utility function for $U_1(t,c)$. An example is given by the class of utility functions $U_1(t,c) = \Delta(t)U(c)$, with $U(c)$ satisfying Assumption 1 and $\Delta(t)>0$ being a general discount function of time—that is, a declining function of time as in Watson and Scott [29].[8] A special case of the utility function in (2)—that is, a utility function of the form $U_1(t,c) = (1-F(t))U(c)$, where $U(c)$ satisfies Assumption 1 and $F(t)$ is the cumulative distribution function of a random time taking value in $[T,T^1]$—belongs to this class. Another example is the class of utility functions with time-varying risk aversion such as, for example, $U_1(t,c) = c^{1-\gamma(t)}/(1-\gamma(t))$, where $\gamma(t)$ is a deterministic function satisfying $\gamma_m \le \gamma(t) \le \gamma_M$, with $0<\gamma_m<\gamma_M$ being constants such that $\gamma_m>1$ or $\gamma_M<1$.

2.2. Financial Market

The financial market consists of one riskless asset and d risky assets. We assume that the risk-free rate is a positive constant and equal to r.

The price P_0 of the riskless asset and the price P_i of the ith risky asset are governed by the following stochastic differential equations (SDEs):

$$\begin{cases} dP_{0,t} = rP_{0,t}\,dt, & P_{0,0} = P_0,\\ dP_{i,t} = \alpha_i P_{i,t}\,dt + \displaystyle\sum_{j=1}^{d} \sigma^{ij} P_{i,t}\,dB_t^j, & P_{i,0} = P_i, \end{cases}$$

where $B_t = (B_t^1,\dots,B_t^d)^{\mathrm{T}}$ is a d-dimensional standard Brownian motion, which represents sources of risk for the asset returns; $\alpha = (\alpha_1,\dots,\alpha_d)^{\mathrm{T}}$ represents the vector of expected rates of return on the risky assets; and $\Sigma = (\sigma^{ij})_{d\times d}$ represents contributions by the risk sources to the volatility of asset returns ($^{\mathrm{T}}$ denotes the transpose of a matrix). We assume that the investment opportunity is constant (i.e., α and Σ are a constant vector and a constant matrix, respectively). The Brownian motion is defined on a probability space $(\Omega, (\mathcal{F}_t)_{t=0}^{T^1}, \mathbb{P})$, and $(\mathcal{F}_t)_{t=0}^{T^1}$ is just the augmented filtration generated by Brownian motion B.[9]

We assume that Σ is positive-definite and $\alpha - r1_d \neq 0_d$, where 1_d and 0_d denote the d-dimensional column vectors of 1's and 0's, respectively. We will assume that all processes are $(\mathcal{F}_t)_{t=0}^{T^1}$-progressively measurable, and all stopping times are $(\mathcal{F}_t)_{t=0}^{T^1}$-stopping times.

2.3. Admissible Choices and Optimization Problem

We now explain admissible choices of the agent. We consider the problem starting at time $t \ge 0$.

The consumption rate, c, and the monetary amounts invested in the risky assets, $\pi = (\pi_1,\dots,\pi_d)^{\mathrm{T}}$, are admissible only if $c\ge 0$. The time, τ, of voluntary retirement is admissible only if it belongs to $\mathcal{U}_{t,T}$, the set of all stopping times taking values in $[t,T]$ (define $[t,T]=\{t\}$ if $t>T$ in this paper). Namely, the agent's choice should be conditional only on currently available information.

The agent's wealth process W is governed by

$$dW_s^{t,w;\tau,c,\pi} = \left[\pi_s^{\mathrm{T}}(\alpha - r1_d) + rW_s^{t,w;\tau,c,\pi} - c_s + \varrho I_{\{s\le\tau\}}\right]ds + \pi_s^{\mathrm{T}}\Sigma\,dB_s, \quad W_t = w. \tag{5}$$

Another requirement for the triplet of choices (τ,c,π) to be admissible is the following:

$$\int_t^{T^1} c_s + |\pi_s|^2\,ds < \infty \text{ a.s. in } \Omega, \quad \text{subject to } c_s \ge 0, \quad \tau\in\mathcal{U}_{t,T}, \qquad W_s^{t,w;\tau,c,\pi} > g(s)I_{\{s<\tau\}}, \quad \forall s\in[t,T^1], \tag{6}$$

where

$$g(s) = \begin{cases} \dfrac{\varrho}{r}(e^{rs-rT}-1) & \forall s<T,\\ 0 & \forall s\in[T,T^1]. \end{cases}$$

The quantity $g(s)$, the negative of which is the present value of labor income at time s under the assumption that the agent does not choose early retirement, is defined as above to facilitate the transformations. The existence of retirement option might cause mismatch of dual value functions at the time τ of retirement when one applies the martingale dual approach separately to the agent's optimization problems before and after retirement. The definition, however, allows us to modify the dual value function after retirement (see (16)), to make the dual value function before retirement match the modified dual value function after retirement (see (24)), and to derive a variational inequality (19) that connects the problems before and after retirement. In this paper we allow the agent to borrow fully against the stream of future income. Consideration of limited borrowing ability as in Dybvig and Liu [6, 7] would be interesting, but it introduces complications to the already challenging finite horizon model. As a first step toward investigating the model of consumption, investment/retirement choice with a mandatory retirement requirement, and a voluntary retirement option, we will confine our attention to this case of the maximal borrowing ability, which is consistent with the agent being able to pay debt back eventually (i.e., $W_{T^1} \geq 0$).

Remark 2. We may consider the case where the agent receives a pension income after retirement. For the moment, let us assume that the rate of labor income is ϱ_l and the rate of pension income is ϱ_p such that $0 \leq \varrho_p < \varrho_l$. Let us denote $\varrho = \varrho_l - \varrho_p$. Then, in the absence of borrowing constraints, it is easy to show that the agent's problem with $W_t = w$ in this case is equivalent to the problem in our model with $W_t = w + (\varrho_p/r)(1 - e^{-r(T^1-t)})$ in which the agent receives labor income at the rate of ϱ before retirement and does not have income when retired. In this sense, our assumption that the agent receives no income after retirement is not restrictive, and the result we have obtained in this paper under the assumption can be extended to the more general case with only a slight modification.

We will denote the set of all strategies satisfying (6) by $\mathscr{A}(t,w)$. Finally, the set of admissible controls, denoted by $\mathscr{A}^1(t,w)$, is defined as follows:

$$\mathscr{A}^1(t,w) \triangleq \left\{(\tau,c,\pi) \in \mathscr{A}(t,w)\colon \mathbb{E}\left[\int_t^{T^1} e^{-\beta(s-t)} U_1^-(s,c_s)\,ds + e^{-\beta(T^1-t)} U_2^-(T^1, W_{T^1}^{t,w;\tau,c,\pi})\right] < +\infty\right\},$$

with $U_i^- = \max\{0, -U_i\}$, $i = 1,2$.

The agent's objective at time t is to maximize the following utility function by choosing $(\tau,c,\pi) \in \mathscr{A}^1(t,w)$:

$$\begin{aligned} J(t,w;\tau,c,\pi) &\triangleq \mathbb{E}\left[\int_t^{T^1} e^{-\beta(s-t)}(U_1(s,c_s) - lI_{\{s\leq\tau\}})\,ds + e^{-\beta(T^1-t)} U_2(T^1, W_{T^1}^{t,w;\tau,c,\pi})\right] \\ &= \mathbb{E}\left\{\int_t^{\tau} e^{-\beta(s-t)}(U_1(s,c_s) - l)\,ds + \mathbb{E}\left[\int_\tau^{T^1} e^{-\beta(s-t)} U_1(s,c_s)\,ds + e^{-\beta(T^1-t)} U_2(T^1, W_{T^1}^{t,w;\tau,c,\pi}) \,\middle|\, \mathscr{F}_\tau\right]\right\}, \\ &\qquad \forall\,(t,w) \in \tilde{\mathscr{M}}_{T^1}, \end{aligned} \tag{7}$$

where we have used the following notation, which we will use throughout this paper:

$$\begin{aligned} &\mathscr{M}_T \triangleq \{(t,w)\colon w > g(t),\ t \in [0,T)\}, \qquad \mathscr{M}_{T^1} \triangleq \{(t,w)\colon w > g(t),\ t \in [0,T^1)\}, \\ &\tilde{\mathscr{M}}_T \triangleq \{(t,w)\colon w > g(t),\ t \in [0,T]\}, \qquad \tilde{\mathscr{M}}_{T^1} \triangleq \{(t,w)\colon w > g(t),\ t \in [0,T^1]\}. \end{aligned} \tag{8}$$

The agent's problem is to find an optimal strategy, $(\tau^*, c^*, \pi^*) \in \mathscr{A}^1(t,w)$, such that

$$J(t,w;\tau^*,c^*,\pi^*) = V(t,w) \triangleq \sup\{J(t,w;\tau,c,\pi)\colon (\tau,c,\pi) \in \mathscr{A}^1(t,w)\}, \quad \forall\,(t,w) \in \tilde{\mathscr{M}}_{T^1}.$$

2.4. Static Budget Constraints and Convex Dual Functions

We will use the martingale and duality methods (see, e.g., Cox and Huang [5], Karatzas and Shreve [16], Karatzas et al. [18]). Define the discount process D, the market price of risk θ, an exponential martingale M, and the state-price-density process H as

$$D(s) = e^{-r(s-t)}, \qquad \theta = \Sigma^{-1}(\alpha - r1_d), \qquad M_s = \exp\left\{-\int_t^s \theta^{\mathrm{T}}\,dB_u - \frac{1}{2}\int_t^s |\theta|^2\,du\right\}, \qquad H_s = D(s)M_s.$$

It is not difficult to deduce that

$$dH_s[W_s - g(s)I_{\{s<\tau\}}] = -H_s c_s\,ds + H_s[\pi_s^{\mathrm{T}}\Sigma - (W_s - g(s)I_{\{s<\tau\}})\theta^{\mathrm{T}}]\,dB_s, \quad \forall\,s \in [t,\tau) \cup (\tau, T^1].$$

Since $W_s - g(s)I_{\{s<\tau\}} \geq 0$ for any $s \in [t,T^1]$, we can deduce that $W_s - g(s)I_{\{s\leq\tau\}} \geq 0$ for any $s \in [t,T^1]$ from the continuity of $W_s - g(s)$ with respect to s. So Fatou's lemma implies that

$$\mathbb{E}\left\{H_s[W_s - g(s)] + \int_t^s H_u c_u\,du\right\} \leq w - g(t), \quad \text{if } 0 \leq t \leq s \leq \tau, \tag{9}$$

$$\mathbb{E}\left[H_s W_s + \int_t^s H_u c_u\,du\right] \leq w, \quad \text{if } 0 \leq t = \tau \leq s \leq T^1. \tag{10}$$

Remark 3. The constraints (9) and (10) are called *static budget constraints*. According to Karatzas and Shreve [16] and Karatzas and Wang [17], the inequalities in (9) and (10) hold as equalities for candidate optimal choices, and the static budget constraints are equivalent to the wealth evolution equation (5).

We denote by $\tilde{U}_i$ the convex dual functions of the concave functions U_i, $i=1,2$. For convenience of exposition, we list properties of $\tilde{U}_i$ (see Karatzas and Shreve [16]).

Lemma 1. *Utility functions U_i and their convex dual functions $\tilde{U}_i$, $i=1,2$ have the following properties:*

$$\tilde{U}_i(t,x) \triangleq \sup_{c>0}[U_i(t,c)-xc] = U_i(t,\mathcal{J}_{U_i}(t,x)) - x\mathcal{J}_{U_i}(t,x), \quad \partial_x \tilde{U}_i(t,x) = -\mathcal{J}_{U_i}(t,x) < 0,$$
$$U_i(t,c) = \inf_{x>0}[\tilde{U}_i(t,x)+xc] = \tilde{U}_i(t,\mathcal{J}_{\tilde{U}_i}(t,-c)) + \mathcal{J}_{\tilde{U}_i}(t,-c)c, \quad \partial_c U_i(t,c) = \mathcal{J}_{\tilde{U}_i}(t,-c) > 0,$$
$$\partial_{xx}\tilde{U}_i(t,x) = -\partial_x \mathcal{J}_{U_i}(t,x) = \frac{-1}{\partial_{cc}U_i(t,\mathcal{J}_{U_i}(t,x))} > 0, \quad \partial_{cc}U_i(t,c) = -\partial_c \mathcal{J}_{\tilde{U}_i}(t,-c) = \frac{-1}{\partial_{xx}\tilde{U}_i(t,\mathcal{J}_{\tilde{U}_i}(t,-c))},$$
$$\partial_{xxx}\tilde{U}_i(t,x) = \frac{\partial_{ccc}U_i(t,\mathcal{J}_{U_i}(t,x))}{(\partial_{cc}U_i(t,\mathcal{J}_{U_i}(t,x)))^3}, \ldots, \quad \forall\, t \in [0,T^1], \ \ x>0, \ \ c>0, \ \ i=1,2,$$

where $\mathcal{J}_{\tilde{U}_i}(t,\cdot)$ is the inverse function of $\partial_x \tilde{U}_i(t,\cdot)$, $i=1,2$. Moreover, $\partial_x\tilde{U}_i(t,x) \to -\infty$ as $x \to 0^+$, and $\partial_x \tilde{U}_i(t,x) \to 0^-$ as $x \to +\infty$ for any $t \in [0,T^1]$, $i=1,2$.

2.5. Notation for Spaces of Stochastic Processes and Function Spaces

To facilitate the exposition, we introduce the following notation.

For some $p \ge 1$, we introduce two spaces of stochastic processes and one space of random variables, which will be useful for our later argument about backward stochastic differential equations (BSDEs) (we refer to Yong and Zhou [32] for the theory of BSDEs):

- $\mathcal{S}_t^p$, the space of continuous $(\mathcal{F}_t)_{t=0}^{T^1}$-progressively measurable stochastic processes with norm $[\mathbb{E}(\sup_{s\in[t,T^1]}|X_s|^p)]^{1/p}$ for process $(X_s)_{s=t}^{T^1}$;
- $\mathcal{L}_t^p$, the space of $(\mathcal{F}_t)_{t=0}^{T^1}$-progressively measurable stochastic processes with norm $[\mathbb{E}(\int_t^{T^1}|X_s|^p\,ds)]^{1/p}$ for process $(X_s)_{s=t}^{T^1}$; and
- $L^p(\mathcal{F}_{T^1})$, the space of $\mathcal{F}_{T^1}$-measurable random variables with norm $[\mathbb{E}(|X_{T^1}|^p)]^{1/p}$ for random variable X_{T^1}.

We next introduce two Sobolev spaces, which will be useful to study variational inequalities (we refer to Krylov [21] for the theory of Sobolev spaces; note that Itô's formula still holds for $V(t,X_t)$ if V belongs to an appropriate Sobolev space):[10]

- $W_p^{2,1}(\mathcal{M}_{T^1})$, $p\ge 1$, the completion of $C^\infty(\mathcal{M}_{T^1})$ under the norm for V,

$$\|V\|_{W_p^{2,1}(\mathcal{M}_{T^1})} \triangleq \left[\int_{\mathcal{M}_{T^1}} (|V|^p + |\partial_t V|^p + |\partial_x V|^p + |\partial_{xx}V|^p)\,dx\,dt\right]^{1/p}; \quad \text{and}$$

- $W_{p,\mathrm{loc}}^{2,1}(\mathcal{M}_{T^1})$, $p\ge 1$, the set of all functions whose restrictions to the domain $\mathcal{M}_{T^1}^*$ belong to $W_p^{2,1}(\mathcal{M}_{T^1}^*)$ for any compact subset $\mathcal{M}_{T^1}^*$ of $\mathcal{M}_{T^1}$.

3. Transformation of the Original Problem into a Variational Inequality and Verification Theorem

In this section we will recast the original optimal stochastic control problem into a variational inequality (VI) by making three successive transformations. Next we will present the verification theorem (Theorem 1), which provides a connection to all three transformations. The verification theorem states that a solution to the VI satisfying suitable regularity conditions gives an optimal strategy for the original optimization problem.

As a preparation for the transformations, we first consider the agent's problem after retirement.

3.1. The Agent's Optimization Problem After Retirement

In this subsection we will consider the agent's optimization problem after retirement. The problem is a standard optimal consumption and investment choice problem similar to Merton [24]. The solution to the problem with a general utility function (in the absence of pension income) has been provided by Karatzas et al. [18].

After retirement, the agent does not face any choice of a stopping time. Thus, the control does not involve stopping time τ. Formally, the model in the previous section accommodates this case if we let $\tau = t$, where t

is the fixed current time. Then, the admissible set is $\mathcal{A}_t^1(t,w) \triangleq \{(c,\pi): (t,c,\pi) \in \mathcal{A}^1(t,w)\}$, where the admissible set is dependent on the initial time t and the initial wealth w, and the subscript t indicates that the stopping time is equal to t, (i.e., $\tau = t$). Let us denote the agent's value function after retirement by $\underline{V}$; that is,

$$\underline{V}(t,w) = \sup_{(c,\pi)\in\mathcal{A}_t^1(t,w)} \mathbb{E}\left[\int_t^{T^1} e^{-\beta(s-t)} U_1(s,c_s)\,ds + e^{-\beta(T^1-t)} U_2(T^1, W_{T^1}^{t,w;t,c,\pi})\right].$$

Now (10) implies that for any $t \in [0,T^1]$, $x > 0$, $w > 0$,

$$\begin{aligned}\underline{V}(t,w) - xw &\le \sup_{(c,\pi)\in\mathcal{A}_t^1(t,w)} \mathbb{E}\left\{\int_t^{T^1} [e^{-\beta(s-t)} U_1(s,c_s) - xH_s c_s]\,ds + [e^{-\beta(T^1-t)} U_2(T^1, W_{T^1}^{t,w;t,c,\pi}) - xH_{T^1} W_{T^1}^{t,w;t,c,\pi}]\right\} \\ &\le \mathbb{E}\left[\int_t^{T^1} e^{-\beta(s-t)} \tilde{U}_1(s,X_s)\,ds + e^{-\beta(T^1-t)} \tilde{U}_2(T^1, X_{T^1})\right] \triangleq \underline{\tilde{V}}(t,x),\end{aligned} \tag{11}$$

with $X_s = xe^{\beta(s-t)} H_s$.

Remark 4. The Lagrange multiplier, x, and the dual variable process, $(X_s)_{s\in[t,T^1]}$, represent the agent's marginal utility of wealth at time t and at time $s \in [t,T^1]$, respectively. We call $\underline{\tilde{V}}(t,x)$ the *dual value function* of the agent's optimization problem after retirement. By the argument below in the proof of Theorem 1, we can show that $X = (X_s)_{s\in[t,T^1]}$, $X^{-1} \in \mathcal{S}_t^p$ for any $p \ge 1$ and $\underline{V} < \infty$. Thus we know by (3) that all assumptions in Karatzas and Shreve [16, theorem 3.6.11] are satisfied. Then, according to Karatzas and Shreve [16, theorem 3.6.11], we deduce that for any $x > 0$, there exists a unique $w > 0$ such that the inequalities in the above hold as equalities, and $\underline{\tilde{V}}$ is the convex dual function of the concave function $\underline{V}$; that is,

$$\underline{\tilde{V}}(t,x) = \sup_{w>0}[\underline{V}(t,w) - xw], \qquad \underline{V}(t,w) = \inf_{x>0}[\underline{\tilde{V}}(t,x) + xw], \quad \forall\, t \in [0,T^1], \;\; x > 0, \;\; w > 0.$$

Thus, it is possible to deduce properties of $\underline{V}$ through those of $\underline{\tilde{V}}$.

Itô's formula implies that X is governed by the following SDE:

$$dX_s = (\beta - r)X_s\,ds - \theta^{\mathrm{T}} X_s\,dB_s, \quad \forall\, s \in [t,T^1], \qquad X_t = x. \tag{12}$$

In view of the Feynman–Kac formula (see Fleming and Soner [10]), $\underline{\tilde{V}}$ is expected to satisfy the following PDE:

$$\begin{cases} -\partial_t \underline{\tilde{V}} - \mathcal{L}\underline{\tilde{V}} = \tilde{U}_1 & \text{in } \mathcal{N}_{T^1}; \\ \underline{\tilde{V}}(T^1,x) = \tilde{U}_2(T^1,x), & \forall\, x \in (0,+\infty), \end{cases} \tag{13}$$

where we have used the following notation, which we will use throughout the paper:

$$\begin{gathered} \mathcal{L} \triangleq \frac{|\theta|^2 x^2}{2}\partial_{xx} + (\beta - r)x\partial_x - \beta, \\ \mathcal{N}_T \triangleq [0,T) \times (0,+\infty), \qquad \tilde{\mathcal{N}}_T \triangleq [0,T] \times (0,+\infty), \\ \mathcal{N}_{T^1} \triangleq [0,T^1) \times (0,+\infty), \qquad \tilde{\mathcal{N}}_{T^1} \triangleq [0,T^1] \times (0,+\infty). \end{gathered}$$

3.2. Transformations

In this subsection we make successive transformations to change the original problem into a VI.

Transformation 1. In the first step, we apply the dynamic programming principle to transform the problem into an optimal consumption, investment/retirement problem where the utility function after retirement is given by $\underline{V}$ the value function of the agent after retirement, which has been discovered in Subsection 3.1.

From the agent's problem in (7), we can deduce that for any $(t,w) \in \tilde{\mathcal{M}}_{T^1}$,

$$V(t,w) \le \sup_{(\tau,c,\pi)\in\mathcal{A}^1(t,w)} \mathbb{E}\left[\int_t^{\tau} e^{-\beta(s-t)}(U_1(s,c_s) - l)\,ds + e^{-\beta(\tau-t)} \underline{V}(\tau, W_\tau^{t,w;\tau,c,\pi})\right] \tag{14}$$

subject to (5).

Remark 5. If $t \in [T, T^1]$, then the agent has already retired, and thus, the agent's problem is just that of a standard optimal investment and consumption choice, and $V(t,w) = \underline{V}(t,w)$ for any $(t,w) \in [T, T^1] \times (0, +\infty)$. Hence, we will mainly focus on the case $t \le T$. Note that even if $t > T$, (14) still holds since we have defined $[t,T] = \{t\}$ and $\tau \in \mathcal{U}_{t,T} = \{t\}$ above.

Remark 6. According to the dynamic programming principle or the result in Koo et al. [20], we can prove that the inequality in (14) is indeed an equality, and the problem in Transformation 1 is equivalent to the original problem. That is, the agent's optimization problem after retirement can be summarized by its value function $\underline{V}$.

Transformation 2. In the second step we transform the original problem, which involves both stochastic control and optimal stopping, into a standard optimal stopping problem that does not involve stochastic control. We use the martingale and duality methods, following the idea in Karatzas and Shreve [16] and Karatzas and Wang [17].

For a Lagrange multiplier $x > 0$, we define $\hat{U}_1$, $\hat{\underline{V}}$ as

$$\hat{U}_1(t,x) = \sup_{c>0}[U_1(t,c) - l - xc] = \sup_{c>0}[U_1(t,c) - xc] - l = \tilde{U}_1(t,x) - l, \tag{15}$$

$$\hat{\underline{V}}(t,x) = \tilde{\underline{V}}(t,x) + xg(t), \tag{16}$$

for any $t \in [0,T]$, $x > 0$.

Remark 7. There is a difference in income before and after retirement, and the term $xg(t)$ is necessary to adjust the dual value function after retirement to the difference, as will be shown in (17).

As a result of the transformation, we will obtain the *dual value function* $\hat{V}$ for the agent's problem. Since the retirement time τ must be no later than the mandatory retirement date T, we apply the transformation only in $\tilde{\mathcal{M}}_T$ rather than in $\tilde{\mathcal{M}}_{T^1}$.

So (14), (9), (15), (11), and (16) imply that for any $(t,w) \in \tilde{\mathcal{M}}_T$, $x > 0$,

$$\begin{aligned}
&V(t,w) - x(w - g(t)) \\
&\quad \le \sup_{(\tau,c,\pi)\in\mathcal{A}^1(t,w)} \mathbb{E}\left\{\int_t^\tau [e^{-\beta(s-t)}U_1(s,c_s) - l]\,ds + e^{-\beta(\tau-t)}\underline{V}(\tau, W_\tau^{t,w;\tau,c,\pi}) - x\left[\int_t^\tau H_s c_s\,ds + H_\tau(W_\tau^{t,w;\tau,c,\pi} - g(\tau))\right]\right\} \\
&\quad = \sup_{(\tau,c,\pi)\in\mathcal{A}^1(t,w)} \mathbb{E}\left\{\int_t^\tau [e^{-\beta(s-t)}U_1(s,c_s) - l - xH_s c_s]\,ds + [e^{-\beta(\tau-t)}\underline{V}(\tau, W_\tau^{t,w;\tau,c,\pi}) - xH_\tau(W_\tau^{t,w;\tau,c,\pi} - g(\tau))]\right\} \\
&\quad \le \sup_{\tau\in\mathcal{U}_{t,T}} \mathbb{E}\left[\int_t^\tau e^{-\beta(s-t)}\hat{U}_1(s,X_s)\,ds + e^{-\beta(\tau-t)}(\tilde{\underline{V}}(\tau,X_\tau) + X_\tau g(\tau))\right] \\
&\quad = \sup_{\tau\in\mathcal{U}_{t,T}} \mathbb{E}\left[\int_t^\tau e^{-\beta(s-t)}\hat{U}_1(s,X_s)\,ds + e^{-\beta(\tau-t)}\hat{\underline{V}}(\tau,X_\tau)\right] \triangleq \hat{V}(t,x),
\end{aligned} \tag{17}$$

where we recall $X_s = xe^{\beta(s-t)}H_s$.

Remark 8. Applying the idea in theorem 8.5, corrollary 8.7 in Karatzas and Wang [17], for any $x > 0$ we conjecture that the inequalities in the above hold as equalities for a unique $w > g(t)$. In addition,

$$\hat{V}(t,x) = \sup_{w>g(t)}[V(t,w) - x(w - g(t))], \qquad V(t,w) = \inf_{x>0}[\hat{V}(t,x) + x(w - g(t))] \tag{18}$$

for any $t \in [0,T]$, $x > 0$, $w > g(t)$. If the conjecture is true, then we can derive properties of V through those of $\hat{V}$. We will show that the conjecture is true in the verification theorem (Theorem 1).

Transformation 3. The optimization problem represented by the right-hand side of the last equality in (17) is a standard optimal stopping problem for $t \in [0,T]$. Thus, in this last step we can use the relationship between optimal stopping problems and VIs to transform the original problem into a VI.

By relying on a standard relationship based on the dynamic programming principle, we derive the following VI for $\hat{V}$ from the optimal stopping problem in (17) (see, e.g., Peskir and Shriyaev [25]):

$$\begin{cases} -\partial_t \hat{V} - \mathcal{L}\hat{V} = \hat{U}_1, & \text{if } \hat{V} > \hat{\underline{V}} \text{ and } (t,x) \in \mathcal{N}_T; \\ -\partial_t \hat{V} - \mathcal{L}\hat{V} \ge \hat{U}_1, & \text{if } \hat{V} = \hat{\underline{V}} \text{ and } (t,x) \in \mathcal{N}_T; \\ \hat{V}(T,x) = \hat{\underline{V}}(T,x), & \forall\, x \in (0,+\infty). \end{cases} \tag{19}$$

Remark 9. If $t \in [T, T^1]$, the agent has already retired. So we consider VI (19) only in $\tilde{\mathcal{N}}_T$ rather than $\tilde{\mathcal{N}}_{T^1}$. But to give a convenient and complete expression for our model, we extend $\hat{V}$ and $\hat{\underline{V}}$ as $\hat{V}(t,x) = \hat{\underline{V}}(t,x) = \tilde{\underline{V}}(t,x)$ for any $(t,x) \in (T, T^1] \times (0, +\infty)$, where $\tilde{\underline{V}}$ is the solution of PDE (13).

3.3. Verification Theorem

In this subsection we will state and prove the verification theorem, which provides a justification as well as a connection to the three transformations. The verification theorem states that if solutions to VI (19) and PDE (13) satisfy suitable regularity conditions, then the value function V of the original optimal control problem is just the concave dual function of the solution of VI (19), and (18) is valid. In the theorem we also construct optimal strategies by using the solution to VI (19). That is, the marginal utility of the wealth process is discovered by using the inverse function of $\partial_x \hat{v}(t,\cdot)$, where $\hat{v}$ is the solution to VI (19) and optimal consumption is expressed in terms of the inverse, $\mathcal{J}_{U_1}$, of the derivative of the felicity function (i.e., the marginal utility of consumption) and the marginal utility of wealth process. The agent's wealth at time T is determined by the inverse, $\mathcal{J}_{\hat{U}_2}$, of the derivative of $\hat{U}_2$ (i.e., the inverse of the marginal utility of wealth function at time T^1). The optimal portfolio is discovered from the unique solution to a BSDE. The optimal stopping time is determined as the first time when $\hat{v}(s, X_s^*)$ hits $\hat{\underline{v}}(s, X_s^*)$.

Theorem 1. *Suppose that $\tilde{\underline{v}}$ is the strong solution to PDE (13), and denote $\hat{\underline{v}} = \tilde{\underline{v}} + xg(t)$ in $\tilde{\mathcal{N}}_{T^1}$. Assume that $\hat{v}$ is the strong solution to VI (19), where the terminal value and lower obstacle $\hat{\underline{V}}$ is replaced by $\hat{\underline{v}}$. Extend $\hat{v} = \hat{\underline{v}}$ in $(T, T^1] \times (0, +\infty)$. Suppose that $\hat{v}$, $\hat{\underline{v}}$ have the following properties:*

(1) *$\hat{v}, \hat{\underline{v}} \in W^{2,1}_{p,\mathrm{loc}}(\mathcal{N}_{T^1}) \cap C(\tilde{\mathcal{N}}_{T^1})$ with some $p \geq 3$, and $\partial_x \hat{v}$, $\partial_x \hat{\underline{v}} \in C(\tilde{\mathcal{N}}_{T^1})$.*

(2) *$\partial_{xx} \hat{v} > 0$ a.e. in $\mathcal{N}_{T^1}$. Moreover, $\partial_x \hat{v}(t,x) \to -\infty$ as $x \to 0^+$, and $\partial_x \hat{v}(t,x) \to 0$ as $x \to +\infty$ for any $t \in [0, T^1]$. There exist positive constants C and K such that*

$$|\partial_x \hat{v}(t,x)| + |\partial_x \hat{\underline{v}}(t,x)| \leq C(1 + x^{-K}), \quad \forall\, (t,x) \in \tilde{\mathcal{N}}_{T^1}.$$

Then,

$$\tilde{\underline{V}} = \tilde{\underline{v}} \text{ in } \tilde{\mathcal{N}}_{T^1}, \qquad \hat{\underline{V}} = \hat{\underline{v}}, \quad \hat{V} = \hat{v} \text{ in } \tilde{\mathcal{N}}_T, \tag{20}$$

where $\tilde{\underline{V}}$, $\hat{\underline{V}}$, and $\hat{V}$ are defined in (11), (16), and (17), respectively. Moreover, for any $(t,w) \in \tilde{\mathcal{M}}_{T^1}$, we have

$$V(t,w) = \inf_{x>0} [\hat{v}(t,x) + x(w - g(t))] = \hat{v}(t, x^*(t,w)) + x^*(t,w)(w - g(t)), \tag{21}$$

where $x^*(t,w) = \mathcal{J}_{\hat{v}}(t, g(t) - w) > 0$, and $\mathcal{J}_{\hat{v}}(t,\cdot)$ is the inverse function of $\partial_x \hat{v}(t,\cdot)$. Moreover, $x^* \in C(\tilde{\mathcal{M}}_{T^1})$, and for any $t \in [0, T^1]$, $x^*(t,w)$ is strictly decreasing with respect to w and has the asymptotic properties: $x^*(t,w) \to +\infty$ as $w \to g(t)^+$, and $x^*(t,w) \to 0^+$ as $w \to +\infty$.

The optimal consumption and the optimal retirement strategy can be described as

$$c_s^* = \mathcal{J}_{U_1}(s, X_s^*), \qquad \tau^* = \inf\{s \in [t,T] \colon \hat{v}(s, X_s^*) = \hat{\underline{v}}(s, X_s^*)\} \quad \text{with } X_s^* = x^*(t,w) e^{\beta(s-t)} H_s,$$

where X^* is the solution of SDE (12) with the initial state $x^*(t,w)$. Moreover, the optimal investment strategy π^* is governed by the following BSDE:

$$W_s^* = W_{T^1}^* - \int_s^{T^1} \left[(\pi_u^*)^{\mathrm{T}} \Sigma \theta + r W_u^* - c_u^* + \varrho I_{\{u \leq \tau^*\}} \right] du - \int_s^{T^1} (\pi_u^*)^{\mathrm{T}} \Sigma \, dB_u, \quad \forall\, s \in [t, T^1], \tag{22}$$

with $W_{T^1}^* = \mathcal{J}_{U_2}(T^1, X_{T^1}^*)$.

The proof of Theorem 1 is based on the martingale method and dual attainment. Concretely, the main idea of the proof is to prove that

$$J(t,w;\tau^*,c^*,\pi^*) \geq \hat{v}(t,x^*) + x^*(w - g(t)) \geq V(t,w) \geq J(t,w;\tau^*,c^*,\pi^*).$$

The first inequality in the above means the strategy (τ^*, c^*, π^*) is optimal. The second and the third inequalities imply that V is the dual concave function of $\hat{v}$ (see (21)) and V is the value function.

Proof of Theorem 1. First, we show that $x^*(t,w)$ in Theorem 1 is well defined and $x^*(t,w)>0$, $x^*\in C(\tilde{\mathcal{M}}_{T^1})$. We also show that $x^*(t,w)$ has the monotonicity and asymptotic properties as in the conclusion of the theorem. In fact, since $\partial_x\hat{v}\in C(\tilde{\mathcal{N}}_{T^1})$ and $\partial_{xx}\hat{v}>0$ a.e. in $\mathcal{N}_{T^1}$, we deduce that $\partial_x\hat{v}(t,x)$ is strictly increasing with respect to x for any $t\in[0,T^1)$. And property (2) in the assumptions in this theorem implies that $\partial_x\hat{v}(t,\cdot)$: $(0,+\infty)\to(-\infty,0)$ for any $t\in[0,T^1)$. Hence, we deduce that $x^*(t,w)=\mathcal{J}_{\hat{v}}(t,g(t)-w)$ exists for any $(t,w)\in\mathcal{M}_{T^1}$, takes values on $(0,+\infty)$, and is continuous and strictly decreasing with respect to w for any $t\in[0,T^1)$. And the asymptotic properties of $x^*(t,w)$ come from those of $\partial_x\hat{v}(t,x)$. Moreover, the properties on $t=T^1$ come from the terminal condition of PDE (13) and Lemma 1.

Second, we show that c^*, $W^*_{T^1}$, and τ^* are well defined. In fact, U_i, $i=1,2$ are strictly concave by Assumption 1; thus, $\mathcal{J}_{U_i}$ are well defined, and hence, c^* and $W^*_{T^1}$ are well defined. In the case of $t\le T$, since $\hat{v}(s,X^*_s)-\hat{\underline{v}}(s,X^*_s)$ is continuous with respect to s, τ^* is a stopping time. Moreover, the terminal value condition of VI (19) implies that $\tau^*\le T$ and $\tau^*\in\mathcal{U}_{t,T}$. In the case of $t>T$, noting that we have defined $[t,T]=\{t\}$, and recalling the fact that $\hat{v}=\hat{\underline{v}}$ in $(T,T^1]\times(0,+\infty)$, we deduce $\tau^*=t\in\mathcal{U}_{t,T}$.

We will show in Lemma 2 that π^* can be constructed from the solution of BSDE (22) and $(\tau^*,c^*,\pi^*)\in\mathcal{A}^1(t,W^*_t)$. From the lemma we continue the proof of the theorem. The optimal investment strategy π^* comes from BSDE (22), and thus it is necessary to prove $W^*=W^{t,w;\tau^*,c^*,\pi^*}$. For this purpose, we compare SDE (5) with BSDE (22), and recall $\Sigma\theta=\alpha-r1_d$. We find that

$$d(e^{-r(s-t)}W^{t,w;\tau^*,c^*,\pi^*}_s)=d(e^{-r(s-t)}W^*_s),\quad \forall\, t\le s\le T^1.$$

Hence, the uniqueness of the solution of the SDE implies that it is sufficient to prove that $W^*_t=w$.

Applying Itô's formula, we have

$$d[e^{-\beta(u-t)}X^*_u(W^*_u-g(u))]=e^{-\beta(u-t)}X^*_u[-c^*_u-\varrho I_{\{\tau^*<u\le T\}}]\,du+e^{-\beta(u-t)}X^*_u[(\pi^*_u)^{\mathrm{T}}\Sigma-\theta^{\mathrm{T}}(W^*_u-g(u))]\,dB_u,\quad \forall\, u\in[t,T^1].$$

So we deduce that

$$x^*[W^*_t-g(t)]=e^{-\beta(T^1-t)}X^*_{T^1}W^*_{T^1}+\int_t^{T^1}e^{-\beta(u-t)}X^*_u[c^*_u+\varrho I_{\{\tau^*<u\le T\}}]\,du-\int_t^{T^1}e^{-\beta(u-t)}X^*_u[(\pi^*_u)^{\mathrm{T}}\Sigma-\theta^{\mathrm{T}}(W^*_u-g(u))]\,dB_u.$$

Taking $\mathcal{F}_t$-conditional expectation in this equality and applying the Markov property, and combining this with the fact that $X^*,W^*\in\mathcal{S}^p_t$, $\pi^*\in\mathcal{L}^p_t$ for any $p\ge1$ (refer to the proof of Lemma 2), we have

$$\begin{aligned}
x^*[W^*_t-g(t)]&=\mathbb{E}\left[e^{-\beta(T^1-t)}X^*_{T^1}W^*_{T^1}+\int_t^{T^1}e^{-\beta(u-t)}X^*_u[c^*_u+\varrho I_{\{\tau^*<u\le T\}}]\,du\right]\\
&=\mathbb{E}\left\{e^{-\beta(T^1-t)}[U_2(T^1,W^*_{T^1})-\tilde{U}_2(T^1,X^*_{T^1})]+\int_t^{T^1}e^{-\beta(u-t)}[U_1(u,c^*_u)-\tilde{U}_1(u,X^*_u)+\varrho X^*_u I_{\{\tau^*<u\le T\}}]\,du\right\}\\
&=J(t,W^*_t;\tau^*,c^*,\pi^*)-\mathbb{E}\left\{e^{-\beta(T^1-t)\underline{\tilde{v}}(T^1,X^*_{T^1})}+\int_t^{\tau^*}e^{-\beta(u-t)}\hat{U}_1(u,X^*_u)\,du\right.\\
&\qquad\left.+\int_{\tau^*}^{T^1}e^{-\beta(u-t)}[\tilde{U}_1(u,X^*_u)-\varrho X^*_u I_{\{u\le T\}}]\,du\right\}.
\end{aligned}\tag{23}$$

Here, we have used the definitions of c^* and $W^*_{T^1}$ in the second equality and used (7) and (15) and the terminal condition of PDE (13) in the third equality.

Since $\hat{v}(\cdot,X^*_\cdot)$ and $\hat{\underline{v}}(\cdot,X^*_\cdot)$ are continuous stochastic processes (see Krylov [21] or Yang and Tang [31]), the definition of τ^* and $\hat{v}=\hat{\underline{v}}=\tilde{\underline{v}}$ in $[T,T^1]\times(0,+\infty)$ imply that

$$\hat{v}(T^1,X^*_{T^1})=\hat{\underline{v}}(T^1,X^*_{T^1})=\tilde{\underline{v}}(T^1,X^*_{T^1}),\qquad \hat{v}(\tau^*,X^*_{\tau^*})=\hat{\underline{v}}(\tau^*,X^*_{\tau^*}).\tag{24}$$

Recalling PDE (13), we have

$$\begin{aligned}
&\partial_t\hat{\underline{v}}(u,X^*_u)+\mathcal{L}\hat{\underline{v}}(u,X^*_u)=\partial_t[\tilde{\underline{v}}(t,x)+xg(t)]_{(t,x)=(u,X^*_u)}+\mathcal{L}[\tilde{\underline{v}}(t,x)+xg(t)]_{(t,x)=(u,X^*_u)}\\
&\quad=-\tilde{U}_1(u,X^*_u)+\varrho X^*_u I_{\{u\le T\}},\quad \forall\, u\in[t,T^1].
\end{aligned}\tag{25}$$

Since $\underline{\hat{v}} \in W^{2,1}_{p,\mathrm{loc}}(\mathcal{N}_{T^1}) \cap C(\tilde{\mathcal{N}}_{T^1})$ with some $p \ge 3$, applying Itô's formula to $\underline{\hat{v}}$ (see Krylov [21] or Yang and Tang [31]), by (23)–(25), we deduce that

$$\begin{aligned}
&J(t, W^*_t; \tau^*, c^*, \pi^*) - x^*[W^*_t - g(t)]\\
&= \mathbb{E}\left\{ e^{-\beta(T^1-t)} \underline{\hat{v}}(T^1, X^*_{T^1}) + \int_t^{\tau^*} e^{-\beta(u-t)} \hat{U}_1(u, X^*_u)\, du + \int_{\tau^*}^{T^1} e^{-\beta(u-t)} [\tilde{U}_1(u, X^*_u) - \varrho X^*_u I_{\{u \le T\}}]\, du \right\}\\
&= \mathbb{E}\left\{ e^{-\beta(\tau^*-t)} \hat{v}(\tau^*, X^*_{\tau^*}) + \int_t^{\tau^*} e^{-\beta(u-t)} \hat{U}_1(u, X^*_u)\, du - \int_{\tau^*}^{T^1} e^{-\beta(u-t)} \partial_x \underline{\hat{v}}(u, X^*_u) \theta^{\mathrm{T}} X^*_u\, dB_u \right\}.
\end{aligned}$$

Property (2) in the assumptions in the theorem implies that

$$|e^{-\beta(u-t)} \partial_x \underline{\hat{v}}(u, X^*_u)\, \theta^{\mathrm{T}} X^*_u| \le C X^*_u + C(X^*_u)^{1-K}, \quad u \in [t, T^1].$$

Recalling X^*, $(X^*)^{-1} \in \mathcal{S}^p_t$ for any $p \ge 1$, we deduce that $e^{-\beta(\cdot-t)} \partial_x \underline{\hat{v}}(\cdot, X^*_\cdot) \theta^{\mathrm{T}} X^*_\cdot \in \mathcal{S}^p_t$ for any $p \ge 1$, too. Hence, we have

$$\begin{aligned}
&J(t, W^*_t; \tau^*, c^*, \pi^*) - x^*[W^*_t - g(t)]\\
&= \mathbb{E}\left\{ e^{-\beta(\tau^*-t)} \hat{v}(\tau^*, X^*_{\tau^*}) + \int_t^{\tau^*} e^{-\beta(u-t)} \hat{U}_1(u, X^*_u)\, du \right\} = \mathbb{E}[e^{-\beta(t-t)} \hat{v}(t, X^*_t)] = \hat{v}(t, x^*).
\end{aligned}$$

In the second equality, we have used a method similar to that in the above and the fact that

$$\partial_t \hat{v}(u, X^*_u) + \mathcal{L}\hat{v}(u, X^*_u) = -\hat{U}_1(u, X^*_u), \quad \text{for any } u \in [t, \tau^*],$$

which can be deduced by the definition of τ^* and VI (19).

Until now, we have proved that for any $(t, w) \in \tilde{\mathcal{M}}_{T^1}$,

$$\begin{aligned}
J(t, W^*_t; \tau^*, c^*, \pi^*) &= \hat{v}(t, x^*) + x^*[W^*_t - g(t)] \ge \inf_{x>0}\{\hat{v}(t, x) + x[W^*_t - g(t)]\}\\
&= \hat{v}(t, \hat{x}^*) + \hat{x}^*[W^*_t - g(t)],
\end{aligned} \tag{26}$$

where $\hat{x}^* = \mathcal{J}_{\hat{v}}(t, g(t) - W^*_t)$.

On the other hand, recalling (16), (11) and (13), and applying Itô's formula, we have

$$\begin{aligned}
\underline{\hat{V}}(t, x) &= \mathbb{E}\left[\int_t^{T^1} e^{-\beta(s-t)} \tilde{U}_1(s, X_s)\, ds + e^{-\beta(T^1-t)} \underline{\tilde{v}}(T^1, X_{T^1}) \right] + x g(t)\\
&= \mathbb{E}\left[e^{-\beta(t-t)} \underline{\tilde{v}}(t, x) - \int_t^{T^1} e^{-\beta(s-t)} \partial_x \underline{\tilde{v}}(s, X_s) \theta^{\mathrm{T}} X_s\, dB_s \right] + x g(t) = \underline{\hat{v}}\,(t, x), \quad \forall\, (t, x) \in \tilde{\mathcal{N}}_{T^1}.
\end{aligned} \tag{27}$$

Combining (17), (27), and VI (19), and applying Itô's formula, we deduce that for any $(t, \tilde{w}) \in \tilde{\mathcal{M}}_{T^1}$, $x > 0$, $(\tau, c, \pi) \in \mathcal{A}^1(t, \tilde{w})$,

$$\begin{aligned}
J(t, \tilde{w}; \tau, c, \pi) &\le V(t, \tilde{w}) \le \hat{V}(t, x) + x[\tilde{w} - g(t)]\\
&\le \sup_{\tau \in \mathcal{U}_{t,T}} \mathbb{E}\left[\int_t^{\tau} e^{-\beta(s-t)} \hat{U}_1(s, X_s)\, ds + e^{-\beta(\tau-t)} \hat{v}(\tau, X_\tau) \right] + x[\tilde{w} - g(t)]\\
&\le \sup_{\tau \in \mathcal{U}_{t,T}} \mathbb{E}\left[e^{-\beta(t-t)} \hat{v}\,(t, x) - \int_t^{\tau} e^{-\beta(s-t)} \partial_x \hat{v}(s, X_s) \theta^{\mathrm{T}} X_s\, dB_s \right] + x[\tilde{w} - g(t)]\\
&= \hat{v}(t, x) + x[\tilde{w} - g(t)].
\end{aligned}$$

Setting $\tilde{w} = W^*_t$, $x = \hat{x}^*$, $(\tau, c, \pi) = (\tau^*, c^*, \pi^*)$ in the inequality, and recalling (26), we deduce that for any $(t, w) \in \tilde{\mathcal{M}}_{T^1}$,

$$\begin{aligned}
J(t, W^*_t; \tau^*, c^*, \pi^*) &= \hat{v}(t, x^*) + x^*[W^*_t - g(t)] = \inf_{x>0}\{\hat{v}(t, x) + x[W^*_t - g(t)]\}\\
&= \hat{v}(t, \hat{x}^*) + \hat{x}^*[W^*_t - g(t)] = \hat{V}(t, \hat{x}^*) + \hat{x}^*[W^*_t - g(t)] = V(t, W^*_t).
\end{aligned} \tag{28}$$

Since $\partial_{xx}\hat{v} > 0$ a.e. in $\mathcal{N}_{T^1}$, we have

$$\mathcal{J}_{\hat{v}}(t, g(t) - w) = x^* = \hat{x}^* = \mathcal{J}_{\hat{v}}(t, g(t) - W_t^*).$$

Since $\mathcal{J}_{\hat{v}}(t, g(t) - w)$ is strictly decreasing with respect to w for any $t \in [0, T^1]$, we conclude that $w = W_t^*$ and $W^* = W^{t,w;\tau^*,c^*,\pi^*}$ for any $(t, w) \in \tilde{\mathcal{M}}_{T^1}$.

Finally, (27) and (28) imply the statements in (20) and (21). This completes the proof. □

Lemma 2. *Suppose that the assumptions in Theorem 1 are satisfied. Then, the strategy π^* in Theorem 1 is well defined and can be constructed from the solution to BSDE (22), and $(\tau^*, c^*, \pi^*) \in \mathcal{A}^1(t, W_t^*)$.*

Proof. We prove the existence of π^*. In fact, SDE (12) implies that $X^* \in \mathcal{S}_t^p$ for any $p \geq 1$. Denote $Y^* \triangleq 1/X^*$; then it is not difficult to deduce that Y^* is governed by

$$Y_s^* = \frac{1}{x^*} + \int_t^s (r - \beta + |\theta|^2) Y_u^* \, du + \int_t^s \theta^{\mathrm{T}} Y_u^* \, dB_u, \quad \forall s \in [t, T^1].$$

Thus, we can claim that $(X^*)^{-1} = Y^* \in \mathcal{S}_t^p$ for any $p \geq 1$. Hence, (3) implies that $W_{T^1}^* = \mathcal{J}_{U_2}(T^1, X_{T^1}^*) \in L^p(\mathcal{F}_{T^1})$ for any $p \geq 1$. Repeating the same argument as in the above, we derive that $c^* \in \mathcal{S}_t^p$. Consider the following BSDE:

$$W_s^* = W_{T^1}^* - \int_s^{T^1} [Z_u^{\mathrm{T}}\theta + rW_u^* - c_u^* + \varrho I_{\{u \leq \tau^*\}}] \, du - \int_s^{T^1} Z_u^{\mathrm{T}} \, dB_u, \quad \forall s \in [t, T^1]. \tag{29}$$

It is clear that BSDE (29) has a unique solution $(W^*, Z) \in \mathcal{S}_t^p \times \mathcal{L}_t^p$ for any $p \geq 1$. Since Σ is positive-definite, we can get $\pi^* = (\Sigma^{-1})^{\mathrm{T}} Z \in \mathcal{L}_t^p$ for any $p \geq 1$.

Next, we prove that $(\tau^*, c^*, \pi^*) \in \mathcal{A}^1(t, W_t^*)$. In fact, since the ranges of the functions $\mathcal{J}_{U_i}$, $i = 1, 2$ are $(0, +\infty)$ and $c^* \in \mathcal{S}_t^p$, $\pi^* \in \mathcal{L}_t^p$ for any $p \geq 1$, we deduce that

$$\int_t^{T^1} c_s^* + \|\pi_s^*\|^2 ds < \infty, \quad W_{T^1}^* > 0 \text{ a.s. in } \Omega, \qquad c_s^* > 0, \quad \text{for any } s \in [t, T^1] \text{ a.s. in } \Omega.$$

Recalling the definitions of c^* and $W_{T^1}^*$, we deduce that

$$\begin{aligned}
&\int_t^{T^1} e^{-\beta(s-t)} U_1(s, c_s^*) \, ds + e^{-\beta(T^1-t)} U_2(T^1, W_{T^1}^*) \\
&\quad = \int_t^{T^1} e^{-\beta(s-t)} [X_s^* c_s^* + \tilde{U}_1(s, X_s^*)] \, ds + e^{-\beta(T^1-t)} [X_{T^1}^* W_{T^1}^* + \tilde{U}_2(T^1, X_{T^1}^*)] \\
&\quad \geq \int_t^{T^1} e^{-\beta(s-t)} [X_s^* c_s^* + U_1(s, c) - X_s^* c] \, ds + e^{-\beta(T^1-t)} [X_{T^1}^* W_{T^1}^* + U_2(T^1, c) - X_{T^1}^* c],
\end{aligned}$$

where we have used that $\tilde{U}_i(t, x) \geq U_i(t, c) - xc$ for any $t \in [0, T^1]$, x, $c > 0$, $i = 1, 2$. Combining this with the fact that X^*, W^*, $c^* \in \mathcal{S}_t^p$ for any $p \geq 1$ and that c is arbitrary, we have

$$\mathbb{E}\left[\int_t^{T^1} e^{-\beta(s-t)} U_1^-(s, c_s^*) \, ds + e^{-\beta(T^1-t)} U_2^-(T^1, W_{T^1}^*)\right] < +\infty.$$

Moreover, it is not difficult to check that $(W^* - g(\cdot) I_{\{s<\tau^*\}}, \Sigma^{\mathrm{T}}\pi^*)$ satisfies the following BSDE:

$$\begin{aligned}
W_u^* - g(u) I_{\{s<\tau^*\}} = W_{T^1}^* &- \int_u^{T^1} \left[(\pi_\xi^*)^{\mathrm{T}} \Sigma\theta + r(W_\xi^* - g(\xi) I_{\{s<\tau^*\}}) - c_\xi^* - \varrho I_{\{\tau^* < \xi \leq T\}} I_{\{s<\tau^*\}}\right] d\xi \\
&- \int_u^{T^1} (\pi_\xi^*)^{\mathrm{T}} \Sigma \, dB_\xi
\end{aligned}$$

for any $t \leq s \leq u \leq T^1$. Since $W_{T^1}^* > 0$, $c_\xi^* + \varrho I_{\{\tau^* < \xi \leq T\}} I_{\{s<\tau^*\}} > 0$, applying the comparison theory for BSDEs, we deduce that $W_u^* - g(u) I_{\{s<\tau^*\}} > 0$ for any $t \leq s \leq u \leq T^1$. In particular, setting $u = s$, then we have $W_s^* - g(s) I_{\{s<\tau^*\}} > 0$ for any $s \in [t, T^1]$. Hence, we have proved that $(\tau^*, c^*, \pi^*) \in \mathcal{A}^1(t, W_t^*)$. □

4. Solution to the Variational Inequality

In this section, we consider PDE (13) and VI (19), and we prove that the unique solution $\hat{V}$ and the lower obstacle $\hat{\underline{V}}$ of VI (19) satisfy the assumptions in Theorem 1. We will use $\tilde{\underline{V}}$, $\hat{V}$ for solutions to PDE (13) and (19), respectively, rather than for the value functions defined in (11) and (17). We will show that $\hat{\underline{V}} \triangleq \tilde{\underline{V}} + xg(t)$ and $\hat{V}$ satisfy the assumptions of $\hat{\underline{v}}$, $\hat{v}$ in Theorem 1, and the theorem implies that $\tilde{\underline{V}}$, $\hat{\underline{V}}$, and $\hat{V}$ are, respectively, equal to those defined in (11), (16), and (17). Thus, our notation does not lead to any confusion.

We first show the properties of the solution to PDE (13), $\tilde{\underline{V}}$, in the following theorem. The properties of the lower obstacle $\hat{\underline{V}}$ will be derived from those of $\tilde{\underline{V}}$ by its definition, $\hat{\underline{V}} = \tilde{\underline{V}} + xg(t)$.

Theorem 2. *PDE (13) has a unique solution $\tilde{\underline{V}}$ satisfying the following properties:*

(1) $\tilde{\underline{V}} \in C^\infty(\tilde{\mathcal{N}}_{T^1})$.

(2) $\partial_{xx}\tilde{\underline{V}} > 0$ *in* $\tilde{\mathcal{N}}_{T^1}$.

(3) $\partial_x\tilde{\underline{V}}(t,x) < 0$ *for any* $(t,x) \in \tilde{\mathcal{N}}_{T^1}$. *Moreover,* $\partial_x\tilde{\underline{V}}(t,x) \to -\infty$ *as* $x \to 0^+$, $\partial_x\tilde{\underline{V}}(t,x) \to 0^-$ *as* $x \to +\infty$ *for any* $t \in [0,T^1]$. *There exist positive constants C and K such that*

$$|\partial_x\tilde{\underline{V}}(t,x)| \le C(1 + x^{-K}), \quad \forall (t,x) \in \tilde{\mathcal{N}}_{T^1}. \tag{30}$$

Proof. (1) Since the differential operator is degenerate parabolic, we need the following transformation to prove the existence of the classical solution of PDE (13):

$$z = \ln x, \qquad \check{\underline{V}}(t,z) = \tilde{\underline{V}}(t,x). \tag{31}$$

Then, it is not difficult to deduce that $\check{\underline{V}}$ is governed by the following PDE:

$$\begin{cases} -\partial_t\check{\underline{V}} - \tilde{\mathcal{L}}\check{\underline{V}} = \tilde{U}_1(t,e^z), & \text{in } [0,T^1)\times\mathbb{R}, \\ \check{\underline{V}}(T^1,z) = \tilde{U}_2(T^1,e^z), & \forall z \in \mathbb{R}, \end{cases} \tag{32}$$

where

$$\tilde{\mathcal{L}} \triangleq \frac{|\theta|^2}{2}\partial_{zz} + \left(\beta - r - \frac{|\theta|^2}{2}\right)\partial_z - \beta,$$

which is a uniformly parabolic differential operator with constant coefficients. Moreover, recalling Lemma 1, we deduce that the inhomogeneous term, $\tilde{U}_1(t,e^z)$, and the terminal value, $\tilde{U}_2(t,e^z)$, belong to $C^\infty([0,T^1]\times\mathbb{R})$. Hence, the regularity theory for PDEs (see, e.g., Lieberman [22]) implies that PDE (32) has a unique solution, $\check{\underline{V}} \in C^\infty([0,T^1]\times\mathbb{R})$. Recalling transformation (31), we deduce that PDE (13) has a unique solution $\tilde{\underline{V}} \in C^\infty(\tilde{\mathcal{N}}_{T^1})$.

(2) We now prove $\tilde{\underline{V}}$ is strictly convex. Differentiating PDE (13) twice with respect to x, we derive that $\partial_{xx}\tilde{\underline{V}}$ satisfies

$$\begin{cases} -\partial_t(\partial_{xx}\tilde{\underline{V}}) - \mathcal{L}_{xx}(\partial_{xx}\tilde{\underline{V}}) = \partial_{xx}\tilde{U}_1 > 0, & \text{in } \mathcal{N}_{T^1}, \\ \partial_{xx}\tilde{\underline{V}}(T^1,x) = \partial_{xx}\tilde{U}_2(T^1,x) > 0, & \forall x \in (0,+\infty), \end{cases}$$

where

$$\mathcal{L}_{xx} \triangleq \frac{|\theta|^2x^2}{2}\partial_{xx} + (2|\theta|^2 + \beta - r)x\partial_x + (|\theta|^2 + \beta - 2r).$$

The strong maximum principle implies that $\partial_{xx}\tilde{\underline{V}} > 0$ in $\tilde{\mathcal{N}}_{T^1}$.

(3) Temporarily denoting $Q = \partial_x\tilde{\underline{V}}$, and differentiating PDE (13) with respect to x, we have

$$\begin{cases} -\partial_t Q - \mathcal{L}_x Q = \partial_x\tilde{U}_1, & \text{in } \mathcal{N}_{T^1}, \\ Q(T^1,x) = \partial_x\tilde{U}_2(T^1,x), & \forall x \in (0,+\infty), \end{cases} \tag{33}$$

where

$$\mathcal{L}_x \triangleq \frac{|\theta|^2x^2}{2}\partial_{xx} + (|\theta|^2 + \beta - r)x\partial_x - r.$$

Since $\partial_x\tilde{U}_i < 0$ in $\tilde{\mathcal{N}}_{T^1}$ for any $i = 1,2$, the strong maximum principle (see, e.g., Lieberman [22]) implies that

$$\partial_x\tilde{\underline{V}} = Q < 0, \quad \text{in } \mathcal{N}_{T^1}.$$

Moreover, we deduce that $\partial_x\tilde{\underline{V}}(T^1,x) < 0$ for any $x > 0$ by using the terminal value of PDE (33) and Lemma 1.

Next, we prove that there exist positive constants C and K such that

$$|\partial_x \underset{\sim}{\tilde{V}}(t,x)| = |Q(t,x)| \le C(x^K + x^{-K}), \quad \forall\,(t,x) \in \tilde{\mathcal{N}}_{T^1}. \tag{34}$$

In fact, it is clear that Q can be approximated by Q_n, $n = 2, \ldots$, which is the unique classical solution to the following PDE:

$$\begin{cases} -\partial_t Q_n - \mathcal{L}_x Q_n = \partial_x \tilde{U}_1, & \text{in } \mathcal{N}^n_{T^1} \triangleq [0,T^1) \times (1/n, n), \\ Q_n(T^1, x) = \partial_x \tilde{U}_2(T^1, x), & \forall\, x \in [1/n, n], \\ Q_n(t, 1/n) = \partial_x \tilde{U}_2(t, 1/n), & Q_n(t,n) = \partial_x \tilde{U}_2(t,n), \quad \forall\, t \in [0, T^1]. \end{cases}$$

It is clear that $Q_n \to Q$ in $C(\overline{\mathcal{N}^m_{T^1}})$ for any $m = 2, \ldots$. Let us temporarily denote

$$\underset{\sim}{Q}(t,x) = -Ce^{K^3(T^1-t)}(x^K + x^{-K}), \quad \forall\,(t,x) \in \tilde{\mathcal{N}}_{T^1},$$

where C and K are positive constants defined later. A straightforward calculation shows

$$\begin{aligned} -\partial_t \underset{\sim}{Q} - \mathcal{L}_x \underset{\sim}{Q} = Ce^{K^3(T^1-t)} \Bigg\{ &\left[-K^3 + \frac{|\theta|^2}{2} K(K-1) + (|\theta|^2 + \beta - r)K - r \right] x^K \\ &+ \left[-K^3 + \frac{|\theta|^2}{2} K(K+1) - (|\theta|^2 + \beta - r)K - r \right] x^{-K} \Bigg\}. \end{aligned}$$

Recalling (3) and Lemma 1, we deduce that $\partial_x \tilde{U}_i(t,x) \ge -C(x^K + x^{-K})$ for any $(t,x) \in \tilde{\mathcal{N}}_{T^1}$, $i = 1,2$, where we have changed the constant C and let K be large enough such that $K \ge 1/k$, and

$$K^3 \ge \frac{|\theta|^2}{2} K(K+1) + (|\theta|^2 + \beta + r)K + 1.$$

Hence, we have

$$\begin{cases} -\partial_t \underset{\sim}{Q} - \mathcal{L}_x \underset{\sim}{Q} \le -C(x^K + x^{-K}) \le \partial_x \tilde{U}_1 = -\partial_t Q_n - \mathcal{L}_x Q_n, & \text{in } \mathcal{N}^n_{T^1}, \\ \underset{\sim}{Q}(T^1, x) = -C(x^K + x^{-K}) \le \partial_x \tilde{U}_2(T^1, x) = Q_n(T^1, x), & \forall\, x \in [1/n, n], \\ \underset{\sim}{Q}(t, 1/n) \le \underset{\sim}{Q}(T^1, 1/n) \le \partial_x \tilde{U}_2(t, 1/n) = Q_n(t, 1/n), \underset{\sim}{Q}(t,n) \le \partial_x \tilde{U}_2(t,n) = Q_n(t,n), & \forall\, t \in [0, T^1]. \end{cases}$$

The comparison principle for PDEs (see Tso [27]) implies that $Q_n \ge \underset{\sim}{Q}$ in $\overline{\mathcal{N}^n_{T^1}}$ for any $n = 2, \ldots$. Note that $\underset{\sim}{Q}$ is independent of n, so we have $Q \ge \underset{\sim}{Q}$ in $\tilde{\mathcal{N}}_{T^1}$. Since $Q < 0$ in $\tilde{\mathcal{N}}_{T^1}$, (34) is obvious.

Moreover, since $\partial_{xx} \underset{\sim}{\tilde{V}} > 0$ in $\tilde{\mathcal{N}}_{T^1}$, we deduce that

$$\partial_x \underset{\sim}{\tilde{V}}(t,x) \ge \min\{\partial_x \underset{\sim}{\tilde{V}}(s,1) \colon s \in [0,T^1]\}, \quad \forall\, t \in [0,T^1], \ \forall\, x \ge 1.$$

Combining (34) and $\partial_x \underset{\sim}{\tilde{V}} < 0$ in $\mathcal{N}_{T^1}$, we can obtain (30) if we change the constant C appropriately.

Next, we prove the asymptotic properties of Q. By slight abuse of notation, we will redefine $\underset{\sim}{Q}$ temporarily as follows:

$$\underset{\sim}{Q}(t,x) = \begin{cases} \dfrac{\lambda_2 \delta}{\lambda_1 - \lambda_2} \left(\dfrac{x}{K\tilde{x}} \right)^{\lambda_1} - \dfrac{\lambda_1 \delta}{\lambda_1 - \lambda_2} \left(\dfrac{x}{K\tilde{x}} \right)^{\lambda_2} - \delta, & (t,x) \in [0,T^1] \times [\tilde{x}, K\tilde{x}], \\ -2\delta, & (t,x) \in [0,T^1] \times (K\tilde{x}, +\infty), \end{cases}$$

where K, $\tilde{x}$,and δ are positive constants that will be determined later, and λ_1 and λ_2 are the positive and negative roots, respectively, of the following algebraic equation:

$$\frac{|\theta|^2}{2} \lambda(\lambda - 1) + (|\theta|^2 + \beta - r)\lambda - r = 0. \tag{35}$$

It is not difficult to check that $\underset{\sim}{Q}(t, K\tilde{x}) = -2\delta$, $\partial_x \underset{\sim}{Q}(t, K\tilde{x}) = 0$ for any $t \in [0, T^1]$, and

$$\underset{\sim}{Q} \in C([0,T^1] \times [\tilde{x}, +\infty)) \cap W^{2,1}_{p,\mathrm{loc}}([0,T^1] \times (\tilde{x}, +\infty)), \qquad \partial_x \underset{\sim}{Q}(t,x) > 0, \quad \text{for any } (t,x) \in [0,T^1] \times [\tilde{x}, K\tilde{x}),$$

and

$$-\partial_t \underset{\sim}{Q} - \mathcal{L}_x \underset{\sim}{Q} = \begin{cases} -r\delta, & \text{in } [0,T^1) \times [\tilde{x}, K\tilde{x}), \\ -2r\delta, & \text{in } [0,T^1) \times (K\tilde{x}, +\infty). \end{cases}$$

Fix $\tilde{x}$, then take $\delta = \max\{-\partial_x \tilde{U}_1(t,\tilde{x})/r, -\partial_x \tilde{U}_2(t,\tilde{x})/2 : t \in [0,T^1]\}$, and choose K large enough such that

$$\underline{Q}(t,\tilde{x}) = \frac{\lambda_2\delta}{\lambda_1-\lambda_2}\left(\frac{1}{K}\right)^{\lambda_1} - \frac{\lambda_1\delta}{\lambda_1-\lambda_2}\left(\frac{1}{K}\right)^{\lambda_2} - \delta \le -\frac{\lambda_1\delta}{\lambda_1-\lambda_2}K^{-\lambda_2} \le \min\{Q(t,\tilde{x}) \colon t \in [0,T^1]\}.$$

By the fact that $\tilde{U}_i$ is convex with respect to x, we deduce that $\underline{Q}$ satisfies

$$\begin{cases} -\partial_t \underline{Q} - \mathscr{L}_x \underline{Q} \le -r\delta \le \partial_x \tilde{U}_1(t,\tilde{x}) \le \partial_x \tilde{U}_1(t,x) = -\partial_t Q - \mathscr{L}_x Q, & \text{in } [0,T^1)\times(\tilde{x},+\infty),\\ \underline{Q}(T^1,x) \le \underline{Q}(T^1,K\tilde{x}) = -2\delta \le \partial_x\tilde{U}_2(T^1,\tilde{x}) \le \partial_x \tilde{U}_2(T^1,x) = Q(T^1,x), & \forall\, x \in (\tilde{x},+\infty),\\ \underline{Q}(t,\tilde{x}) \le \min\{Q(t,\tilde{x}) \colon t\in[0,T^1]\} \le Q(t,\tilde{x}), & \forall\, t \in [0,T^1]. \end{cases}$$

Thus, the comparison principle for PDEs implies that $Q \ge \underline{Q}$ in $[0,T^1]\times[\tilde{x},+\infty)$. Letting $x \to +\infty$, we have

$$\liminf_{x\to+\infty} Q(t,x) \ge \lim_{x\to+\infty} \underline{Q}(t,x) = -2\delta = \min\{2\partial_x \tilde{U}_1(t,\tilde{x})/r, \partial_x\tilde{U}_2(t,\tilde{x}) \colon t\in[0,T^1]\}, \quad \forall\, t \in [0,T^1], \quad \tilde{x} > 0.$$

Recalling Lemma 1, and letting $\tilde{x} \to +\infty$, we deduce that

$$\liminf_{x\to+\infty} Q(t,x) \ge 0, \quad \forall\, t\in[0,T^1].$$

Combining this with the fact that $Q \le 0$ in $\tilde{\mathcal{N}}_{T^1}$, we derive that for any $t \in [0,T^1]$,

$$\lim_{x\to+\infty} \partial_x \underline{\tilde{V}}(t,x) = \lim_{x\to+\infty} Q(t,x) = 0.$$

Let us temporarily denote

$$\bar{Q}(t,x) = \begin{cases} \dfrac{\delta}{\lambda_1-\lambda_2}[(\lambda_1-\lambda_2) - (\lambda_1 K^{\lambda_2} - \lambda_2 K^{\lambda_1})], & (t,x)\in[0,T^1]\times(0,\tilde{x}),\\ \dfrac{\delta}{\lambda_1-\lambda_2}\left\{\left[\lambda_1\left(\dfrac{x}{\tilde{x}}\right)^{\lambda_2} - \lambda_2\left(\dfrac{x}{\tilde{x}}\right)^{\lambda_1}\right] - (\lambda_1K^{\lambda_2} - \lambda_2K^{\lambda_1})\right\}, & (t,x)\in[0,T^1]\times[\tilde{x},K\tilde{x}], \end{cases}$$

where, by slight abuse of notation, K is a constant large enough such that $(\lambda_1-\lambda_2)-(\lambda_1K^{\lambda_2}-\lambda_2K^{\lambda_1}) < 0$, $\tilde{x}$ and δ are positive constants that will be determined later, and λ_1 and λ_2 are the same as previously defined.

It is not difficult to check that

$$\bar{Q} \in C([0,T^1]\times(0,K\tilde{x}]) \cap W^{2,1}_{p,\mathrm{loc}}([0,T^1]\times(0,K\tilde{x})), \partial_x\bar{Q}(t,x) > 0, \quad \text{for any } (t,x)\in[0,T^1]\times(\tilde{x},K\tilde{x}],$$

and

$$-\partial_t\bar{Q} - \mathscr{L}_x\bar{Q} = \begin{cases} \dfrac{-r\delta}{\lambda_1-\lambda_2}[(\lambda_1K^{\lambda_2}-\lambda_2K^{\lambda_1}) - (\lambda_1-\lambda_2)], & \text{in } [0,T^1)\times(0,\tilde{x}),\\ \dfrac{-r\delta}{\lambda_1-\lambda_2}(\lambda_1K^{\lambda_2}-\lambda_2K^{\lambda_1}), & \text{in } [0,T^1)\times(\tilde{x},K\tilde{x}]. \end{cases}$$

Fix $\tilde{x}$, then take

$$\delta = \frac{\lambda_1-\lambda_2}{\lambda_1K^{\lambda_2}-\lambda_2K^{\lambda_1}}\min\left\{\frac{-\partial_x\tilde{U}_1(t,K\tilde{x})}{r}, -\partial_x\tilde{U}_2(t,K\tilde{x}) \colon t\in[0,T^1]\right\}.$$

Combining this with the fact that $\tilde{U}_i$ is convex with respect to x, we deduce that $\bar{Q}$ satisfies

$$\begin{cases} -\partial_t\bar{Q} - \mathscr{L}_x\bar{Q} \ge \partial_x\tilde{U}_1(t,K\tilde{x}) \ge \partial_x\tilde{U}_1(t,x) = -\partial_tQ - \mathscr{L}_xQ, & \text{in } [0,T^1)\times(0,K\tilde{x}),\\ \bar{Q}(T^1,x) \ge \bar{Q}(T^1,\tilde{x}) \ge \dfrac{-\delta(\lambda_1K^{\lambda_2}-\lambda_2K^{\lambda_1})}{\lambda_1-\lambda_2} \ge \partial_x\tilde{U}_2(T^1,K\tilde{x}) \ge \partial_x\tilde{U}_2(T^1,x) = Q(T^1,x), & \forall\, x\in(0,K\tilde{x}],\\ \bar{Q}(t,K\tilde{x}) = 0 \ge Q(t,K\tilde{x}), & \forall\, t\in[0,T^1]. \end{cases}$$

Thus, the comparison principle for PDEs implies that $Q \le \bar{Q}$ in $[0,T^1]\times(0,K\tilde{x}]$. Letting $x\to0^+$, we have

$$\limsup_{x\to0^+} Q(t,x) \le \lim_{x\to0^+}\bar{Q}(t,x) = \frac{(\lambda_1-\lambda_2)-(\lambda_1K^{\lambda_2}-\lambda_2K^{\lambda_1})}{\lambda_1K^{\lambda_2}-\lambda_2K^{\lambda_1}}\min\left\{\frac{-\partial_x\tilde{U}_1(t,K\tilde{x})}{r}, -\partial_x\tilde{U}_2(t,K\tilde{x}) \colon t\in[0,T^1]\right\}$$

for any $t\in[0,T^1]$.

Recalling Lemma 1, and letting $\tilde{x}\to0^+$, we deduce that

$$K\tilde{x}\to0^+, \qquad \lim_{x\to0^+}\partial_x\underline{\tilde{V}}(t,x) = \lim_{x\to0^+}Q(t,x) = -\infty, \quad \forall\, t\in[0,T^1]. \quad \square$$

Next, we state and prove the properties of the unique solution $\hat{V}$ of VI (19) in the following theorem.

Theorem 3. *VI (19) has a unique strong solution $\hat{V}$ satisfying the following properties:*

(1) $\hat{V} \in W^{2,1}_{p,\text{loc}}(\mathcal{N}_T) \cap C(\tilde{\mathcal{N}}_T)$ *for any* $p \geq 1$ *and* $\partial_x \hat{V}$, $\partial_t \hat{V} \in C(\tilde{\mathcal{N}}_T)$.

(2) $\partial_x \hat{V} < 0$ *in* $\tilde{\mathcal{N}}_T$. *Moreover,* $\partial_x \hat{V}(t,x) \to -\infty$ *as* $x \to 0^+$, $\partial_x \hat{V}(t,x) \to 0$ *as* $x \to +\infty$ *for any* $t \in [0,T]$. *There exist positive constants* C, K *such that*

$$|\partial_x \hat{V}(t,x)| \leq C(1 + x^{-K}), \quad \forall (t,x) \in \tilde{\mathcal{N}}_T. \tag{36}$$

(3) $\partial_{xx} \hat{V} > 0$ *a.e. in* $\tilde{\mathcal{N}}_T$.

To prove the theorem, we study properties of the following function:

$$P \triangleq \hat{V} - \hat{\underline{V}} = \hat{V} - \underline{\tilde{V}} - xg(t).$$

Recalling that $\underline{\tilde{V}}$ satisfies PDE (13), we transform VI (19) into that of P:

$$\begin{cases} -\partial_t P - \mathcal{L}P = \varrho x - l, & \text{if } P > 0 \text{ and } (t,x) \in \mathcal{N}_T, \\ -\partial_t P - \mathcal{L}P \geq \varrho x - l, & \text{if } P = 0 \text{ and } (t,x) \in \mathcal{N}_T, \\ P(T,x) = 0, & \forall x > 0. \end{cases} \tag{37}$$

In VI (37), the lower obstacle becomes 0; thus we have transformed the problem into a problem where the continuation value after retirement is 0. We will show the existence and uniqueness of a strong solution to VI (37) and properties of the solution in the lemmas below. The following proof of Theorem 3 will be based on the lemmas.

Proof of Theorem 3. (1) Since $\hat{V} = P + \hat{\underline{V}} = P + \underline{\tilde{V}} + g(t)x$ in $\tilde{\mathcal{N}}_T$ and $g(t) \in C^\infty([0,T])$, $\hat{V} \in W^{2,1}_{p,\text{loc}}(\mathcal{N}_T) \cap C(\tilde{\mathcal{N}}_T)$ for any $p \geq 1$. Conclusion 1 in Theorem 2 and the fact $\partial_x P \in C(\tilde{\mathcal{N}}_T)$ (proved in Lemma 3) imply $\partial_x \hat{V} \in C(\tilde{\mathcal{N}}_T)$. Also, $\partial_t \hat{V} \in C(\tilde{\mathcal{N}}_T)$ follows from conclusion 1 in Theorem 2 and the fact $\partial_t P \in C(\tilde{\mathcal{N}}_T)$ (proved in Lemma 4).

(2) The fact that $\partial_x \hat{V} < 0$ in $\tilde{\mathcal{N}}_T$ is deduced from conclusion 3 in Theorem 2 and the fact that $\partial_x P(t,x) \leq -g(t)$ for any $(t,x) \in \tilde{\mathcal{N}}_T$ (proved in Lemma 5). By the fact that $P \equiv 0$ in the domain $[0,T) \times (0, x^\infty]$ (proved in Lemma 5), we deduce that $\partial_x P(t,x) \to 0$ as $x \to 0^+$. Moreover, the fact that $\partial_x P(t,x) \geq 0$ and $\partial_x P(t,x) \leq -g(t)$ for any $(t,x) \in \tilde{\mathcal{N}}_T$ (proved in Lemmas 3 and 5) implies that $\partial_x P$ is bounded in $\tilde{\mathcal{N}}_T$. Hence, conclusion 3 in Theorem 2 and the fact that $\partial_x P(t,x) \to -g(t)$ as $x \to +\infty$ for any $t \in [0,T]$ (proved in Lemma 5) imply the asymptotic properties of $\partial_x \hat{V}$ and (36).

(3) $\partial_{xx} \hat{V} > 0$ comes from conclusion 2 in Theorem 2 and the fact that $\partial_{xx} P \geq 0$ a.e. in $\tilde{\mathcal{N}}_T$ (proved in Lemma 5). □

Lemma 3. *VI (37) has a unique strong solution P satisfying the following properties:*

(1) $P \in W^{2,1}_{p,\text{loc}}(\mathcal{N}_T) \cap C(\tilde{\mathcal{N}}_T)$ *for any* $p \geq 1$ *and* $\partial_x P \in C(\tilde{\mathcal{N}}_T)$.

(2) $\partial_x P \geq 0$ *in* $\tilde{\mathcal{N}}_T$ *and* $\partial_t P \leq 0$ *a.e. in* $\tilde{\mathcal{N}}_T$.

Proof. (1) Repeating the same transformation as in (31), the degenerate parabolic differential operator $\mathcal{L}$ is transformed into a constant coefficient nondegenerate parabolic differential operator, $\tilde{\mathcal{L}}$. Note that the inhomogeneous term $\varrho x - l$, the lower obstacle 0, and the terminal condition 0, are smooth. Thus, it is not difficult to prove that VI (37) has a unique solution $P \in W^{2,1}_{p,\text{loc}}(\mathcal{N}_T) \cap C(\tilde{\mathcal{N}}_T)$ for any $p \geq 1$, and $\partial_x P \in C(\tilde{\mathcal{N}}_T)$ (see, e.g., Friedman [13] or Yan et al. [30]).

(2) Next, we prove that $\partial_x P \geq 0$ in $\tilde{\mathcal{N}}_T$. Let us temporarily denote $\tilde{P}(t,x) = P(t, \delta x)$ with $\delta > 1$; then we deduce that

$$\partial_t \tilde{P}(t,x) = \partial_t P(t,\delta x), \qquad x\partial_x \tilde{P}(t,x) = (\delta x)\partial_x P(t,\delta x), \qquad x^2 \partial_{xx} \tilde{P}(t,x) = (\delta x)^2 \partial_{xx} P(t,\delta x),$$

and

$$(\partial_t \tilde{P} + \mathcal{L}\tilde{P})(t,x) = (\partial_t P + \mathcal{L}P)(t,\delta x).$$

So $\tilde{P}$ satisfies

$$\begin{cases} -\partial_t \tilde{P} - \mathcal{L}\tilde{P} = \varrho \delta x - l, & \text{if } \tilde{P} > 0 \text{ and } (t,x) \in \mathcal{N}_T, \\ -\partial_t \tilde{P} - \mathcal{L}\tilde{P} \geq \varrho \delta x - l, & \text{if } \tilde{P} = 0 \text{ and } (t,x) \in \mathcal{N}_T, \\ \tilde{P}(T,x) = 0, & \forall x > 0. \end{cases}$$

Since $\varrho \delta x - l \geq \varrho x - l$ for any $\delta > 1$, $x > 0$, and the terminal values and the lower obstacles in VIs of $\tilde{P}$ and P are the same, the comparison principle for VIs (see, e.g., Friedman [13] or Yan et al. [30]) implies that $P(t,\delta x) = \tilde{P}(t,x) \geq P(t,x)$ for any $\delta > 1$ and $(x,t) \in \tilde{\mathcal{N}}_T$. Hence, we deduce that $\partial_x P \geq 0$ in $\tilde{\mathcal{N}}_T$.

Let us temporarily denote $\hat{P}(t,x) = P(t-\delta,x)$ with $\delta > 0$ being sufficiently small. Then, it is not difficult to deduce that $\hat{P}$ satisfies

$$\begin{cases} -\partial_t \hat{P} - \mathcal{L}\hat{P} = \varrho x - l, & \text{if } \hat{P} > 0 \text{ and } (t,x) \in \mathcal{N}_{T,\delta} \triangleq [\delta,T) \times (0,+\infty), \\ -\partial_t \hat{P} - \mathcal{L}\hat{P} \geq \varrho x - l, & \text{if } \hat{P} = 0 \text{ and } (t,x) \in \mathcal{N}_{T,\delta}, \\ \hat{P}(T,x) = P(T-\delta,x) \geq 0, & \forall x > 0. \end{cases}$$

Since $\hat{P}(T,x) \geq P(T,x)$ for any $\delta > 0$, $x > 0$, and the lower obstacles and the inhomogeneous terms in VIs of $\hat{P}$ and P are the same, the comparison principle for VIs implies that $P(t-\delta,x) = \hat{P}(t,x) \geq P(t,x)$ for any $\delta > 0$, $(x,t) \in \mathcal{N}_{T,\delta}$. Hence, we deduce that $\partial_t P \leq 0$ a.e. in $\tilde{\mathcal{N}}_T$. □

Note, in particular, that $\partial_t P \leq 0$ in Lemma 3. It is monotonically decreasing in time.

Since $\partial_x P \geq 0$ in $\tilde{\mathcal{N}}_T$, P is increasing with respect to x, and the following free boundary (the optimal retirement boundary) is well defined:

$$R_x(t) \triangleq \inf\{x \geq 0 \colon P(t,x) > 0\}, \quad \forall t \in [0,T).$$

We will show that R_x is increasing in the interval $[0,T)$ in Lemma 6. Thus the left-hand limit of $R_x(t)$ as $t \to T$ exists, and $R_x(T)$ is well defined as the limit,

$$R_x(T) \triangleq \lim_{t \to T^-} R_x(t).$$

Moreover, we can define the working region ($\mathbf{WR}_x$) and the retirement region ($\mathbf{RR}_x$) as follows:

$$\mathbf{WR}_x = \{(t,x) \colon x > R_x(t), t \in [0,T)\}, \qquad \mathbf{RR}_x = \{(t,x) \colon 0 < x \leq R_x(t), t \in [0,T)\}.$$

Note that if the initial $(t,x) \in \mathbf{WR}_x$, then $P(t,x) > 0$ (i.e., $\hat{V}(t,x) > \hat{\underline{V}}(t,x)$). The definition of τ^* in Theorem 1 implies that $\tau^* > t$ a.s. in Ω (i.e., the agent chooses to work). As time passes, $s > t$, before the trajectory of the dual variable process X_s^* first hits the optimal retirement boundary $R_x(s)$, $(s, X_s^*) \in \mathbf{WR}_x$, and $\tau^* > s$ (i.e., the agent keeps working). If X_s^* hits $R_x(s)$, then $(s, X_s^*) \in \mathbf{RR}_x$, $\tau^* = s$, and the agent chooses to retire. If the initial $(t,x) \in \mathbf{RR}_x$, however, then $P(t,x) = 0$ (i.e., $\hat{V}(t,x) = \hat{\underline{V}}(t,x)$). The definition of τ^* in Theorem 1 implies that $\tau^* = t$ a.s. in Ω (i.e., the agent chooses to retire).

Since we have proved that $\partial_t P \leq 0$ a.e. in $\tilde{\mathcal{N}}_T$; the coefficient functions in the parabolic differential operator $\mathcal{L}$; and the terminal value function, the lower obstacle function, and the inhomogeneous term are all smooth, it is not difficult to deduce the following regularity results by using a standard method as explained in Friedman [12]. We omit its proof.

Lemma 4. *The optimal retirement boundary is smooth (i.e., $R_x \in C^\infty[0,T) \cap C[0,T]$). Moreover, the solution $P \equiv 0$ in $\mathbf{RR}_x$, and $P \in C^\infty(\{(t,x) \colon x \geq R_x(t), t \in [0,T]\})$, and $\partial_t P \in C(\tilde{\mathcal{N}}_T)$.*

Next, we state and prove the following lemma about P, which will lead us to the proof of Theorem 3.

Lemma 5. (1) *$P \equiv 0$ in the domain $[0,T) \times (0, x^\infty]$, where x^∞ is defined in (38). And $P(t,x) > 0$ in the domain $[0,T) \times (x^T, +\infty)$ with $x^T \triangleq l/\varrho$. Hence, $x^\infty \leq R_x \leq x^T$ in $[0,T]$.*

(2) *$\partial_x P(t,x) \leq -g(t)$ for any $(t,x) \in \tilde{\mathcal{N}}_T$. Moreover, $\partial_x P(t,x) \to -g(t)$ as $x \to +\infty$ for any $t \in [0,T]$.*

(3) *$\partial_{xx} P \geq 0$ a.e. in $\tilde{\mathcal{N}}_T$.*

Proof. (1) First, we prove that $P \equiv 0$ in the domain $[0,T) \times (0, x^\infty]$. Let us denote

$$P^\infty(t,x) \triangleq \begin{cases} 0, & (t,x) \in [0,T] \times (0, x^\infty), \\ \dfrac{l}{\beta(1-\lambda_4)} \left(\dfrac{x}{x^\infty}\right)^{\lambda_4} + \dfrac{\varrho x}{r} - \dfrac{l}{\beta}, & (t,x) \in [0,T] \times [x^\infty, +\infty), \end{cases}$$

where

$$x^\infty \triangleq \frac{-r l \lambda_4}{\varrho \beta (1-\lambda_4)}, \tag{38}$$

and λ_3 and λ_4 are, respectively, the positive and negative roots of the following algebraic equation:

$$\frac{|\theta|^2}{2}\lambda(\lambda-1) + (\beta - r)\lambda - \beta = 0.$$

It is not difficult to check that

$$P^\infty \in C(\tilde{\mathcal{N}}_T) \cap W^{2,1}_{p,\text{loc}}(\mathcal{N}_T), \qquad \partial_{xx} P^\infty(t,x) > 0, \quad \text{for any } (t,x) \in [0,T] \times (x^\infty, +\infty),$$

and

$$-\partial_t P^\infty - \mathscr{L}P^\infty = \begin{cases} 0, & \text{in } [0,T)\times(0,x^\infty), \\ \varrho x - l, & \text{in } [0,T)\times(x^\infty,+\infty). \end{cases}$$

Since $\lambda_4 < 0$ and

$$\beta(\lambda_4 - 1) - r\lambda_4 = (\beta - r)\lambda_4 - \beta = \frac{-|\theta|^2}{2}\lambda_4(\lambda_4 - 1) < 0,$$

we have $x^\infty < l/\varrho$ and

$$-\partial_t P^\infty - \mathscr{L}P^\infty \geq \varrho x - l, \quad \text{in } \mathcal{N}_T.$$

Recalling $\partial_x P^\infty(t,x^\infty) = P^\infty(t,x^\infty) = 0$ for any $t\in[0,T]$, and $\partial_{xx}P^\infty > 0$ in $[0,T]\times(x^\infty,+\infty)$, we deduce that $P^\infty > 0$ in $[0,T]\times(x^\infty,+\infty)$.

Hence, P^∞ satisfies the following VI:

$$\begin{cases} -\partial_t P^\infty - \mathscr{L}P^\infty = \varrho x - l, & \text{if } P^\infty > 0 \text{ and } (t,x)\in\mathcal{N}_T, \\ -\partial_t P^\infty - \mathscr{L}P^\infty \geq \varrho x - l, & \text{if } P^\infty = 0 \text{ and } (t,x)\in\mathcal{N}_T, \\ P^\infty(T,x) \geq 0, & \forall\, x > 0. \end{cases}$$

Thus, the comparison principle for VIs implies that $P \leq P^\infty$ in $\tilde{\mathcal{N}}_T$. In particular,

$$0 \leq P \leq P^\infty \leq 0, \quad \text{in } [0,T]\times(0,x^\infty).$$

Hence, the definition of the optimal retirement boundary R_x implies that $R_x \geq x^\infty$ in $[0,T]$.

Next, we prove that $P > 0$ in the domain $[0,T)\times(x^T,+\infty)$. In fact, VI (37) implies that in the domain $\mathbf{RR}_x$,

$$P = 0, \quad 0 = -\partial_t P - \mathscr{L}P \geq \varrho x - l.$$

So we deduce that $\mathbf{RR}_x \subset [0,T)\times(0,x^T]$. Hence, it is clear that $\{(t,x)\colon P(t,x) > 0\} = \mathbf{WR}_x \supset [0,T)\times(x^T,+\infty)$. The definition of the optimal retirement boundary R_x implies that $R_x \leq x^T$ in $[0,T]$.

(2) Next, we consider the property of $\partial_x P$. It is clear that P satisfies

$$\begin{cases} -\partial_t P - \mathscr{L}P = \varrho x - l, & \text{in } \mathbf{WR}_x, \\ P(T,x) = 0, \quad \forall\, x \geq R_x(T), & P(t,R_x(t)) = 0, \quad \forall\, t\in[0,T]. \end{cases} \tag{39}$$

Temporarily denoting $Q = \partial_x P + g(t)$, we have

$$\begin{cases} -\partial_t Q - \mathscr{L}_x Q = 0, & d\text{in } \mathbf{WR}_x, \\ Q(T,x) = 0, \quad \forall\, x \geq R_x(T), & Q(t,R_x(t)) = g(t) \leq 0, \quad \forall\, t\in[0,T], \end{cases} \tag{40}$$

where we have used the fact $\partial_x P$ is continuous on $[0,T]\times(0,+\infty)$, and, in particular, it continuously crosses the optimal retirement boundary $x = R_x(t)$.

Applying the comparison principle for PDEs, we deduce that $Q \leq 0$ and $\partial_x P \leq -g(t)$ in $\mathbf{WR}_x$. Moreover, it is clear that $\partial_x P = 0 \leq -g(t)$ in $\mathbf{RR}_x$. Hence, we conclude that $\partial_x P \leq -g(t)$ in $\mathcal{N}_T$. In addition, $\partial_x P(T,x) = 0 = -g(T)$ follows from the terminal condition of VI (37). We thus have proved that $\partial_x P \leq -g(t)$ in $\tilde{\mathcal{N}}_T$.

Let us temporarily denote

$$\underset{\sim}{Q}(t,x) = g(0)\left(\frac{x}{x^T}\right)^{\lambda_2}, \quad \forall\, x > 0,$$

where λ_2 is the negative root of (35).

Then we can check that

$$\begin{cases} -\partial_t \underset{\sim}{Q} - \mathscr{L}_x \underset{\sim}{Q} = 0 = -\partial_t Q - \mathscr{L}_x Q, & \text{in } \mathbf{WR}_x, \\ \underset{\sim}{Q}(T,x) \leq 0 = Q(T,x), & \forall\, x \geq R_x(T), \\ \underset{\sim}{Q}(t,R_x(t)) = g(0)\left(\dfrac{R_x(t)}{x^T}\right)^{\lambda_2} \leq g(0) \leq g(t) = Q(t,R_x(t)), & \forall\, t\in[0,T], \end{cases}$$

where we have used the fact that $R_x \leq x^T$. Thus, the comparison principle for PDEs implies that $Q \geq \underset{\sim}{Q}$ in $\mathbf{WR}_x$. Combining this with the fact that $Q \leq 0$ in $\mathbf{WR}_x$, we conclude that $\underset{\sim}{Q} \leq Q \leq 0$ and

$$Q(t,x) \to 0, \quad \partial_x P(t,x) = Q(t,x) - g(t) \to -g(t), \quad \text{as } x \to +\infty, \;\; \forall\, t\in[0,T].$$

(3) Differentiating PDE (39) twice with respect to x, we deduce that $\partial_{xx}P$ satisfies

$$\begin{cases} -\partial_t\partial_{xx}P - \mathscr{L}_{xx}\partial_{xx}P = 0 & \text{in } \mathbf{WR}_x, \\ \partial_{xx}P(T,x) = 0, & \forall\, x \geq R_x(T). \end{cases}$$

Moreover, we have P, $\partial_x P$, and $\partial_t P \in C(\tilde{\mathcal{N}}_T)$ from Lemmas 3 and 4. So on the optimal retirement boundary $x = R_x(t)$, $t \in [0,T)$, we have $P(t,x) = \partial_x P(t,x) = \partial_t P(t,x) = 0$ and

$$\partial_{xx}P(t,x) = \frac{2}{|\theta|^2 x^2}(\partial_t P + \mathscr{L}P)(t,x) = \frac{2(l-\varrho x)}{|\theta|^2 x^2} \geq \frac{2(l-\varrho x^T)}{|\theta|^2 x^2} = 0.$$

Applying the comparison principle for PDEs, we deduce that $\partial_{xx}P \geq 0$ in $\mathbf{WR}_x$. Combining this with the fact that $P = 0$ in $\mathbf{RR}_x$, we conclude that $\partial_{xx}P \geq 0$ a.e. in $\tilde{\mathcal{N}}_T$. □

Remark 10. By Theorem 2 and 3, all the assumptions in Theorem 1 are satisfied by $\hat{V}$ and $\underline{\hat{V}}$. Hence, the agent's optimal strategy and value function are given as in Theorem 1. It turns out that the optimal consumption is a *continuous* stochastic process, since $\partial_x \hat{V}$ is continuous. This implies, in particular, that the agent's optimal consumption does not jump at the time of retirement.

Next, we finish the section with a lemma about the optimal retirement boundary $R_x(t)$.

Lemma 6. *The optimal retirement boundary, $x = R_x(t)$, $t \in [0,T]$, is strictly increasing with the terminal point $R_x(T) \triangleq \lim_{t\to T^-} R_x(t) = x^T$ (see Figure 1). In addition, $x^\infty < R_x(t) < x^T$ for any $t \in [0,T)$, where x^T and x^∞ are defined in Lemma 5.*

Proof. By Lemma 3, we have

$$\partial_x P \geq 0, \quad \partial_t P \leq 0 \quad \text{a.e. in } \mathcal{N}_T, \quad P \in C([0,T]\times(0,+\infty)).$$

So we deduce that P is decreasing with respect to t and increasing with respect to x.

For any fixed $t_0 \in [0,T)$ and any $x \in [0, R_x(t_0)]$, $t \in (t_0, T]$, we deduce that

$$0 \leq P(t,x) \leq P(t, R_x(t_0)) \leq P(t_0, R_x(t_0)) = 0,$$

where we have used that $P = 0$ on the optimal retirement boundary $x = R_x(t)$. Hence, the definition of the optimal retirement boundary implies that $R_x(t) \geq R_x(t_0)$ for any $0 \leq t_0 \leq t \leq T$. Hence, R_x is increasing on $[0,T]$.

Since R_x is increasing, the limit, $R_x(T)$, of $R_x(t)$ as $t \to T^-$ exists. Next, we prove that the limit is equal to x^T. Recalling property 1 in Lemma 5, we know that $R_x \leq x^T$ in $[0,T]$. So it is sufficient to prove that $R_x(T) \geq x^T$. Otherwise, $R_x(T) < x^T$, and $[0,T)\times(R_x(T), x^T) \subset \mathbf{WR}_x$ (see Figure 2). Recalling (39), we can show the following by computation:

$$\partial_t P(T,x) = -\mathscr{L}P(T,x) - \varrho x + l = -\varrho x + l > 0, \quad \forall\, x \in (R_x(T), x^T).$$

On the other hand, we know that $\partial_t P \leq 0$ a.e. in $\mathcal{N}_T$ from Lemma 3, and $P \in C^\infty(\{(t,x)\colon x \geq R_x(t), t \in [0,T]\})$ from Lemma 4. So we have $\partial_t P(T,x) \leq 0$ for any $x \in (R_x(T), x^T)$. Hence, we obtain a contradiction and thus have proved that $R_x(T) \geq x^T$, and $R_x(T) = x^T$.

Figure 1. The Optimal Retirement Boundary $x = R_x(t)$

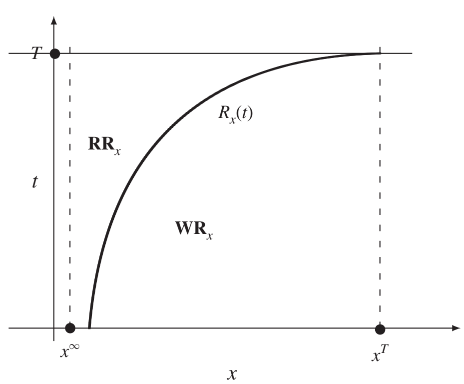
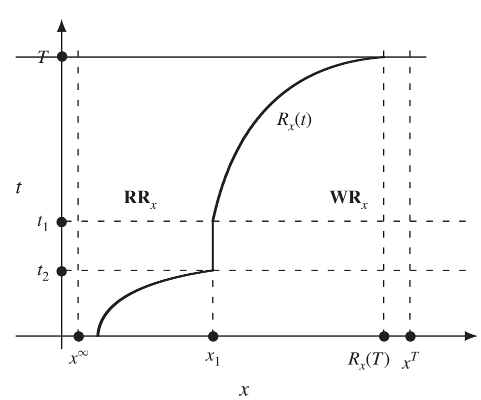


Figure 2. Nonstrictly-Increasing Optimal Retirement Boundary



Next, we prove that R_x is strictly increasing on $[0,T]$. Otherwise, there exist constants x_1, t_1, and t_2 such that $x_1 \in [x^\infty, x^T]$, $0 \le t_2 < t_1 \le T$, and $R_x(t) = x_1$ for any $t \in [t_2, t_1]$ (see Figure 2). It is clear that $P(t,x) = 0$ for any $(t,x) \in [t_2,t_1] \times (0, x_1]$. Since $\partial_x P$ continuously crosses the optimal retirement boundary, $\partial_x P(t, x_1) = 0$ for any $t \in [t_2, t_1]$. We then deduce that

$$\partial_t P(t,x_1) = 0, \qquad \partial_t(\partial_x P)(t,x_1) = 0, \quad \forall\, t \in [t_2,t_1]. \tag{41}$$

On the other hand, in the domain $[t_2,t_1) \times (x_1,+\infty)$, P and $\partial_t P$ respectively satisfy

$$-\partial_t P - \mathscr{L}P = \varrho x - l \text{ in } [t_2,t_1)\times(x_1,+\infty), \qquad P(t,x_1) = 0, \quad \forall\, t \in (t_2,t_1),$$

$$\begin{cases} -\partial_t\partial_t P - \mathscr{L}\partial_t P = 0,\ \partial_t P \le 0, & \text{in } [t_2,t_1)\times(x_1,+\infty),\\ \partial_t P(t,x_1) = 0, & \forall\, t\in(t_2,t_1). \end{cases}$$

By applying the Hopf lemma (see Evans [8]), we deduce $\partial_x(\partial_t P)(t,x_1) < 0$, which contradicts the second equality in (41).

Since $x^\infty \le R_x \le x^T = R_x(T)$, and R_x is strictly increasing, we conclude that $x^\infty < R_x(t) < x^T$ for any $t \in [0,T)$. □

Remark 11. The proof of Lemma 6 shows that the optimal retirement boundary $x = R_x(t)$ is independent of the utility function after retirement. In other words, if we look at the agent's problem through his or her marginal utility of wealth, then the optimal retirement boundary, expressed in marginal utility, does not depend on the agent's utility function after retirement. The utility function after retirement influences the agent's choice only by affecting his or her marginal utility of wealth at initial time, not through other channels.

5. Optimal Retirement Threshold

In the previous section, we studied the properties of the convex dual function $\hat{V}$ as a solution to PDE/VI. We used (t,x), where t denotes time and x denotes marginal utility, as the coordinate system for the study. In this section we will come back to study the value function V in the original coordinate system (t,w), where w denotes the wealth of the agent. For this purpose, we redefine the working domain and the retirement domain, and we define the optimal retirement threshold in the (t,w)-coordinate system.

Recalling Theorem 1, we know that $x^*(t,w) = \mathcal{J}_{\hat{V}}(t, -w + g(t))$ is continuous and strictly decreasing with respect to w in $\tilde{\mathcal{M}}_{T^1}$, and it is a bijection from $(g(t),+\infty)$ to $(0,+\infty)$ for any $t \in [0,T^1]$. So for any $t \in [0,T^1]$, $x^*(t,\cdot)$ has an inverse function $w^*(t,\cdot)$, which is continuous, is strictly decreasing, and maps $(0,+\infty)$ to $(g(t),+\infty)$.

Let us define

$$\begin{aligned} &R_w(t) \triangleq w^*(t, R_x(t)), \quad \forall\, t\in[0,T], \\ &\mathbf{RR}_w \triangleq \{(t,w)\in\mathcal{M}_T\colon (t,x^*(t,w))\in\mathbf{RR}_x\}, \qquad \mathbf{WR}_w \triangleq \{(t,w)\in\mathcal{M}_T\colon (t,x^*(t,w))\in\mathbf{WR}_x\}. \end{aligned} \tag{42}$$

Then, $\mathbf{RR}_w$, $\mathbf{WR}_w$, and $w = R_w(t)$ represent the retirement region, the working region, and the optimal retirement threshold in the (t,w)-coordinate system, respectively. Moreover,

$$\mathbf{WR}_w = \{(t,w)\colon g(t) < w < R_w(t), t\in[0,T)\}, \qquad \mathbf{RR}_w = \{(t,w)\colon w \ge R_w(t), t\in[0,T)\}.$$

If the trajectory of the wealth process W_s^* stays in $\mathbf{WR}_w$, then the trajectory of the dual process X_s^* stays in $\mathbf{WR}_x$ and the agent chooses to work. If, however, the trajectory of the wealth process W_s^* reaches $\mathbf{RR}_w$, then the trajectory of the dual variable process X_s^* reaches $\mathbf{RR}_x$ and the agent chooses to retire. The optimal retirement threshold is such that the agent retires as soon as the wealth process reaches the threshold.

We now state and prove properties of the value function in the following theorem.

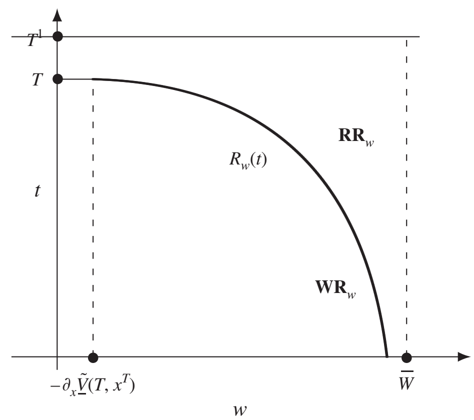
Theorem 4. *The value function V satisfies the following properties*:

(1) *$V \in W^{2,1}_{p,\mathrm{loc}}(\mathcal{M}_{T^1})$ for any $p \ge 1$, and V, $\partial_w V \in C(\tilde{\mathcal{M}}_{T^1})$, $\partial_t V \in C(\tilde{\mathcal{M}}_T)$. Moreover, V is piecewise smooth—that is, $V \in C^\infty(\{(t,w)\colon g(t) < w \le R_w(t), t\in[0,T)\}) \cap C^\infty(\mathbf{RR}_w) \cap C^\infty([T,T^1]\times(0,+\infty))$, where $\mathcal{M}_{T^1}$, $\tilde{\mathcal{M}}_{T^1}$, and $\tilde{\mathcal{M}}_T$ are defined in* (8).

(2) *$\partial_w V > 0$ in $\tilde{\mathcal{M}}_{T^1}$. Moreover, $\partial_w V(t,w) \to +\infty$ as $w \to g(t)^+$, and $\partial_w V(t,w) \to 0^+$ as $w\to+\infty$ for any $t\in[0,T^1]$.*

(3) *$\partial_{ww} V < 0$ a.e. in $\tilde{\mathcal{M}}_{T^1}$.*

Figure 3. The Optimal Retirement Threshold $w = R_w(t)$



Proof. (1) Recalling conclusion 3 in Theorem 3, conclusion 2 in Theorem 2, and $\hat{V} = \tilde{\underline{V}}$ in $[T, T^1] \times (0, +\infty)$, we claim that $\partial_{xx}\hat{V} > 0$ a.e. in $\tilde{\mathcal{N}}_{T^1}$. By conclusion 1 in Theorem 3, we conclude that $\hat{V} \in W^{2,1}_{p,\text{loc}}(\mathcal{N}_T)$ for any $p \geq 1$, and $\hat{V}$, $\partial_x\hat{V} \in C(\tilde{\mathcal{N}}_T)$, $\partial_t\hat{V} \in C(\tilde{\mathcal{N}}_T)$.

Recalling $\tilde{\underline{V}} \in C^\infty(\tilde{\mathcal{N}}_{T^1})$ (by conclusion 1 in Theorem 2) and the fact that

$$\hat{V} = \tilde{\underline{V}} + P + xg(t) \text{ in } \tilde{\mathcal{N}}_T, \qquad \hat{V} = \tilde{\underline{V}} \text{ in } [T, T^1] \times (0, +\infty), \qquad P(T, x) = g(T) = 0,$$

we deduce that $\hat{V}$, $\partial_x\hat{V} \in C(\tilde{\mathcal{N}}_{T^1})$. Moreover, combining this fact with Lemma 4, we have that $\hat{V} \in C^\infty(\{(t, x): x \geq R_x(t), t \in [0, T)\}) \cap C^\infty(\mathbf{RR}_x) \cap C^\infty([T, T^1] \times (0, +\infty))$.

Combining the above regularity properties with (21) allows us to compute the following partial derivatives:

$$\begin{aligned}
\partial_t V(t, w) &= \partial_t\hat{V}(t, x^*(t, w)) + [\partial_x\hat{V}(t, x^*(t, w)) - g(t) + w]\partial_t x^*(t, w) - x^*(t, w)g'(t)\\
&= \partial_t\hat{V}(t, x^*(t, w)) - x^*(t, w)g'(t),\\
\partial_w V(t, w) &= x^*(t, w), \qquad (43)\\
\partial_{ww} V(t, w) &= \partial_w x^*(t, w) = \frac{-1}{\partial_{xx}\hat{V}(t, x^*(t, w))}, \ldots, \qquad (44)
\end{aligned}$$

where we have used the fact that $\partial_x\hat{V}(t, x^*(t, w)) = g(t) - w$. Hence, the results about the regularity of V are obvious.

(2) Since $\partial_w V(t, w) = x^*(t, w) = \mathcal{J}_{\hat{V}}(t, g(t) - w)$ and $g(t) \equiv 0$ for any $t \in [T, T^1]$, Theorem 1 implies all the results in conclusion 2.

(3) Recalling (44) and $\partial_{xx}\hat{V} > 0$ a.e. in $\tilde{\mathcal{N}}_{T^1}$, we deduce that $\partial_{ww}V < 0$ a.e. in $\tilde{\mathcal{M}}_{T^1}$. □

The optimal retirement threshold in the primal (t, w)-coordinate system is illustrated in Figure 3.

We now state and prove properties of the optimal retirement threshold in the (t, w)-coordinate system.

Theorem 5. (1) *The optimal retirement threshold is given by* $w = R_w(t) = -\partial_x\tilde{\underline{V}}(t, R_x(t))$, *and* $R_w \in C^\infty([0, T)) \cap C([0, T])$ *with the terminal point* $R_w(T) \triangleq \lim_{t\to T^-} R_w(t) = -\partial_x\tilde{\underline{V}}(T, x^T)$. *And* $\underline{W} < R_w < \bar{W}$ *in* $[0, T)$, *where*

$$\underline{W} = -\max\{\partial_x\tilde{\underline{V}}(t, x^T): t \in [0, T]\}, \qquad \bar{W} = -\min\{\partial_x\tilde{\underline{V}}(t, x^\infty): t \in [0, T]\}, \tag{45}$$

and x^T *and* x^∞ *are defined in Lemma 5. Moreover,* R_w *is strictly decreasing near the terminal time* T *with* $\lim_{t\to T^-} R'_w(t) = -\infty$.

(2) *The optimal consumption* $c^*_t(t, w) = \mathcal{J}_{U_1}(t, x^*(t, w))$ *for any* $(t, w) \in \tilde{\mathcal{M}}_{T^1}$, *where* $x^*(t, w)$ *is defined in Theorem 1. The optimal consumption function* $c^*_t \in C(\tilde{\mathcal{M}}_{T^1})$ *and is strictly increasing with respect to* w. *Moreover, for any* $t \in [0, T^1)$, $c^*_t(t, w) \to 0^+$ *as* $w \to g(t)^+$, *and* $c^*_t(t, w) \to +\infty$ *as* $w \to +\infty$.

(3) *The optimal investment is*

$$\pi^*_t(t, w) = x^*(t, w)\partial_{xx}\hat{V}(t, x^*(t, w))(\Sigma^{\mathrm{T}})^{-1}\theta, \qquad \text{and} \qquad \theta^{\mathrm{T}}\Sigma^{\mathrm{T}}\pi^*_t > 0 \text{ in } \tilde{\mathcal{M}}_{T^1}.$$

Proof. (1) Recalling (42), we know that $R_w(t) = w^*(t, R_x(t))$, where $w^*(t,\cdot)$ is the inverse function of $x^*(t,\cdot)$.

Since $x^*(t,w) = \mathcal{J}_{\hat{V}}(t, -w + g(t))$, we have $\partial_x \hat{V}(t, x^*(t,w)) = -w + g(t)$. Taking $w = w^*(t,x)$, we can show by computation

$$\partial_x \hat{V}(t,x) = \partial_x \hat{V}(t, x^*(t, w^*(t,x))) = -w^*(t,x) + g(t), \quad \forall (t,x) \in \tilde{\mathcal{N}}_{T^1}.$$

Hence, we deduce that $w^*(t,x) = g(t) - \partial_x \hat{V}(t,x)$ for any $(t,x) \in \tilde{\mathcal{N}}_{T^1}$. So we have

$$\begin{aligned} R_w(t) &= w^*(t, R_x(t)) = g(t) - \partial_x \hat{V}(t, R_x(t)) = g(t) - \partial_x (P + \hat{\underline{V}})(t, R_x(t)) \\ &= g(t) - \partial_x [P + \tilde{\underline{V}} + g(t)x]_{x=R_x(t)} = -\partial_x \tilde{\underline{V}}(t, R_x(t)), \quad \forall t \in [0,T], \end{aligned}$$

where we have used the fact that $\partial_x P(t, R_x(t)) = 0$ for any $t \in [0,T]$.

Since $R_x \in C^\infty([0,T)) \cap C([0,T])$ and $\tilde{\underline{V}} \in C^\infty(\tilde{\mathcal{N}}_{T^1})$, we deduce that $R_w \in C^\infty([0,T)) \cap C([0,T])$, too.

It is clear that

$$R_w(T) = -\partial_x \tilde{\underline{V}}(T, R_x(T)) = -\partial_x \tilde{\underline{V}}(T, x^T).$$

From Lemma 6, we have that $x^T < R_x(t) < x^\infty$ for any $t \in [0,T)$. Combining this with the fact that $\partial_x \tilde{\underline{V}}(t,x)$ is strictly increasing with respect to x, we deduce that

$$-\partial_x \tilde{\underline{V}}(t, x^T) < R_w(t) < -\partial_x \tilde{\underline{V}}(t, x^\infty), \quad \forall t \in [0,T).$$

And (45) is obvious.

Next, we prove that $R_w(t)$ is strictly decreasing near the terminal time T with $\lim_{t\to T^-} R'_w(t) = -\infty$. Differentiating (39) with respect to x, we can show that

$$\partial_{tx} P = -\mathcal{L}_x(\partial_x P) - \varrho \text{ in } \{(t,x) \colon x \geq R_x(t), t \in [0,T]\}.$$

Recalling (33), we have

$$-\partial_{tx} \tilde{\underline{V}} = \partial_x \tilde{U}_1 + \mathcal{L}_x \partial_x \tilde{\underline{V}} \text{ in } \{(t,x) \colon x \geq R_x(t), t \in [0,T]\}.$$

Since $\partial_x P$ continuously crosses the optimal retirement boundary $x = R_x(t)$ and $P = 0$ in $\mathbf{RR}_x$, we have $\partial_x P(t, R_x(t)) = 0$ for any $t \in [0,T]$. So we have

$$\partial_{tx} P(t, R_x(t)) + \partial_{xx} P(t, R_x(t)) R'_x(t) = 0 \quad \Rightarrow \quad R'_x(t) = -\frac{\partial_{tx} P(t, R_x(t))}{\partial_{xx} P(t, R_x(t))} = \frac{\mathcal{L}_x(\partial_x P)(t, R_x(t)) + \varrho}{\partial_{xx} P(t, R_x(t))}.$$

Then, it is clear that for any $t \in [0,T)$,

$$\begin{aligned} R'_w(t) &= -\partial_{tx} \tilde{\underline{V}}(t, R_x(t)) - \partial_{xx} \tilde{\underline{V}}(t, R_x(t)), R'_x(t) \\ &= \partial_x \tilde{U}_1(t, R_x(t)) + \mathcal{L}_x(\partial_x \tilde{\underline{V}})(t, R_x(t)) - \partial_{xx} \tilde{\underline{V}}(t, R_x(t)) \frac{\mathcal{L}_x(\partial_x P)(t, R_x(t)) + \varrho}{\partial_{xx} P(t, R_x(t))}. \end{aligned}$$

Letting $t \to T^-$, we have

$$R'_w(t) \to \partial_x \tilde{U}_1(T, R_x(T)) + \mathcal{L}_x(\partial_x \tilde{\underline{V}})(T, R_x(T)) - \partial_{xx} \tilde{\underline{V}}(T, R_x(T)) \frac{\mathcal{L}_x 0 + \varrho}{0^+} = -\infty,$$

where we have used the terminal conditions of P, $\partial_x \tilde{U}_1 < 0$, $\partial_{xx} \tilde{\underline{V}} > 0$, and $\partial_{xx} P \geq 0$ in $\tilde{\mathcal{N}}_{T^1}$ and $\tilde{\underline{V}} \in C^\infty(\tilde{\mathcal{N}}_{T^1})$. Thus, $R'_w(t)$ is negative for every t sufficiently close to terminal time T, and the optimal retirement threshold R_w is strictly decreasing near the terminal time.

(2) Recalling Theorem 1, we know that optimal consumption is given by $c^* = \mathcal{J}_{U_1}(\cdot, X^*_\cdot) \in \mathcal{S}^p_t$. So we have $c^*_t = \mathcal{J}_{U_1}(t, x^*)$, and thus, $c^*_t(t,w) = \mathcal{J}_{U_1}(t, x^*(t,w))$ in the (t,w)-coordinate system.

Since $x^*(t,w) \in C(\tilde{\mathcal{M}}_{T^1})$ and $\mathcal{J}_{U_1}(t,x) \in C(\tilde{\mathcal{N}}_{T^1})$, we know that $c^*_t(t,w) \in C(\tilde{\mathcal{M}}_{T^1})$.

Note that $x^*(t,w)$ is strictly decreasing with respect to w, and $\mathcal{J}_{U_1}(t,x)$ is strictly decreasing with respect to x. Then, we can deduce that $c^*_t(t,w)$ is strictly increasing with respect to w.

Since $x^*(t,w) \to +\infty$ as $w \to g(t)^+$, and $\mathcal{J}_{U_1}(t,x) \to 0^+$ as $x \to +\infty$, $c^*_t(t,w) \to 0^+$ as $w \to g(t)^+$.

Since $x^*(t,w) \to 0^+$ as $w \to +\infty$, and $\mathcal{J}_{U_1}(t,x) \to +\infty$ as $x \to 0^+$, we deduce that $c^*_t(t,w) \to +\infty$ as $w \to +\infty$.

(3) It is clear that the value function V satisfies the following HJB equation (see, e.g., Yong and Zhou [32]):

$$\sup_{\pi\in\mathbb{R},\,c>0}\Big\{\partial_t V+\tfrac{1}{2}\pi^{\mathrm{T}}\Sigma\Sigma^{\mathrm{T}}\pi\partial_{ww}V+[\pi^{\mathrm{T}}\Sigma\theta+rw-c+\varrho I_{\{w<R_w(t)\}}]\partial_w V-\beta V+U_1(t,c)-l\,I_{\{w<R_w(t)\}}\Big\}=0.$$

So the optimal investment is given by

$$\pi^*_t(t,w)=\frac{-\partial_w V(t,w)}{\partial_{ww}V(t,w)}(\Sigma^{\mathrm{T}})^{-1}\theta=x^*(t,w)\partial_{xx}\hat{V}(t,x^*(t,w))(\Sigma^{\mathrm{T}})^{-1}\theta \text{ in } \tilde{\mathcal{M}}_{T^1},$$

where we have used (43) and (44) in the second equality.

Finally, a straightforward compuation shows that

$$\theta^{\mathrm{T}}\Sigma^{\mathrm{T}}\pi^*_t(t,w)=x^*(t,w)\partial_{xx}\hat{V}(t,x^*(t,w))|\theta|^2>0,\quad \forall\,(t,w)\in\tilde{\mathcal{M}}_{T^1}.\quad\square$$

Remark 12. It is possible to prove that $R_w(\cdot)$ is decreasing with respect to t in $[0,T]$ under a restrictive assumption—for example, the assumption that $\partial_{tx}\tilde{U}_1\le 0$ in $\tilde{\mathcal{N}}_{T^1}$ and $\partial_x\tilde{U}_1(T,x)+\mathcal{L}_x\partial_x\tilde{U}_2(T,x)\le 0$ for any $x>0$. Such an assumption, however, is rather technical, and hence, it is an interesting open question left for future research to find an essential condition to ensure that $R_w(\cdot)$ is monotonic.

6. Comparative Static Analysis

In this section we conduct comparative static analysis with respect to important model parameters. First, we analyze the effect of the rate, ϱ, of labor income on the agent's optimal strategy.

Theorem 6. *The value function V, the optimal retirement threshold R_w, and the optimal consumption c^*_t are increasing with respect to ϱ.*

Proof. Suppose $\varrho_1>\varrho_2$. Denote $V,\hat{V},\dots$ for the case $\varrho=\varrho_i$ by $V_i,\hat{V}_i,\dots,\ i=1,2$.

From VI (19) and (16), we know that $\hat{V}-xg(t)$ satisfies the following VI:

$$\begin{cases}-\partial_t(\hat{V}-xg(t))-\mathcal{L}(\hat{V}-xg(t))=\tilde{U}_1-l+\varrho x, & \text{if } \hat{V}-xg(t)>\underline{\tilde{V}} \text{ and } (t,x)\in\mathcal{N}_T,\\ -\partial_t(\hat{V}-xg(t))-\mathcal{L}(\hat{V}-xg(t))\ge\tilde{U}_1-l+\varrho x, & \text{if } \hat{V}-xg(t)=\underline{\tilde{V}} \text{ and } (t,x)\in\mathcal{N}_T,\\ \hat{V}(T,x)-xg(T)=\underline{\tilde{V}}(T,x), & \forall\,x\in(0,+\infty),\end{cases}\tag{46}$$

where $\tilde{U}_1$ is independent of ϱ, and $\underline{\tilde{V}}$ satisfies PDE (13), which is independent of ϱ. So we deduce that $\underline{\tilde{V}}_1=\underline{\tilde{V}}_2$ in $\tilde{\mathcal{N}}_{T^1}$. Since $\tilde{U}_1-l+\varrho_1x>\tilde{U}_1-l+\varrho_2x$, the comparison principle for VIs implies that $\hat{V}_1-xg_1(t)\ge\hat{V}_2-xg_2(t)$ in $\tilde{\mathcal{N}}_T$. Recalling the extension of $\hat{V}$ to $(T,T^1]\times(0,+\infty)$ in Remark 9, we deduce that

$$\hat{V}_1(t,x)-xg_1(t)=\hat{V}_1(t,x)=\underline{\tilde{V}}_1(t,x)=\underline{\tilde{V}}_2(t,x)=\hat{V}_2(t,x)=\hat{V}_2(t,x)-xg_2(t),\quad\forall\,(t,x)\in(T,T^1]\times(0,+\infty).$$

Hence, $\hat{V}_1-xg_1(t)\ge\hat{V}_2-xg_2(t)$ in $\tilde{\mathcal{N}}_{T^1}$.

Since $V_i(t,w)=\inf_{x>0}[\hat{V}_i(t,x)-x(g_i(t)-w)]$ in $\tilde{\mathcal{M}}_{T^1,i}$, $i=1,2$, we have $V_1\ge V_2$ in $\tilde{\mathcal{M}}_{T^1,2}=\tilde{\mathcal{M}}_{T^1,1}\cap\tilde{\mathcal{M}}_{T^1,2}$.

Consider VI (37). Since $\varrho_1x-l>\varrho_2x-l$, the comparison principle for VIs implies that $P_1\ge P_2$, and $\{P_1>0\}\supset\{P_2>0\}$. Hence, the definition of the optimal retirement boundary implies that $R_{x,1}\le R_{x,2}$.

Consider PDE (33), which is independent of ϱ. So we deduce that $-\partial_x\underline{\tilde{V}}_1=-\partial_x\underline{\tilde{V}}_2$ in $\tilde{\mathcal{N}}_{T^1}$. Since $-\partial_x\underline{\tilde{V}}_1(t,x)$ is decreasing with respect to x, we derive that $R_{w,1}\ge R_{w,2}$ by Theorem 5.

Recalling (39), we have

$$\begin{cases}-\partial_t\partial_xP_1-\mathcal{L}_x\partial_xP_1=\varrho_1, & \text{in } \mathbf{WR}_{x,2},\\ \partial_xP_1(T,x)=0,\quad\forall\,x\ge R_{x,2}(T), & \partial_xP_1(t,R_{x,2}(t))\ge 0,\quad\forall\,t\in[0,T],\end{cases}$$

and

$$\begin{cases}-\partial_t\partial_xP_2-\mathcal{L}_x\partial_xP_2=\varrho_2, & \text{in } \mathbf{WR}_{x,2},\\ \partial_xP_2(T,x)=0,\quad\forall\,x\ge R_{x,2}(T), & \partial_xP_2(t,R_{x,2}(t))=0,\quad\forall\,t\in[0,T],\end{cases}$$

where we have used the fact that $\partial_xP\ge 0$, $R_{x,1}\le R_{x,2}$ and $\mathbf{WR}_{x,1}\supset\mathbf{WR}_{x,2}$. So the comparison principle for VIs implies that $\partial_xP_1\ge\partial_xP_2$ in $\tilde{\mathcal{N}}_T$.

Combining this with the fact that $-\partial_x\underline{\tilde{V}}_1=-\partial_x\underline{\tilde{V}}_2$ in $\tilde{\mathcal{N}}_{T^1}$ and $\hat{V}_i=P_i+g_i(t)x+\underline{\tilde{V}}_i$ in $\tilde{\mathcal{N}}_T$, $i=1,2$, we know that

$$g_1(t)-\partial_x\hat{V}_1=-\partial_xP_1-\underline{\tilde{V}}_1\le-\partial_xP_2-\underline{\tilde{V}}_2=g_2(t)-\partial_x\hat{V}_2 \text{ in } \tilde{\mathcal{N}}_T.$$

Moreover, recalling Remark 9 and the definition of $g(t)$, we deduce that $g_1(t) - \partial_x \hat{V}_1 \le g_2(t) - \partial_x \hat{V}_2$ in $\tilde{\mathcal{N}}_{T^1}$. From $x_i^*(t,w) = \mathcal{I}_{\hat{V}_i}(-w + g_i(t)), i = 1,2$, we know by straightforward computation that

$$g_1(t) - \partial_x \hat{V}_1(t, x_1^*(t,w)) = w = g_2(t) - \partial_x \hat{V}_2(t, x_2^*(t,w)) \ge g_1(t) - \partial_x \hat{V}_1(t, x_2^*(t,w)).$$

Since $-\partial_x \hat{V}_i(t,x), i = 1,2$ are decreasing with respect to x, $x_1^*(t,w) \le x_2^*(t,w)$ in $\tilde{\mathcal{M}}_{T^1,2}$. Combining this with the fact that $\mathcal{I}_{U_1}(t,x)$ is decreasing with respect to x, we conclude that $c_{t,1}^*(t,w) \ge c_{t,2}^*(t,w)$ in $\tilde{\mathcal{M}}_{T^1,2}$. □

Remark 13. Theorem 6 says that as the wage increases, the optimal retirement threshold tends to go higher and optimal consumption tends to increase. Thus, the agent consumes more and retires later if the wage increases. This is an intuitive result and extends a similar result by Choi and Shim [3] originally derived in an infinite horizon model.

In the next theorem we show the effect of the utility cost of labor, l, on the agent's optimal strategy.

Theorem 7. *The value function V, the optimal retirement threshold R_w, and the optimal consumption c_t^* are decreasing with respect to l.*

Proof. Suppose $l_1 > l_2$. Denote $V, \hat{V}, \ldots$ for the case $l = l_i$ by $V_i, \hat{V}_i, \ldots$, $i = 1,2$.

Recalling VI (46) and $\tilde{U}_1 - l_1 + \varrho x < \tilde{U}_1 - l_2 + \varrho x$, and repeating an argument similar to that in the proof of Theorem 6, we can deduce that $\hat{V}_1 - x g_1(t) \le \hat{V}_2 - x g_2(t)$ in $\tilde{\mathcal{N}}_{T^1}$, and $V_1 \le V_2$ in $\tilde{\mathcal{M}}_{T^1}$.

Again, repeating an argument similar to that in the proof of Theorem 6, and using $\varrho x - l_1 < \varrho x - l_2$, we can show that $P_1 \le P_2$ in $\tilde{\mathcal{N}}_T$, and $R_{x,1} \ge R_{x,2}$, and $R_{w,1} \le R_{w,2}$.

Repeating an argument similar to that in the proof of Theorem 6, we deduce that $\partial_x P_1 \le \partial_x P_2$ in $\tilde{\mathcal{N}}_T$, $g_1(t) - \partial_x \hat{V}_1 \ge g_2(t) - \partial_x \hat{V}_2$ in $\tilde{\mathcal{N}}_{T^1}$, $x_1^*(t,w) \ge x_2^*(t,w)$ in $\tilde{\mathcal{M}}_{T^1}$, and $c_{t,1}^*(t,w) \le c_{t,2}^*(t,w)$ in $\tilde{\mathcal{M}}_{T^1}$. □

Remark 14. Theorem 7 says that as the utility cost of labor increases, the optimal retirement threshold tends to go lower and optimal consumption tends to decrease. Thus, the agent consumes less and retires earlier if the utility cost of labor increases. This result also extends a similar result by Choi and Shim [3] originally derived in an infinite horizon model.

Next, we compare the optimal strategy in our model with that in the model where the agent does not have an early retirement option and every other feature is the same as in our model. We will refer to the latter as the *nonretirement option* model.

Theorem 8. *Denote the value function and the optimal consumption of the nonretirement option model by $\mathcal{V}$ and $\mathcal{C}_t^*$, respectively. Then,*

$$\mathcal{V}(t,w) \le V(t,w), \qquad \mathcal{C}_t^*(t,w) \ge c_t^*(t,w), \quad \forall (t,w) \in \tilde{\mathcal{M}}_{T^1}.$$

Proof. It is clear that the nonretirement option model has the following admissible set:

$$\mathcal{A}^2(t,w) \triangleq \{(\tau, c, \pi) \in \mathcal{A}^1(t,w) \colon \tau = T\}.$$

Let $\hat{\mathcal{V}}$ be the smooth solution of the following PDE:

$$\begin{cases} -\partial_t \hat{\mathcal{V}} - \mathcal{L}\hat{\mathcal{V}} = \hat{U}_1 = \tilde{U}_1 - l, & \text{in } \mathcal{N}_T, \\ \hat{\mathcal{V}}(T,x) = \underline{\hat{V}}(T,x) = \underline{\tilde{V}}(T,x), & \forall x \in (0,+\infty). \end{cases} \tag{47}$$

As previously, extend $\hat{\mathcal{V}}$ as $\hat{\mathcal{V}} = \underline{\tilde{V}}$ in $[T, T^1] \times (0,+\infty)$. Then, for any $(t,w) \in \tilde{\mathcal{M}}_{T^1}$, we have

$$\mathcal{V}(t,w) = \inf_{x>0} [\hat{\mathcal{V}}(t,x) - x g(t) + x w].$$

The optimal consumption is given by $\mathcal{C}_t^*(t,w) = \mathcal{I}_{U_1}(t, \mathcal{X}^*(t,w))$, where $\mathcal{X}^*(t,w) = \mathcal{I}_{\hat{\mathcal{V}}}(t, -w + g(t))$ for any $(t,w) \in \tilde{\mathcal{M}}_{T^1}$, and the optimal investment is given by

$$\pi_t^*(t,w) = \mathcal{X}^*(t,w) \partial_{xx} \hat{\mathcal{V}}(t, \mathcal{X}^*(t,w)) (\Sigma^{\mathrm{T}})^{-1} \theta \text{ in } \tilde{\mathcal{M}}_{T^1}.$$

Recalling (19), we deduce that $\hat{V}$ satisfies

$$\begin{cases} -\partial_t \hat{V} - \mathcal{L}\hat{V} \ge \hat{U}_1, & \text{in } \mathcal{N}_T, \\ \hat{V}(T,x) = \underline{\tilde{V}}(T,x), & \forall x \in (0,+\infty). \end{cases}$$

Hence, the comparison principle implies that $\hat{V} \geq \hat{\mathscr{V}}$ in $\tilde{\mathscr{N}}_T$. So, $V \geq \mathscr{V}$ in $\tilde{\mathscr{M}}_{T^1}$ is obvious.

Comparing (47) with (13), we deduce that

$$\hat{\mathscr{V}}(t,x) = \underline{\tilde{V}}(t,x) + \frac{l}{r}(e^{rt-rT} - 1)I_{\{t\leq T\}}, \qquad \partial_x \hat{\mathscr{V}}(t,x) = \partial_x \underline{\tilde{V}}(t,x), \quad \forall (t,x) \in \tilde{\mathscr{N}}_{T^1}.$$

Thus, by (16), we have

$$\partial_x \underline{\hat{V}} \leq \partial_x \underline{\tilde{V}} = \partial_x \hat{\mathscr{V}} \text{ in } \tilde{\mathscr{N}}_{T^1}, \qquad \partial_x \hat{V} = \partial_x \underline{\hat{V}} \leq \partial_x \hat{\mathscr{V}} \text{ in } \mathbf{RR}_x.$$

Moreover, from (19) and (47), it is not difficult to check that

$$\begin{cases} -\partial_t \partial_x \hat{V} - \mathscr{L}_x \partial_x \hat{V} = \partial_x \hat{U}_1 = -\partial_t \partial_x \hat{\mathscr{V}} - \mathscr{L}_x \partial_x \hat{\mathscr{V}}, & \text{in } \mathbf{WR}_x, \\ \partial_x \hat{V}(T,x) = \partial_x \underline{\hat{V}}(T,x) = \partial_x \hat{\mathscr{V}}(T,x), & \forall x \in (0,+\infty), \\ \partial_x \hat{V}(t, R_x(t)) \leq \partial_x \hat{\mathscr{V}}(t, R_x(t)), & \forall t \in [0,T]. \end{cases}$$

Hence, the comparison principle implies that $\partial_x \hat{\mathscr{V}} \geq \partial_x \hat{V}$ in $\mathbf{WR}_x$, too. So we have proved that $\partial_x \hat{\mathscr{V}} \geq \partial_x \hat{V}$ in $\mathscr{N}_T$. Since $\hat{V} = \underline{\hat{V}} = \hat{\mathscr{V}}$ in $[T, T^1] \times (0,+\infty)$, we have $\partial_x \hat{\mathscr{V}} \geq \partial_x \hat{V}$ in $\mathscr{N}_{T^1}$. Thus, we know that

$$\partial_x \hat{V}(t, x^*(t,w)) = -w + g(t) = \partial_x \hat{\mathscr{V}}(t, \mathscr{X}^*(t,w)) \geq \partial_x \hat{V}(t, \mathscr{X}^*(t,w)), \quad \forall (t,w) \in \tilde{\mathscr{M}}_{T^1}.$$

Combining this with $\partial_{xx} \hat{V} > 0$ in $\tilde{\mathscr{N}}_{T^1}$, we can show that $\mathscr{X}^*(t,w) \leq x^*(t,w)$ for any $(t,w) \in \tilde{\mathscr{M}}_{T^1}$, and $\mathscr{C}_t^*(t,w) \geq c_t^*(t,w)$ for any $(t,w) \in \tilde{\mathscr{M}}_{T^1}$. □

Remark 15. Theorem 8 says that the agent consumes less when he or she has an early retirement option than when he or she does not. Indeed, the agent gives up future labor income if he or she chooses early retirement, and thus the agent's consumption is lower when there exists an early retirement option. This result also extends a similar result in Choi and Shim [3]. Choi and Shim [3] also show that the agent tends to take higher risk when there is a retirement option than when the agent is forced to work forever in a model without a mandatory retirement date (see also a similar result in Dybvig and Liu [6]). It is possible to prove the same result in our model with a mandatory retirement date under the assumption that the absolute risk aversion coefficient of U_i, $i = 1, 2$ is increasing with respect to consumption or wealth. The assumption, however, is too restrictive, and we leave it as an interesting open question to find the necessary and sufficient conditions for the result to hold in a model with a mandatory retirement date.

7. Conclusion

We have studied a retirement/consumption and investment choice problem of an agent who faces the trade-off between receiving labor income and suffering the utility cost caused by hardship or lost leisure as a result of labor. We have derived a solution to the problem by successive transformations, which ultimately result in a pure optimal stopping problem, and provided a verification theorem. We have also derived properties of optimal strategy and conducted comparative static analysis, which studies the effects of parameter values on the optimal strategy.

The method we have proposed in this paper combines transformations and the PDE approach to analyze properties of the optimal strategy, which has not yet been widely used in the study of financial models. We expect that the approach proposed in this paper will have wide applicability in future studies.

In this paper we have assumed a constant investment opportunity set for simplicity of analysis; the consideration of time-varying investment opportunity is left as a topic for future research. Also, consideration of the borrowing constraints as in Fahri and Panageas [9] and Dybvig and Liu [6] will be an important extension of the research of this paper.

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Endnotes

[1] Bakshi and Chen [1] have shown that population aging had significant effects on capital markets. We expect that retirement decisions also have similar effects on the markets.

[2] This paper can be regarded as continuation of the study by Karatzas and Wang [17]. We work on the conjecture presented in appendix B of their work and consider the case where the stream of consumption and investment extends beyond the moment of stopping.

[3] If the final time T^1 is random and $U_2 \neq 0$, or there is a nonzero probability that $T^1 < T$, then the financial market is essentially incomplete, and the method in this paper does not apply. The market can be made complete by introducing appropriate insurance contracts (see, e.g., Dybvig and Liu [6]). We leave market incompleteness or the introduction of insurance for future research.

[4] It turns out that this assumption is not restrictive. In the absence of borrowing constraints, the case where the agent has pension income is equivalent to the case where the agent has no income but his or her wealth is augmented by the present value of pension income. See Remark 2 in the next subsection.

[5] The limiting value $U_i(t,0)$ maybe be equal to $-\infty$, for example, if U_i is the constant relative risk aversion (CRRA) utility function with $\gamma > 1$ or the logarithmic function in (4).

[6] We need $U_2(T^1,\cdot)$ defined only at $t = T^1$. For convenience of exposition, we will assume that it is defined for all $t \in [0,T^1]$. Indeed, if it is defined only for $t = T^1$, then we can extend it by letting $U_2(t,c) = U_2(T^1,c)$ for $t \in [0,T^1)$.

[7] Treatment of the case $\partial_c U_i(t,0) < +\infty$ is available from the authors upon request.

[8] In this case, the discount rate for utility of consumption is deterministic but time varying. The discount rate for utility of labor, however, is assumed to be constant and equal to β.

[9] We considered an extension of our model to the case where the final time is random in Section 2.1. In such an extension, we assume the random time is independent of $(\mathcal{F}_t)_{t=0}^{T^1}$.

[10] For details, one can refer to Krylov [21] or Yang and Tang [31].

References

[1] Bakshi G, Chen Z (1994) Baby boom, population aging, and capital markets. *J. Bus.* 67(2):165–202.
[2] Bensoussan A, Friedman A (1974) Nonlinear variational inequalities and differential games with stopping times. *J. Functional Anal.* 16(3):305–352.
[3] Choi K, Shim G (2006) Disutility, optimal retirement, and portfolio selection. *Math. Finance* 16(2):443–467.
[4] Choi K, Shim G, Shin Y (2008) Optimal portfolio, consumption-leisure and retirement choice problem with CES utility. *Math. Finance* 18(3):445–472.
[5] Cox J, Huang C (1989) Optimal consumption and portfolio policies when asset prices follow a diffusion process. *J. Econom. Theory* 49(1):33–83.
[6] Dybvig P, Liu H (2010) Lifetime consumption and investment: Retirement and constrained borrowing. *J. Econom. Theory* 145(3):885–907.
[7] Dybvig P, Liu H (2011) Verification theorems for models of optimal consumption and investment with retirement and constrained borrowing. *Math. Oper. Res.* 36(4):620–635.
[8] Evans LC (2010) *Partial Differential Equations* (American Mathematical Society, Providence, RI).
[9] Fahri E, Panageas S (2007) Saving and investing for early retirement: A theoretical analysis. *J. Financial Econom.* 83(1):87–121.
[10] Fleming W, Soner H (2006) *Controlled Markov Processes and Viscosity Solutions* (Springer-Verlag, New York).
[11] Friedman A (1973) Stochastic games and variational inequalities. *Arch. Rational Mech. Anal.* 51(5):321–346.
[12] Friedman A (1975) Parabolic variational inequalities in one space dimension and smoothness of the free boundary. *J. Functional Anal.* 18(2):151–176.
[13] Friedman A (1982) *Variational Principles and Free-Boundary Problems* (John Wiley & Sons, New York).
[14] Hamadene S, Rotenstein E, Zalinescu A (2009) A generalized mixed zero-sum stochastic differential game and double barrier reflected BSDEs with quadratic growth coefficient. *Anal. Stiin. Ale UNIV. Al I Cuza Din Iasi-Serie Noua-Matematica* 55(f2):419–444.
[15] Inada K (1963) On a two-sector model of economic growth: Comments and a generalization. *Rev. Econom. Stud.* 30(2):119–127.
[16] Karatzas I, Shreve S (1998) *Methods of Mathematical Finance* (Springer-Verlag, New York).
[17] Karatzas I, Wang H (2000) Utility maximization with discretionary stopping. *SIAM J. Control Optim.* 39(1):306–329.
[18] Karatzas I, Lehoczky J, Shreve S (1987) Optimal portfolio and consumption decisions for a "small investor" on a finite horizon. *SIAM J. Control Optim.* 25(6):1557–1586.
[19] Karatzas I, Lehoczky J, Sethi S, Shreve S (1986) Explicit solution of a general consumption/investment problem. *Math. Oper. Res.* 11(2):209–216.
[20] Koo H, Tang S, Yang Z (2015) A Dynkin game under Knightian uncertainty. *Discrete Continuous Dynamical Systems* 35(11):5467–5498.
[21] Krylov N (1980) *Controlled Diffusion Processes* (Springer-Verlag, New York).
[22] Lieberman G (1996) *Second Order Parabolic Differential Equations* (World Scientific, Singapore).
[23] Liu H, Loewenstein M (2002) Optimal portfolio selection with transaction costs and finite horizons. *Rev. Financial Stud.* 15(3):805–835.
[24] Merton R (1969) Lifetime portfolio selection under uncertainty: The continuous-time case. *Rev. Econom. Statist.* 51(3):247–257.
[25] Peskir G, Shiryaev A (2006) *Optimal Stopping and Free-Boundary Problems* (Birkhäuser Verlag, Basel).
[26] Touzi N, Vieille N (2002) Continuous-time Dynkin game with mixed strategies. *SIAM J. Control Optim.* 41(4):1073–1088.
[27] Tso K (1985) On an Aleksandrov-Bakel'man type maximum principle for second-order parabolic equations. *Comm. Partial Differential Equations* 10(5):543–553.
[28] Uzawa H (1963) On a two-sector model of economic growth II. *Rev. Econom. Stud.* 30(2):105–118.
[29] Watson J, Scott J (2014) Ratchet consumption over finite and infinite planning horizons. *J. Math. Econom.* 54(October):74–96.
[30] Yan H, Liang G, Yang Z (2015) Indifference pricing and hedging in a multiple-priors model with trading constraints. *Sci. China Ser. A: Math.* 58(4):689–714.
[31] Yang Z, Tang S (2013) Dynkin game of stochastic differential equations with random coefficients, and associated backward stochastic partial differential variational inequality. *SIAM J. Control Optim.* 51(1):64–95.
[32] Yong J, Zhou X (1999) *Stochastic Control: Hamiltonian Systems and HJB Equations* (Springer, New York).