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An Integral Equation Representation for Optimal Retirement Strategies in Portfolio Selection Problem

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Abstract
In this paper we study the consumption and portfolio selection problem of a finitely-lived economic agent with an early retirement option, that is, the agent can choose her/his early retirement time before a mandatory retirement time. Based on the theoretical results in Yang and Koo (Math Oper Res, 43(4):1378–1404, 2018), we derive an integral equation satisfied by the optimal retirement boundary or free boundary using the Mellin transform technique. We also derive integral equation representations for the optimal consumption-portfolio strategies and the optimal wealth process. By using the recursive integration method, we obtain the numerical solutions for the integral equations and discuss economic implications for the optimal retirement strategies by using numerical solutions.

Keywords Portfolio selection · Mandatory retirement · Early retirement · Free boundary · Mellin transform · Integral equation

1 Introduction

In this paper we study the optimal retirement decision of an agent in a continuous time model. We derive an integral equation for optimal policies and provide numerical schemes for the theoretical model proposed by Yang and Koo (2018).

Currently, life expectancy is increasing and population aging is prevalent, and hence retirement is a crucially important issue from a social as well as an individual

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perspective. Understanding the joint problem of optimal retirement and consumption and portfolio decision is a first step to tackle the issue.

In the literature Choi and Shim (2006) initiated investigation into the joint problem by studying a model where an agent chooses her/his retirement time as well as a portfolio of assets and consumption. In their model the decision to retire comes from the trade-off between labor income and the utility cost of labor. Farhi and Panageas (2007) and Choi et al. (2008) have proposed a model where an agent chooses labor, leisure and the retirement time. Dybvig and Liu (2010) and Lim and Shin (2011) have studied a model with borrowing constraints. The time horizon of these models, however, is infinite, and hence, the models do not allow investigation of the effects of changes in mandatory retirement dates or of increasing life expectancy.[1] Recently, Yang and Koo (2018) have proposed a model with mandatory retirement date and early retirement option and extended results in Choi and Shim (2006) to a finite-time horizon model. They have investigated the dual problem and shown the duality relationship that the value function of the primal problem is the concave conjugate of the dual value function by relying on the theory of partial differential equations and backward stochastic differential equations. They have established the existence and uniqueness of the value function and shown that an agent optimally retires if the agent's wealth hits a boundary. All their results, however, are theoretical and do not include methods to calculate optimal strategies explicitly.

In this paper we focus on the numerical implementation of the optimal retirement and consumption/portfolio strategies based on the model proposed by Yang and Koo (2018). Firstly, we apply the Mellin transform to the variational inequality satisfied by the dual value function and derive an integral equation for the optimal retirement boundary. By using the integral equation, we obtain analytic representations of optimal consumption, wealth and portfolio processes. The Mellin transform has been applied to option pricing as one among numerous integral transform methods. For example, Panini and Srivastav (2004) and Jeon et al. (2016) have derived integral equations satisfied by the American and Russian options by using the Mellin transform. Utilizing the Mellin transform, we are able to transform the variational inequality in Yang and Koo (2018) into an ordinary differential equation and to obtain an analytic solution to the variational inequality through the inverse Mellin transform. Secondly, we obtain numerical solutions to the integral equation by applying the recursive integration method proposed by Huang et al. (1996). We study properties of the optimal retirement boundary, consumption and portfolio selection, investigating the numerical solutions.

The rest of this paper is structured as follows: In Sect. 2 we explain the model and briefly review the theoretical results in Yang and Koo (2018). In Sect. 3 we use the variational inequality derived by Yang and Koo (2018) and obtain the integral equation representation of the optimal retirement boundary by the Mellin transform. In Sect. 4 we provide analytic solutions for optimal consumption, wealth and portfolio processes. In Sect. 5, we explain how to solve the integral equation using

[1] Dybvig and Liu (2010) have considered a model with a mandatory retirement date, but the model does not consider both the mandatory retirement date and early retirement option.

the recursive integration method and derive economic implications by investigating numerical solutions. In Sect. 6 we conclude. All the detailed proofs are in Appendix.

2 Preliminaries

2.1 Model

In this section we explain the model of consumption and portfolio selection with early retirement option proposed by Yang and Koo (2018).

2.1.1 Objective Function and Financial Market

We consider an economy with a financial market in which an agent optimizes over a lifetime consumption profile. There is one consumption good. In reality there exist a large number of consumption goods. The one consumption good in our model thus represents the composite of all consumption goods consumed by the agent. Modern theory of portfolio selection and asset pricing has mostly been developed based on the single consumption good assumption (see e.g. Merton 1969; Cochrane 2005 etc). Time is continuous and runs from 0 to T_1. The agent's preference is represented by the following objective function:

$$U \triangleq \mathbb{E}\left[\int_0^{T_1} e^{-\beta t}\left(U_1(t, c_t) - l\mathbf{1}_{\{t \le \tau\}}\right)dt + e^{-\beta T_1} U_2(T_1, W_{T_1})\right], \tag{1}$$

where c_t denotes the rate of consumption at time t, $\beta > 0$ is a subjective discount rate, $l > 0$ is a constant describing the utility cost of labor, τ is the time of retirement, and $\mathbf{1}_A$ denotes the characteristic function of a set A, i.e., $\mathbf{1}_A$ can take two values, 0 and 1, and $\mathbf{1}_A(x) = 1$ if and only if $x \in A$. It is implicit in the utility specification that the agent works until the retirement time τ and suffers the utility cost of labor and she/he will not bear the cost after retirement.

There exists a mandatory retirement time $T < T_1$. The agent can choose the retirement time $\tau \le T$. The agent receives income at a rate equal to $\rho > 0$, which we will assume to be constant for simplicity of the model. The retirement decision is irreversible. In general the wage rate and the utility cost of labor are expected to be time-varying, depending on the agent's productivity, labor supply, and health condition, etc.[2] In order to focus on the effects of the fixed mandatory retirement date on optimal decisions we make the simplifying assumption of constant utility cost and wage rate.

To guarantee existence of a solution to the agent's optimization problem, we assume that the utility function satisfies the following conditions:

[2] See Choi et al. (2008) for study of time varying labor supply and retirement decision in an infinite horizon.

Assumption 1 For $i = 1, 2$, the functions $U_i(t, c) \in C^\infty([0, T_1] \times (0, \infty))$ are strictly increasing and strictly concave with respect to c and take values in $\mathbb{R}$, the set of real numbers. Also there exist C, k such that for any $t \in [0, T_1]$, $U_i(t, c), i = 1, 2$ satisfy the following conditions:

$$\lim_{c \to 0^+} \partial_c U_i(t, c) = +\infty, \quad \lim_{c \to +\infty} \partial_c U_i(t, c) = 0, \quad \limsup_{c \to +\infty} \max_{t \in [0, T_1]} \partial_c U_i(t, c) c^k \leq C.$$

The first two conditions are the Inada conditions and the last condition is a growth condition for marginal utility. As mentioned in Yang and Koo (2018), the constant relative risk aversion (CRRA) cardinal utility functions satisfy Assumption 1.

The financial market consists of two assets: a riskless asset and a risky asset. We assume that the risk-free rate $r > 0$ is constant and the price S_t of the risky asset follows the geometric Brownian motion (GBM):

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \quad S_0 = S > 0,$$

where μ, σ are positive constants, B is a standard Brownian motion defined on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ endowed with the augmented filtration $\mathbb{F} \equiv \{\mathcal{F}_t\}_{t \geq 0}$ generated by the Brownian motion.

2.1.2 Optimization Problem

We consider the agent's optimization problem at $t \in [0, T_1]$.

The agent's wealth $(W_s)_{s=t}^{T_1}$ evolves according to the following equation:

$$dW_s = \left(rW_s + (\mu - r)\pi_s - c_s + \rho \mathbf{1}_{\{s \leq \tau\}}\right) + \sigma \pi_s dB_s \quad \text{with} \quad W_t = w, \tag{2}$$

where π_s is the amount invested in the risky asset at time s.

We transform the wealth process (2) into a static budget constraint by the martingale approach developed by Karatzas et al. (1987) and Cox and Huang (1989). For this purpose, we define, for $s \geq t \in [0, T_1]$,

$$\theta \triangleq \frac{\mu - r}{\sigma}, \quad Z_s \triangleq e^{-\theta(B_s - B_t) - \frac{1}{2}\theta^2(s-t)}, \quad H_s \triangleq e^{-r(s-t)} Z_s.$$

We obtain the following static budget constraint from the wealth evolution Eq. (2)[3]:

$$\begin{aligned} \mathbb{E}_t\left[H_\tau(W_\tau - G(\tau)) + \int_t^\tau H_s c_s ds\right] &\leq W_t - G(t), \quad \text{if } t \leq \tau, \\ \mathbb{E}_t\left[H_{T_1} W_{T_1} + \int_t^{T_1} H_s c_s ds\right] &\leq W_t, \qquad \text{if } t(= \tau) \leq T_1, \end{aligned} \tag{3}$$

where $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot | \mathcal{F}_t]$ is the conditional expectation at time t on the filtration $\mathcal{F}_t$, and

[3] The equation follows from the optional sampling theorem and Fatou's lemma. See e.g., Karatzas and Shreve (1998).

$$G(t) \triangleq -\mathbb{E}_t\left[\int_{t\wedge T}^{T} \rho H_s ds\right] = -\frac{\rho}{r}\left(1 - e^{-r\{T-(t\wedge T)\}}\right),$$

i.e., $G(t)$ is the negative of the present value of labor income under the assumption that the agent does not retire before T.

We describe the admissible strategies in Appendix A. Denoting the set of admissible strategies by $\mathcal{A}^1(t,w)$, the agent's objective function can now be described as follows: for all $(t,w) \in \widetilde{\mathcal{M}}_{T_1}$ and $(\tau, c, \pi) \in \mathcal{A}^1(t,w)$ (the definitions of $\mathcal{A}^1(t,w)$ and $\widetilde{\mathcal{M}}_{T_1}$ are referred to Appendix A),

$$J(t,w;\tau,c,\pi) \triangleq \mathbb{E}_t\left[\int_t^{T_1} e^{-\beta(s-t)}\left(U_1(s,c_s) - l\mathbf{1}_{\{s\le\tau\}}\right)ds + e^{-\beta(T_1-t)}U_2(T_1, W_{T_1})\right]. \tag{4}$$

We now state the agent's optimization problem.

Problem 1 *The agent chooses an optimal strategy,* $(\tau^*, c^*, \pi^*) \in \mathcal{A}^1(t,w)$ *to maximize the objective function J in* (4)*: namely,*

$$V(t,w) \triangleq J(t,w;\tau^*,c^*,\pi^*) = \sup_{(\tau,c,\pi)\in\mathcal{A}^1(t,w)} J(t,w;\tau,c,\pi), \quad \forall\,(t,w) \in \widetilde{\mathcal{M}}_{T_1}, \tag{5}$$

subject to the budget constraint (3).

3 Analytic Solution of the Model

3.1 Derivation of a Non-homogeneous PDE for $P(t,x)$

By the budget constraint (3) and the martingale method, we can define the dual value functions $\underline{\widetilde{V}}(t,x)$ and $\widehat{V}(t,x)$ of the agent's problem after retirement and before retirement, respectively, as follows (for details, see Appendix B):

$$\underline{\widetilde{V}}(t,x) \triangleq \mathbb{E}_t\left[\int_t^{T_1} e^{-\beta(s-t)}\widetilde{U}_1(s,X_s)ds + e^{-\beta(T_1-t)}\widetilde{U}_2(T_1,X_{T_1})\right],$$
$$\widehat{V}(t,x) \triangleq \sup_{\tau\in\mathcal{U}_{t,T}} \mathbb{E}_t\left[\int_t^{\tau} e^{-\beta(s-t)}\widehat{U}_1(s,X_s)ds + e^{-\beta(\tau-t)}\underline{\widehat{V}}(\tau,X_\tau)\right],$$

where $\mathcal{U}_{t,T}$ is defined in Appendix A, $X_s \triangleq xe^{\beta(s-t)}H_s$, $x > 0$, and, for $i = 1,2$,

$$\widetilde{U}_i(t,x) \triangleq \sup_{c>0}[U_i(t,c) - xc] = U_i(t,\mathcal{J}_{U_i}(t,x)) - x\mathcal{J}_{U_i}(t,x),$$
$$\widehat{U}_1(t,x) \triangleq \sup_{c>0}\left[U_1(t,c) - xc\right] - l = \widetilde{U}_1(t,x) - l,$$

$\mathcal{J}_{U_i}(t,\cdot)$ is the inverse function of $\partial_x U_i(t,\cdot)$, $i = 1,2$ and

$$\underline{\widehat{V}}(t,x) = \underline{\widetilde{V}}(t,x) + xG(t).$$

As in Yang and Koo (2018) we study properties of the following function

$$P(t,x) \triangleq \widehat{V}(t,x) - \underline{\widehat{V}}(t,x) = \widehat{V}(t,x) - \underline{\widetilde{V}}(t,x) - xG(t).$$

By Lemma 3 in Yang and Koo (2018), the function $P(t, x)$ is an increasing function with respect to x and the following free boundary (or the optimal retirement boundary) is well-defined:

$$R_x(t) \triangleq \inf\{x \geq 0 \mid P(t,x) > 0\}, \quad \forall t \in [0,T).$$

The following two regions, the working region ($\mathbf{WR}_x$) and the retirement region ($\mathbf{RR}_x$) are also well-defined:

$$\begin{aligned}\mathbf{WR}_x &= \{(t,x) \mid P(t,x) > 0\} = \{(t,x) \mid x > R_x(t),\ t \in [0,T)\},\\ \mathbf{RR}_x &= \{(t,x) \mid P(t,x) = 0\} = \{(t,x) \mid 0 < x \leq R_x(t),\ t \in [0,T)\}.\end{aligned}$$

Yang and Koo (2018) have shown that the function $P(t, x)$ satisfies the following non-homogeneous parabolic PDE:

$$\begin{cases} \partial_t P(t,x) + \mathcal{L}P(t,x) = F(t,x), & \forall (t,x) \in \mathcal{N}_T, \\ P(T,x) = 0, \end{cases} \tag{6}$$

where the domain $\mathcal{N}_T$ and the differential operator $\mathcal{L}$ are defined in Appendix B, and $F(t,x) = -(\rho x - l)\mathbf{1}_{\{x > R_x(t)\}}$.

Yang and Koo (2018) showed that there exists a unique strong solution to Eq. (6), the dual value functions can be constructed from the solution and the value function and the dual value function satisfy a duality relationship. See Appendix C for summary of their results.

3.2 Analytic Representation of $P(t, x)$

We will now derive an analytic representation of $P(t, x)$ by using the Mellin transform. The following Mellin convolution theorem is the key to derive the integral equation for $P(t, x)$[4].

Proposition 1 (The Mellin Convolution Theorem) *Suppose that $f(x)$ and $g(x)$ are locally integrable functions in $(0,\infty)$ and the Mellin transforms $M[f](y)$ and $M[g](y)$ exist for $a_1 < \mathcal{R}(y) < a_2$, where $\mathcal{R}(y)$ denotes the real part of complex number y. Then,*

[4] In Appendix D, we briefly review the definition and properties of the Mellin transformation.

$$f(x) * g(x) \triangleq \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M[f](y)M[g](y)x^{-y}dy$$
$$= \int_0^\infty f\left(\frac{x}{u}\right)g(u)\frac{du}{u}, \quad \text{where} \quad a_1 < c < a_2.$$

Proposition 2 *The function $P(t, x)$ in (6) has the following integral equation representation:*

$$\begin{aligned} P(t,x) = \rho x \int_t^T e^{-r(\eta-t)}\mathcal{N}\left(d^+\left(\eta - t, \frac{x}{R_x(\eta)}\right)\right)d\eta \\ - l\int_t^T e^{-\beta(\eta-t)}\mathcal{N}\left(d^-\left(\eta - t, \frac{x}{R_x(\eta)}\right)\right)d\eta, \end{aligned} \tag{7}$$

where

$$d^{\pm}(t,x) = \frac{\log x + \left(\beta - r \pm \frac{1}{2}\theta^2\right)t}{\theta\sqrt{t}},$$

and $\mathcal{N}(\cdot)$ is a standard normal distribution function.

Proof See Appendix E. □

By *smooth-pasting* condition for $P(t, x)$ (see Lemma 3 in Yang and Koo 2018), we deduce the following corollary:

Corollary 1 *The free boundary $R_x(t)$ satisfies the following integral equation:*

$$\begin{aligned} 0 = \rho R_x(t)\int_t^T e^{-r(\eta-t)}\mathcal{N}\left(d^+(\eta - t, \frac{R_x(t)}{R_x(\eta)})\right)d\eta \\ - l\int_t^T e^{-\beta(\eta-t)}\mathcal{N}\left(d^-(\eta - t, \frac{R_x(t)}{R_x(\eta)})\right)d\eta. \end{aligned} \tag{8}$$

4 Optimal Consumption and Portfolio Strategies

In this section we derive integral equation representations for optimal consumption, wealth and portfolio processes. In order to derive concrete results we will focus on the specific case where the felicity functions have constant relative risk aversion, i.e. we will assume that $U_i, i = 1, 2$ take the following forms:

$$U_1(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad U_2(w) = A^\gamma\frac{w^{1-\gamma}}{1-\gamma}, \quad \gamma > 0, \ \gamma \neq 1, \tag{9}$$

where γ is the agent's coefficient of relative risk aversion and A is a positive constant denoting the strength of the agent's bequest motive.

We will make the following assumption to guarantee the existence of a solution to the standard Merton problem (i.e. the consumption and portfolio selection problem in an infinite horizon without retirement (see Merton 1969)).

$$K \triangleq r + \frac{\beta - r}{\gamma} + \frac{\gamma - 1}{2\gamma^2}\theta^2 > 0.$$

Assumption 2

For the CRRA felicity functions, we have

$$\widetilde{U}_1(x) = \frac{\gamma}{1-\gamma}x^{-\frac{1-\gamma}{\gamma}} \quad \text{and} \quad \widetilde{U}_2(x) = A\frac{\gamma}{1-\gamma}x^{-\frac{1-\gamma}{\gamma}},$$

and the dual value function $\underline{\widetilde{V}}$ of the agent's problem after retirement is given by

$$\begin{aligned}\underline{\widetilde{V}}(t,x) &= \mathbb{E}_t\left[\int_t^{T_1} e^{-\beta(s-t)}\frac{\gamma}{1-\gamma}X_s^{-\frac{1-\gamma}{\gamma}}\,ds + e^{-\beta(T_1-t)}A\frac{\gamma}{1-\gamma}X_{T_1}^{-\frac{1-\gamma}{\gamma}}\right] \\ &= \frac{\gamma}{1-\gamma}x^{-\frac{1-\gamma}{\gamma}}\left(A\cdot e^{-K(T_1-t)} + \frac{1-e^{-K(T_1-t)}}{K}\right).\end{aligned} \tag{10}$$

We state an integral equation representation of the value function $V(t, w)$ in the following proposition:

Proposition 3 *The value function $V(t, w)$ defined in (5) has the following integral equation representation:*

$$\begin{aligned}V(t,w) = {} & \frac{1}{1-\gamma}(x^*)^{-\frac{1-\gamma}{\gamma}}\left(A\cdot e^{-K(T_1-t)} + \frac{1-e^{-K(T_1-t)}}{K}\right) \\ & - \frac{\rho}{\theta\sqrt{2\pi}}x^*\int_t^T \exp\left\{-r(\eta-t) - \frac{d^+(\eta-t,\frac{x^*}{R_x(\eta)})^2}{2}\right\}\frac{1}{\sqrt{\eta-t}}d\eta \\ & + \frac{l}{\theta\sqrt{2\pi}}\int_t^T \exp\left\{-\beta(\eta-t) - \frac{d^-(\eta-t,\frac{x^*}{R_x(\eta)})^2}{2}\right\}\frac{1}{\sqrt{\eta-t}}d\eta \\ & - l\int_t^T e^{-\beta(\eta-t)}\mathcal{N}\left(d_-\left(\eta-t,\frac{x^*}{R_x(\eta)}\right)\right)d\eta,\end{aligned}$$

where $x^* = x^*(t, w)$ is a unique solution to the following integral equation:

$$
\begin{aligned}
w = {} & (x^*)^{-\frac{1}{\gamma}}\left(A\cdot e^{-K(T_1-t)} + \frac{1-e^{-K(T_1-t)}}{K}\right) \\
& + \frac{l}{\theta x^*\sqrt{2\pi}}\int_t^T \exp\left\{-\beta(\eta-t) - \frac{d^-(\eta-t,\frac{x^*}{R_x(\eta)})^2}{2}\right\}\frac{1}{\sqrt{\eta-t}}d\eta \\
& - \frac{\rho}{\theta\sqrt{2\pi}}\int_t^T \exp\left\{-r(\eta-t) - \frac{d^+(\eta-t,\frac{x^*}{R_x(\eta)})^2}{2}\right\}\frac{1}{\sqrt{\eta-t}}d\eta \\
& - \rho\int_t^T e^{-r(\eta-t)}\mathcal{N}\left(d^+\left(\eta-t,\frac{x^*}{R_x(\eta)}\right)\right)d\eta.
\end{aligned}
\tag{11}
$$

Proof See Appendix F. □

Theorem 1 (*Main Theorem*) *Before retirement, the optimal consumption c_s^*, the optimal wealth W_s^*, and the optimal portfolio π_s^* at time $s \geq t$ are given by*

$$
\begin{aligned}
c_s^* &= (X_s^*)^{-\frac{1}{\gamma}}, \\
W_s^* &= (X_s^*)^{-\frac{1}{\gamma}}\left(A\cdot e^{-K(T_1-s)} + \frac{1-e^{-K(T_1-s)}}{K}\right) \\
&\quad + \frac{l}{\theta X_s^*\sqrt{2\pi}}\int_s^T \exp\left\{-\beta(\eta-s) - \frac{d^-(\eta-s,\frac{X_s^*}{R_x(\eta)})^2}{2}\right\}\frac{1}{\sqrt{\eta-s}}d\eta \\
&\quad - \frac{\rho}{\theta\sqrt{2\pi}}\int_s^T \exp\left\{-r(\eta-s) - \frac{d^+(\eta-s,\frac{X_s^*}{R_x(\eta)})^2}{2}\right\}\frac{1}{\sqrt{\eta-s}}d\eta \\
&\quad - \rho\int_s^T e^{-r(\eta-s)}\mathcal{N}\left(d^+\left(\eta-s,\frac{X_s^*}{R_x(\eta)}\right)\right)d\eta, \\
\pi_s^* &= \frac{\theta}{\sigma}\left[\frac{1}{\gamma}(X_s^*)^{-\frac{1}{\gamma}}\left(A\cdot e^{-K(T_1-s)} + \frac{1-e^{-K(T_1-s)}}{K}\right)\right. \\
&\quad + \frac{l}{\theta X_s^*\sqrt{2\pi}}\int_s^T \exp\left\{-\beta(\eta-s) - \frac{d^-(\eta-s,\frac{X_s^*}{R_x(\eta)})^2}{2}\right\}\left(\frac{1}{\sqrt{\eta-s}} + \frac{d^-(\eta-s,\frac{X_s^*}{R_x(\eta)})}{\theta(\eta-s)}\right)d\eta \\
&\quad \left. + \frac{\rho}{\theta\sqrt{2\pi}}\int_s^T \exp\left\{-r(\eta-s) - \frac{d^+(\eta-s,\frac{X_s^*}{R_x(\eta)})^2}{2}\right\}\left(\frac{1}{\sqrt{\eta-s}} - \frac{d^+(\eta-s,\frac{X_s^*}{R_x(\eta)})}{\theta(\eta-s)}\right)d\eta\right],
\end{aligned}
$$

where $X_s^* = x^*(t,w)e^{\beta(s-t)}H_s$.

Proof See Appendix G. □

Theorem 1 in Yang and Koo (2018) implies that $\mathcal{J}_{\hat{\gamma}}(t, -w + G(t)) = x^*(t, w)$ is continuous and strictly decreasing with respect to w in $\widetilde{M}_{T_1}$, and maps $(G(t), +\infty)$ onto $(0, +\infty)$ for any $t \in [0, T_1]$. So for any $t \in [0, T_1]$, $x^*(t, \cdot)$ has the inverse function $w^*(t, \cdot)$, which is continuous, strictly decreasing and maps $(0, +\infty)$ onto $(G(t), +\infty)$.

Thus, in the (t, w)-domain, we can define

$$\begin{aligned} &R_w(t) \triangleq w^*(t, R_x(t)), \ \forall\, t \in [0, T], \\ &\mathbf{RR}_w \triangleq \{(t, w) \in \mathcal{M}_T \mid (t, x^*(t, w)) \in \mathbf{RR}_x\}, \\ &\mathbf{WR}_w \triangleq \{(t, w) \in \mathcal{M}_T \mid (t, x^*(t, w)) \in \mathbf{WR}_x\}, \end{aligned}$$

where $\mathbf{RR}_w$, $\mathbf{WR}_w$, and $R_w(t)$ represent the retirement region, the working region, and the optimal retirement threshold in the (t, w)-coordinate system, respectively.

Clearly, the two regions $\mathbf{RR}_w$ and $\mathbf{WR}_w$ can be represented by

$$\mathbf{RR}_w = \{(t, w)] \mid w \geq R_w(t),\ t \in [0, T)\}, \quad \mathbf{WR}_w = \{(t, w)] \mid G(t) < w < R_w(t),\ t \in [0, T)\}.$$

Moreover, the optimal retirement threshold in the (t, w)-domain is given by

$$R_w(t) = w(t, R_x(t)) = -\partial_x \underline{\widetilde{V}}(t, R_x(t)) - \partial_x P(t, R_x(t)) = -\partial_x \underline{\widetilde{V}}(t, R_x(t)), \tag{12}$$

and hence, from (12) and (26), we see that

$$\begin{aligned} R_w(t) &= (R_x(t))^{-\frac{1}{\gamma}} \left(A \cdot e^{-K(T_1 - t)} + \frac{1 - e^{-K(T_1 - t)}}{K} \right) \\ &= (R_x(t))^{-\frac{1}{\gamma}} \left[\frac{1}{K} + e^{-K(T_1 - t)} \left(A - \frac{1}{K} \right) \right]. \end{aligned} \tag{13}$$

Since the optimal retirement boundary $R_x(t)$ in (t, x)-domain is not affected by the final time T_1, we can deduce the following theorem from the relation in (13).

Theorem 2 *The following statements are true:*

(a) *If the agent's bequest motive is weak (i.e., $0 \leq A < 1/K$), the optimal retirement threshold $R_w(t)$ in (t, w)-domain increases as the final time T_1 increases.*
(b) *If the agent's bequest motive is very strong (i.e., $A > 1/K$), the optimal retirement threshold $R_w(t)$ in (t, w)-domain decreases as the final time T_1 increases.*
(c) *Otherwise (i.e., $A = 1/K$), the final time T_1 does not affect the optimal retirement threshold $R_w(t)$ in (t, w)-domain.*

Theorem 2 provides a theoretical relationship between the life expectancy and the optimal retirement decision. Part (a) of the theorem shows that as the life expectancy increases the optimal retirement time tends to increase, since the optimal retirement threshold increases, if the bequest motive is modest. The result is consistent with earlier theoretical results by Bloom et al. (2007a, b) and Prettner and Canning (2014), which are obtained without consideration of bequest motive. Parts (b) and (c) provide other possibilities if the bequest motive is substantially large. Given the

fact that empirical estimates of bequest motive is not large (Hurd 1989, 1990), the theoretical result is in conflict with the stylized fact that the retirement age has fallen substantially over the last 60 years in the industrialized world (see e.g., Hurd 1990; Blondal and Scarpetta 1997; Gruber and Wise 1998). Blondal and Scarpetta (1997) and Gruber and Wise (1998) explain the discrepancy by the financial incentives in the pension and public income support programs in the countries.

5 Numerical Results

We have obtained the optimal strategies of the agent's optimization problem in Sect. 4. We need to find the free boundary $R_x(t),\ t \in [0, T]$, which is a solution to the integral equation given in (8), but the forms of optimal strategies are not fully explicit. However, using the recursive integration method (RIM) proposed by Huang et al. (1996), the solution $R_x(t)$ to the integral equation in (8) can be obtained numerically.

5.1 Recursive Integration Method (RIM)

From the integral Eq. (8) we have

$$0 = \int_0^{\vartheta} \mathcal{I}(\vartheta, \xi, \widetilde{R}_x(\vartheta), \widetilde{R}_x(\vartheta - \xi)) d\xi, \tag{14}$$

where $\vartheta = T - t$, $\widetilde{R}_x(\vartheta) = R_x(T - \vartheta)$, and

$$\begin{aligned}\mathcal{I}(\vartheta, \xi, \widetilde{R}_x(\vartheta), \widetilde{R}_x(\vartheta - \xi)) = \rho \widetilde{R}_x(\vartheta) e^{-r\xi} \mathcal{N}\left(d^+(\xi, \frac{\widetilde{R}_x(\vartheta)}{\widetilde{R}_x(\vartheta - \xi)}) \right) \\ - l e^{-\beta\xi} \mathcal{N}\left(d^-(\xi, \frac{\widetilde{R}_x(\vartheta)}{\widetilde{R}_x(\vartheta - \xi)}) \right).\end{aligned}$$

We will now explain how to calculate a numerical solution of the free boundary $\widetilde{R}_x(\vartheta)$ in the integral Eq. (14) by using the RIM.

First, we divide the interval $[0, \vartheta]$ into n time steps with end points $\vartheta_i,\ i = 0, 1, \ldots, n$, where $\vartheta_0 = 0,\ \vartheta_n = \vartheta$ and $\Delta\vartheta = \vartheta/n$. Let $\widetilde{R}_i$ denote a numerical approximation to $\widetilde{R}_x(\vartheta_i),\ i = 0, 1, \ldots, n$.

For $\vartheta = \vartheta_1$, by the trapezoidal rule, we can approximate the integral Eq. (14) by the following algebraic equation:

$$0 = \frac{\Delta\vartheta}{2} \left[\mathcal{I}(\vartheta_1, \vartheta_0, \widetilde{R}_1, \widetilde{R}_1) + \mathcal{I}(\vartheta_1, \vartheta_1, \widetilde{R}_1, \widetilde{R}_0) \right]. \tag{15}$$

Remark 1 Since Eq. (14) contains the whole paths of the free boundary $\widetilde{R}_x(\xi)$ from 0 to ϑ, it is difficult to apply the Gaussian quadrature rule when we discretize the integral equation. For this reason, when we apply the RIM, we use the trapezoidal rule.

Table 1 Algorithm to find optimal strategies by the RIM

Algorithm
Step 0: Set $(n+1)$ to be the number of time points dividing the interval $[0, T-t]$ into n equal subintervals.
Step 1: Approximate the optimal retirement boundary $(\widetilde{R}_x(\vartheta))_{\vartheta=0}^{T-t}$ in (14) by the RIM.
Step 1-1: For $\widetilde{R}_0 = l/\rho$, obtain $\widetilde{R}_1$ in Eq. (15) by a numerical root-finding method.
Step 1-2: Calculate $\widetilde{R}_j$ $(j = 2, 3, \cdots, n)$, by solving Eq. (17) recursively.
Step 2: Find the value $x^*(t, w)$ for a given $w > G(t)$ by applying the root-finding method to (11).
Step 3: Calculate the optimal strategy (c_s^*, π_s^*) and optimal wealth W_s^*, for $s \in [0, T_1]$ by simulating X_s^* and applying the RIM to Theorem 1.

By Lemma 6 in Yang and Koo (2018), we see that

$$\widetilde{R}_x(0+) = \lim_{t \to T-} R_x(t) = \frac{l}{\rho}.$$

Thus, $\widetilde{R}_0 = l/\rho$ and only $\widetilde{R}_1$ is unknown in Eq. (15). Using the bisection method as a root-finding scheme, we can find a solution for $\widetilde{R}_1$ in (15).

For $\vartheta = \vartheta_2$, we have

$$0 = \frac{\Delta\vartheta}{2}\left[\mathcal{I}(\vartheta_2, \vartheta_0, \widetilde{R}_2, \widetilde{R}_2) + 2\mathcal{I}(\vartheta_2, \vartheta_1, \widetilde{R}_2, \widetilde{R}_1) + \mathcal{I}(\vartheta_2, \vartheta_2, \widetilde{R}_2, \widetilde{R}_0)\right]. \tag{16}$$

Since we know the $\widetilde{R}_0$ and $\widetilde{R}_1$, we also obtain a solution for $\widetilde{R}_2$ in (16).

Hence, we can obtain a solution for $\widetilde{R}_j$ $(j = 1, 2, \ldots, n)$ recursively by solving the following equations:

$$0 = \frac{\Delta\vartheta}{2}\left[\mathcal{I}(\vartheta_j, \vartheta_0, \widetilde{R}_j, \widetilde{R}_j) + 2\sum_{i=1}^{j-1}\mathcal{I}(\vartheta_j, \vartheta_{j-i}, \widetilde{R}_j, \widetilde{R}_i) + \mathcal{I}(\vartheta_j, \vartheta_j, \widetilde{R}_j, \widetilde{R}_0)\right]. \tag{17}$$

Similarly, the function $P(t, x)$ in (7) can be approximated as follows:

$$P(t, x) \approx \frac{\Delta\vartheta}{2}\left[\mathcal{I}(\vartheta_n, \vartheta_0, x, \widetilde{R}_n) + 2\sum_{i=1}^{n-1}\mathcal{I}(\vartheta_n, \vartheta_{n-i}, x, \widetilde{R}_i) + \mathcal{I}(\vartheta_n, \vartheta_n, x, \widetilde{R}_0)\right]. \tag{18}$$

We summarize the procedures of simulating the optimal strategies by using the RIM in Table 1.

5.2 Comparison with the Binomial Tree Method

In this subsection, we compare our numerical solution $P(t, x)$ with the binomial tree method (BTM) developed by Cox et al. (1979). When the number of time steps is increased the BTM converges to the true value (see Theorem 6.18 in Jiang 2003), and hence with by selecting a large number of time steps one can

Table 2 Comparison of Binomial Tree Model and Recursive Integration Method

T	β	r	μ	σ	ρ	l	x	RIM	BTM	Relative Error(%)
20	0.04	0.0086	0.0784	0.2016	0.5	0.5	0.5	0.3776	0.3773	0.07
	0.04	0.0086	0.0784	0.2016	0.5	0.5	0.6	0.8801	0.8779	0.25
	0.04	0.0086	0.0784	0.2016	0.5	0.5	0.7	1.4840	1.4863	0.15
30	0.04	0.0086	0.0784	0.2016	0.5	0.5	0.5	1.1244	1.1197	0.42
	0.04	0.0086	0.0784	0.2016	0.5	0.5	0.6	2.0052	2.0057	0.02
	0.04	0.0086	0.0784	0.2016	0.5	0.5	0.7	2.9943	2.9988	0.15
40	0.04	0.0086	0.0784	0.2016	0.5	0.5	0.5	2.0666	2.0579	0.42
	0.04	0.0086	0.0784	0.2016	0.5	0.5	0.6	3.3129	3.3146	0.05
	0.04	0.0086	0.0784	0.2016	0.5	0.5	0.7	4.6702	4.6756	0.11
40	0.04	0.0086	0.0784	0.2016	0.5	0.4	0.5	2.7985	2.7967	0.06
	0.04	0.0086	0.0784	0.2016	0.4	0.5	0.5	6.6938	6.6927	0.01
	0.02	0.0086	0.0784	0.2016	0.4	0.5	0.5	1.4281	1.4263	0.12

use the result obtained by the BTM as a benchmark with which other methods can be compared. We use the binomial tree model with the number of time steps equal to 10, 000.

Since $P(t,x) = \widehat{V}(t,x) - \underline{\widehat{V}}(t,x)$, $P(t,\ x)$ can be represented as the following optimal stopping problem:

$$\begin{aligned}
P(t,x) &= \sup_{\tau\in\mathcal{U}_{t,T}} \mathbb{E}_t\left[\int_t^\tau e^{-\beta(s-t)}(\rho X_s - l)ds\right] \\
&= \sup_{\tau\in\mathcal{U}_{t,T}} \mathbb{E}_t\left[\int_\tau^T e^{-\beta(s-t)}(l - \rho X_s)ds\right] + \mathbb{E}_t\left[\int_t^T e^{-\beta(s-t)}(\rho X_s - l)ds\right] \\
&= \sup_{\tau\in\mathcal{U}_{t,T}} \mathbb{E}_t\left[\int_\tau^T e^{-\beta(s-t)}(l - \rho X_s)ds\right] + \left(\frac{1-e^{-r(T-t)}}{r}\rho x - \frac{1-e^{-\beta(T-t)}}{\beta}l\right) \\
&= \underbrace{\sup_{\tau\in\mathcal{U}_{t,T}} \mathbb{E}_t\left[e^{-\beta(\tau-t)}\left(\frac{1-e^{-\beta(T-\tau)}}{\beta}l - \frac{1-e^{-r(T-\tau)}}{r}\rho X_\tau\right)\right]}_{(A)} \\
&\quad + \left(\frac{1-e^{-r(T-t)}}{r}\rho x - \frac{1-e^{-\beta(T-t)}}{\beta}l\right).
\end{aligned} \tag{19}$$

Here, the value of part (A) in (19) can be obtained through the BTM.

Table 2 gives comparison between our numerical results obtained by the RIM and those by the BTM. The differences are small, the maximum relative error being 0.42%. We can conclude from the comparison that the RIM is fairly accurate.

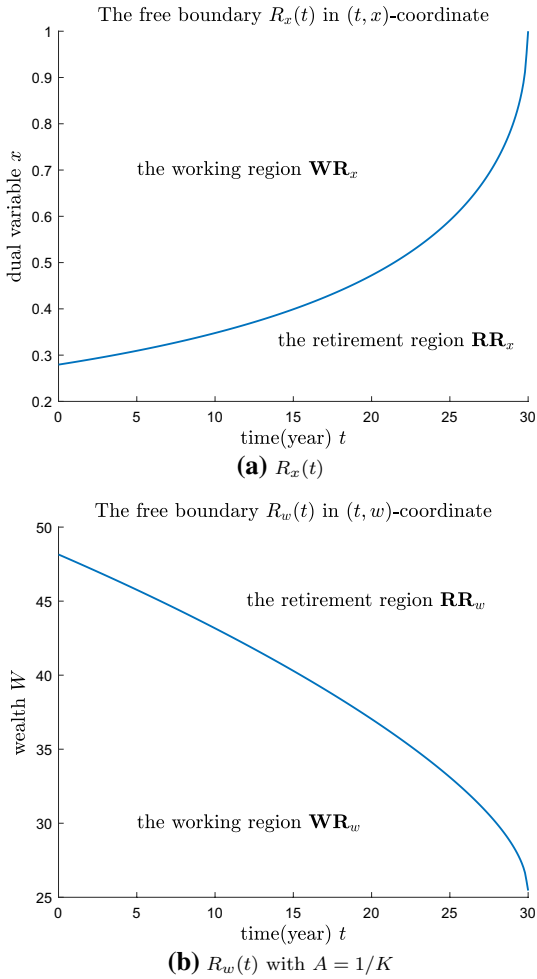


Fig. 1 The free boundaries $R_x(t)$ and $R_w(t)$

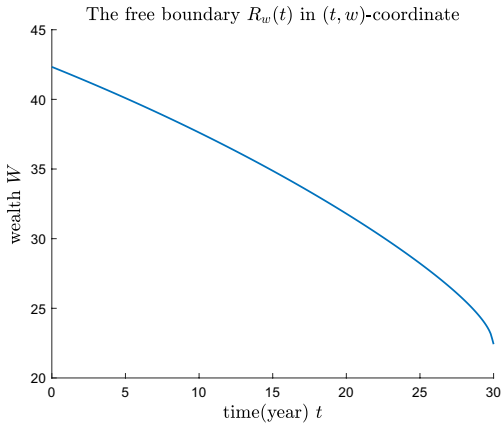
5.3 Implications

In this subsection we explain behavior of solutions to the model by using numerical solutions obtained by the RIM. We choose the values of market parameters following the empirical work by Bansal et al. (2012) and use the following parameter values as benchmark to obtain the numerical solutions:

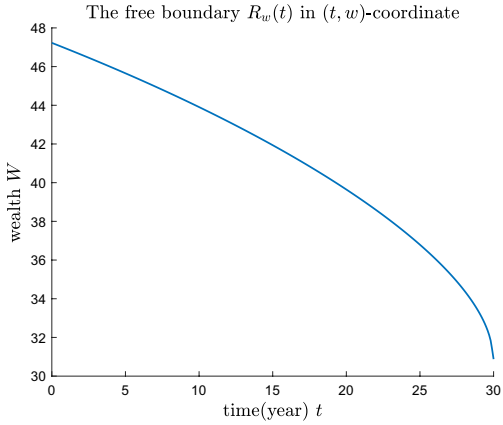
$$\beta = 0.04,\ r = 0.0086,\ \mu = 0.0784,\ \sigma = 0.2016,\ T = 30,\ T_1 = 70,\ \gamma = 3,$$
$$l = 0.5 \ \text{ and } \ \rho = 0.5.$$

Figure 1 illustrates the optimal retirement boundary $R_x(t)$ in the (t, x)- domain and the optimal retirement threshold $R_w(t)$ in the (t, w)-domain, respectively. $R_x(t)$ is strictly increasing in t consistent with the theoretical results in Lemma 6 in

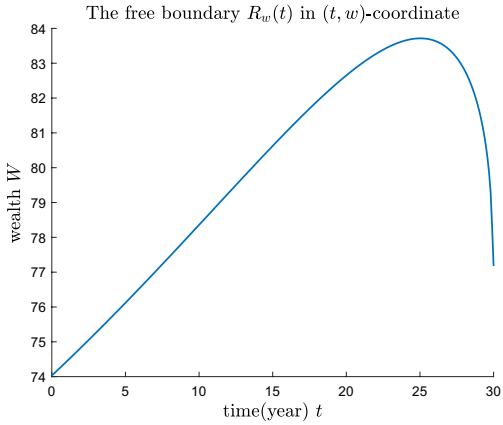
Fig. 2 The behavior of the optimal retirement threshold $R_w(t)$ with respect to A



(a) No bequest motive ($A = 0$)



(b) Weak bequest motive ($A = 1/K$)



(c) Strong bequest motive ($A = 200$)

Yang and Koo (2018). However, the monotonicity of $R_w(t)$ has been unknown (see Remark 12 in Yang and Koo 2018).

In the case of the CRRA felicity functions, the relationship between the optimal retirement boundary $R_x(t)$ and the optimal retirement threshold $R_w(t)$ is given in (13). If A, the constant denoting the bequest motive in (9), is large enough, we can observe that $R_w(t)$ is not monotone from (13) (see Fig. 2c).

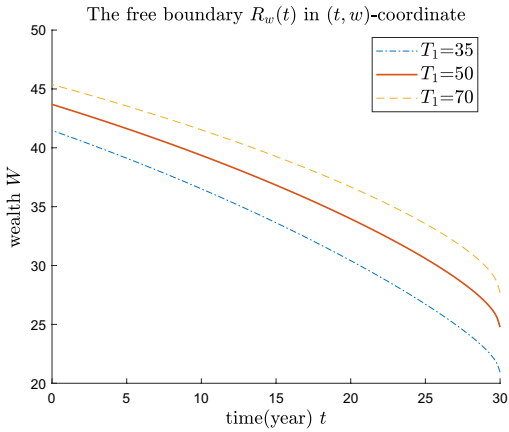
Figure 2 shows the optimal retirement threshold $R_w(t)$ for three different values of A, and we can see that $R_w(t)$ is not monotone for $A = 200$. Namely, if the agent's bequest motive is very strong, the threshold level of wealth for early retirement increases when the agent is young, however, it declines after the agent has become sufficiently old. If the agent has a very strong bequest motive the agent would like to accumulate a large amount wealth to bequeath and thus would like to work longer, and this explains the increasing behavior of the threshold when the agent is young; the threshold reverts to the declining pattern as those of agents with modest bequest motive only when the agent is old enough and the mandatory retirement is near.

Figure 3 shows the effect of the final time T_1 to the optimal retirement threshold $R_w(t)$. Depending on the agent's bequest motive, the effect of final time T_1 on the retirement strategies are different. In Fig. 3a, since the agent's bequest motive is weak, the agent's retirement decision is more affected by the wealth required for her/his consumption until the final time T_1 than her/his bequest motive. This implies that the optimal retirement threshold $R_w(t)$ increases as the final time T_1 increases. In Fig. 3c, since the agent's bequest motive is very strong, more wealth is required at retirement as the final time T_1 decreases. Thus, the optimal retirement threshold $R_w(t)$ increases as the final time T_1 decreases. These results are consistent with Theorem 2.

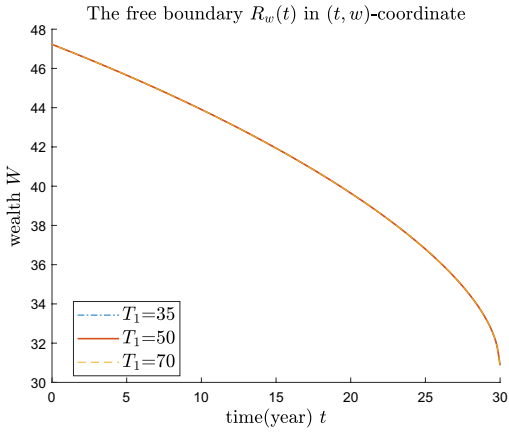
Figure 4 shows simulation results of dual process X_s^*, the optimal wealth W_s^* and optimal strategies (c^*, π^*), using Algorithm in Table 1. As shown in Fig. 4a, b, if the dual process x_s^* stays in the working region $\mathbf{WR}_x$, then the optimal wealth process W_s^* also stays in the working region $\mathbf{WR}_w$. However, if the dual process X_s^* hits the optimal retirement boundary $R_x(\cdot)$, then the wealth process W_s^* hits the optimal retirement threshold $R_w(\cdot)$ and the agent chooses to retire early. Figure 4c shows that the agent tends to invest in the risky asset aggressively just before the early retirement time. This is because the agent would like to increase the expected growth rate of wealth to approach the optimal retirement threshold fast enough to retire early.

The behavior shown by the simulation path is consistent with the optimal consumption and portfolio shown in Fig. 5. The figure shows that the marginal propensity to consume tends to decline as wealth reaches the retirement threshold and increases to a constant level after retirement. Optimal investment in the risky asset sharply increases before retirement and jumps downward after retirement. The behavior of optimal consumption is consistent with a theoretical result in Yang and Koo (2018) which says that optimal consumption in the presence of early retirement option is smaller than it would in the absence of the option (Theorem 8). The behavior of the optimal portfolio is consistent with a theoretical result by Choi and Shim (2006) in an infinite horizon model which states that an agent takes higher risk in the presence of retirement option than in its absence. We partially extend their result

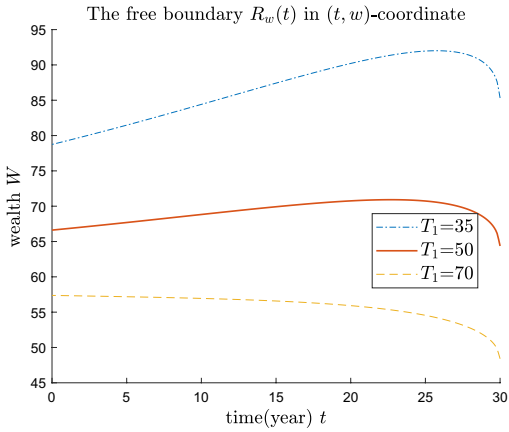
Fig. 3 The behavior of the optimal retirement threshold $R_w(t)$ with respect to T_1 when $T = 30$



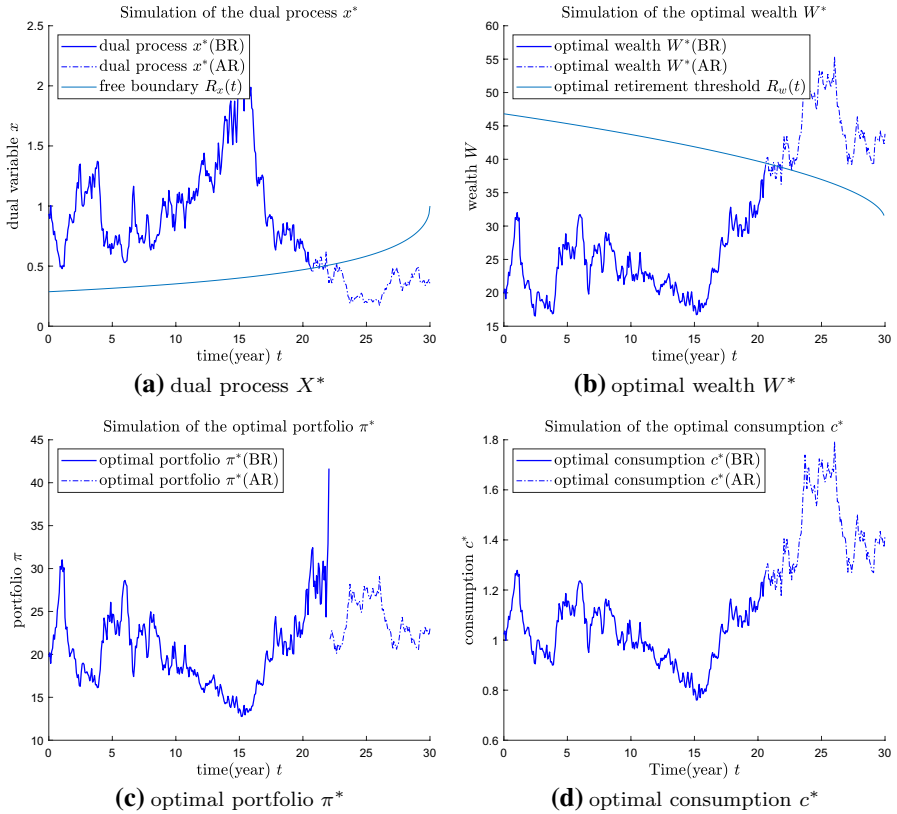
(a) No bequest motive ($A = 0$)



(b) Weak bequest motive ($A = 1/K$)



(c) Strong bequest motive ($A = 200$)



(a) dual process X^* **(b)** optimal wealth W^*

(c) optimal portfolio π^* **(d)** optimal consumption c^*

Fig. 4 Simulation results of the dual process, optimal wealth, optimal portfolio and optimal consumption (BR: before retirement, AR: after retirement)

and show that optimal portfolio jumps downward at the time of retirement in the following proposition.

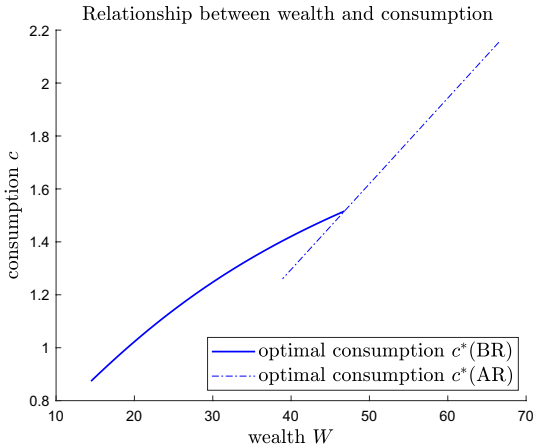
Proposition 4 *If* $\mu - r > 0$, *the optimal investment* π^* *jumps downward after retirement.*

Proof See Appendix H. □

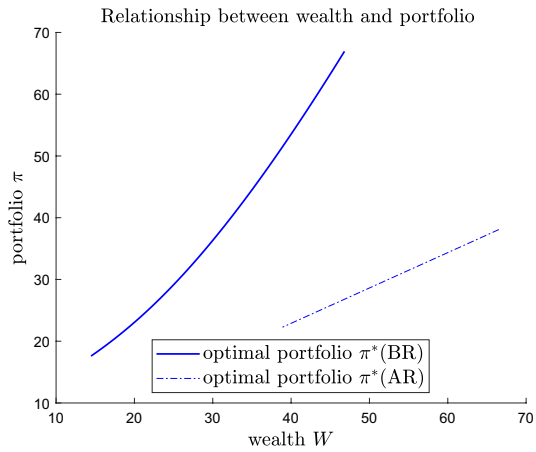
6 Concluding Remarks

In this paper we have investigated an optimal consumption/investment problem with mandatory retirement date and early retirement option. By applying the Mellin transform to the variational inequality derived in Yang and Koo (2018), we have obtained the integral equation representation for optimal retirement boundary. We

Fig. 5 Optimal consumption, portfolio and wealth



(a) Relationship between W and c



(b) Relationship between W and π

have numerically solved the integral equation by using the recursive integration method and discussed economic implications for optimal retirement strategies.

We have assumed that the utility cost of labor and wage rate are constant. Relaxation of this assumption to reflect the dependence of these on productivity, labor supply and health condition is an interesting direction of future research.

Appendix

Admissible Strategies

We require that the consumption rate c_s and the amount invested in the risky asset π_s be $\{\mathcal{F}_s\}_{s\geq t}$-adapted and the early retirement time τ be an $\mathcal{F}$-stopping time. We will denote the set of all $\mathcal{F}$-stopping times taking values in $[t, T]$ by $\mathcal{U}_{t,T}$.

We also require the following integrability condition:

$$\int_t^{T_1} (c_s + \pi_s^2) ds < \infty, \quad \text{a.s.} \quad \text{subject to} \;\; c \geq 0,$$

and for $\tau \in \mathcal{U}_{t,T}$

$$W_s > G(s)\mathbf{1}_{\{s<\tau\}}, \quad \forall s \in [t, T_1].$$

We denote the set of all strategies (τ, c, π) satisfying the conditions stated above by $\mathcal{A}(t, w)$. We define the set $\mathcal{A}^1(t, w)$ of admissible controls as follows:

$$\mathcal{A}^1(t,w) \triangleq \left\{ (\tau, c, \pi) \in \mathcal{A}(t,w) \middle| \mathbb{E}_t\left[\int_t^{T_1} e^{-\beta(s-t)} U_1^-(s, c_s) ds + e^{-\beta(T_1-t)} U_2^-(T_1, W_{T_1})\right] < \infty \right\},$$

with $U_i^- = \max\{0, -U_i\}$, $i = 1, 2$. Thus, the objective function takes a value greater than $-\infty$ with an admissible strategy.

We will use the following notations:

$$\mathcal{M}_T \triangleq \{(t,w) \mid w > G(t),\ t \in [0,T)\}, \quad \mathcal{M}_{T_1} \triangleq \{(t,w) \mid w > G(t),\ t \in [0,T_1)\},$$
$$\widetilde{\mathcal{M}}_T \triangleq \{(t,w) \mid w > G(t),\ t \in [0,T]\}, \quad \widetilde{\mathcal{M}}_{T_1} \triangleq \{(t,w) \mid w > G(t),\ t \in [0,T_1]\}.$$

Derivation of the Dual Value Function

Martingale method For any $(t, w) \in \widetilde{\mathcal{M}}_{T_1}$ and the value function $V(t, w)$ defined in (5) satisfies the following inequality:

$$V(t,w) \leq \sup_{(\tau,c,\pi)\in\mathcal{A}^1(t,w)} \mathbb{E}_t\left[\int_t^{\tau} e^{-\beta(s-t)}(U_1(s,c_s) - l) ds + e^{-\beta(\tau-t)} \underline{V}(\tau, W_\tau)\right], \quad (20)$$

where $\underline{V}(t, w)$ is the agent's value function after retirement defined by

$$\underline{V}(t,w) = \sup_{(c,\pi)\in\mathcal{A}_t^1(t,w)} \mathbb{E}_t\left[\int_t^{T_1} e^{-\beta(s-t)} U_1(s,c_s) ds + e^{-\beta(T_1-t)} U_2(T_1, W_{T_1})\right]$$

with the admissible set $\mathcal{A}_t^1 \triangleq \{(c,\pi) : (t,c,\pi) \in \mathcal{A}^1(t,w)\}$.

From the budget constraint (3) we can deduce that for any $t \in [0, T_1]$, $x > 0,\ w > 0$

$$\begin{aligned}\underline{V}(t,w) - xw &\le \sup_{(c,\pi)\in\mathcal{A}_t^1(t,w)} \mathbb{E}_t\left[\int_t^{T_1} e^{-\beta(s-t)}U_1(s,c_s)ds + e^{-\beta(T_1-t)}U_2(T_1,W_{T_1})\right] \\ &\quad - x\,\mathbb{E}_t\left[H_{T_1}W_{T_1} + \int_t^{T_1} H_s c_s ds\right] \\ &\le \mathbb{E}_t\left[\int_t^{T_1} e^{-\beta(s-t)}\widetilde{U}_1(s,X_s)ds + e^{-\beta(T_1-t)}\widetilde{U}_2(T_1,X_{T_1})\right] \triangleq \underline{\widetilde{V}}(t,x),\end{aligned} \tag{21}$$

where $X_s = xe^{\beta(s-t)}H_s$ and, for $i = 1, 2$,

$$\widetilde{U}_i(t,x) \triangleq \sup_{c>0}\left[U_i(t,c) - xc\right] = U_i(t, \mathcal{J}_{U_i}(t,x)) - x\mathcal{J}_{U_i}(t,x).$$

Here, $\mathcal{J}_{U_i}(t,\cdot)$ is the inverse function of $\partial_x U_i(t,\cdot),\ i = 1, 2$.

The function $\underline{\widetilde{V}}(t,x)$ is called the dual value function of the agent's problem after retirement. According to Karatzas and Shreve (1998) or Yang and Koo (2018), for any $x > 0$, there exists a unique $w > 0$ such that the inequalities in (21) hold as equalities, and $\underline{V}(t,w)$ and $\underline{\widetilde{V}}(t,x)$ satisfy the following duality relationship: for any $t \in [0, T_1]$, $x > 0$, $w > 0$,

$$\underline{\widetilde{V}}(t,x) = \sup_{w>0}\left(\underline{V}(t,w) - xw\right),\quad \underline{V}(t,w) = \inf_{x>0}\left(\underline{\widetilde{V}}(t,x) + xw\right).$$

Since the process X_s follows

$$dX_s = (\beta - r)X_s ds - \theta X_s dB_s,\quad \forall s \in [t, T_1],\ X_t = x,$$

the Feynman-Kac formula implies that $\underline{\widetilde{V}}$ satisfies the following partial differential equation (PDE):

$$\begin{cases} -\partial_t\underline{\widetilde{V}}(t,x) - \mathcal{L}\underline{\widetilde{V}}(t,x) = \widetilde{U}_1(t,x) & \text{in } \mathcal{N}_{T_1}, \\ \underline{\widetilde{V}}(T_1,x) = \widetilde{U}_2(T_1,x), & \forall x \in (0,+\infty), \end{cases}$$

where

$$\mathcal{L} \triangleq \frac{1}{2}\theta^2 x^2\partial_{xx} + (\beta - r)x\partial_x - \beta,$$

and

$$\begin{aligned}\mathcal{N}_T &\triangleq [0,T)\times(0,+\infty), & \widetilde{\mathcal{N}}_T &\triangleq [0,T]\times(0,+\infty), \\ \mathcal{N}_{T_1} &\triangleq [0,T_1)\times(0,+\infty), & \widetilde{\mathcal{N}}_{T_1} &\triangleq [0,T_1]\times(0,+\infty).\end{aligned}$$

For any $(t,w) \in \widetilde{\mathcal{M}}_T$, $x > 0$, inequalities (3) and (20) imply that

$$
\begin{aligned}
&V(t,w) - x(w - G(t)) \\
&\le \sup_{(\tau,c,\pi)\in\mathcal{A}^1(t,w)} \mathbb{E}_t\left[\int_t^\tau e^{-\beta(s-t)}(U_1(s,c_s) - l) + e^{-\beta(\tau-t)}\underline{V}(\tau, W_\tau)\right] \\
&\quad - x\,\mathbb{E}_t\left[H_\tau(W_\tau - G(\tau)) + \int_t^\tau H_s c_s ds\right] \\
&\le \sup_{\tau\in\mathcal{U}_{t,T}} \mathbb{E}_t\left[\int_t^\tau e^{-\beta(s-t)}\widehat{U}_1(s,X_s)ds + e^{-\beta(\tau-t)}(\widetilde{\underline{V}}(\tau,X_\tau) + X_\tau G(\tau))\right] \\
&= \sup_{\tau\in\mathcal{U}_{t,T}} \mathbb{E}_t\left[\int_t^\tau e^{-\beta(s-t)}\widehat{U}_1(s,X_s)ds + e^{-\beta(\tau-t)}\widehat{\underline{V}}(\tau,X_\tau)\right] \triangleq \widehat{V}(t,x),
\end{aligned}
\tag{22}
$$

where

$$
\begin{aligned}
\widehat{U}_1(t,x) &= \sup_{c>0}\left[U_1(t,c) - xc\right] - l = \widetilde{U}_1(t,x) - l, \\
\widehat{\underline{V}}(t,x) &= \widetilde{\underline{V}}(t,x) + xG(t).
\end{aligned}
$$

Summary of Results in Yang and Koo (2018)

In this section, we summarize the theoretical results in Yang and Koo (2018). They have reformulated the agent's optimization problem into a variational inequality (VI) by using the martingale method.

By the budget constraint (3) and the martingale method, we can define the dual value functions $\widetilde{\underline{V}}(t,x)$ and $\widetilde{V}(t,x)$ of the agent's problem after retirement and before retirement, respectively, as follows (for details, see Appendix B):

$$
\begin{aligned}
\widetilde{\underline{V}}(t,x) &\triangleq \mathbb{E}_t\left[\int_t^{T_1} e^{-\beta(s-t)}\widetilde{U}_1(s,X_s)ds + e^{-\beta(T_1-t)}\widetilde{U}_2(T_1,X_{T_1})\right], \\
\widehat{V}(t,x) &\triangleq \sup_{\tau\in\mathcal{U}_{t,T}} \mathbb{E}_t\left[\int_t^\tau e^{-\beta(s-t)}\widehat{U}_1(s,X_s)ds + e^{-\beta(\tau-t)}\widehat{\underline{V}}(\tau,X_\tau)\right],
\end{aligned}
$$

where $\mathcal{U}_{t,T}$ is defined in Appendix A, $X_s = xe^{\beta(s-t)}H_s$, and, for $i = 1, 2$,

$$
\begin{aligned}
\widetilde{U}_i(t,x) &\triangleq \sup_{c>0}\left[U_i(t,c) - xc\right] = U_i(t,\mathcal{J}_{U_i}(t,x)) - x\mathcal{J}_{U_i}(t,x), \\
\widehat{U}_1(t,x) &\triangleq \sup_{c>0}\left[U_1(t,c) - xc\right] - l = \widetilde{U}_1(t,x) - l,
\end{aligned}
$$

$\mathcal{J}_{U_i}(t,\cdot)$ is the inverse function of $\partial_x U_i(t,\cdot)$, $i = 1, 2$ and

$$
\widehat{\underline{V}}(t,x) = \widetilde{\underline{V}}(t,x) + xG(t).
$$

According to Yang and Koo (2018), the following duality relationship is established.

Theorem 3 [*Theorem* 1 *in Yang and Koo* (2018)] *For any* $t \in [0, T]$, $x > 0$, $w > G(t)$, *the following duality relationship holds*:

$$\widehat{V}(t,x) = \sup_{w>G(t)} (V(t,w) - x(w - G(t)), \quad V(t,w) = \inf_{x>0} \Big(\widehat{V}(t,x) + x(w - G(t)) \Big).$$

They have also shown that the dual value function $\widehat{V}(t,x)$ defined in (22) is the unique strong solution of the following variational inequality (VI): (we refer to Yang and Koo (2018) for details)

$$\begin{cases} -\partial_t \widehat{V}(t,x) - \mathcal{L}\widehat{V}(t,x) = \widehat{U}_1(t,x), & \text{if } \widehat{V}(t,x) > \underline{\widehat{V}}(t,x) \text{ and } (t,x) \in \mathcal{N}_T, \\ partial_t \widehat{V}(t,x) - \mathcal{L}\widehat{V}(t,x) \geq \widehat{U}_1(t,x), & \text{if } \widehat{V}(t,x) = \underline{\widehat{V}}(t,x) \text{ and } (t,x) \in \mathcal{N}_T, \\ \widehat{V}(T,x) = \underline{\widehat{V}}(T,x), & \forall x \in (0,+\infty), \end{cases}$$

where the domain $\mathcal{N}_T$ and the differential operator $\mathcal{L}$ are defined in Appendix B. Moreover, $\widehat{V}(t,x)$ is piecewise smooth.

Review of the Mellin Transformation

In this appendix we briefly review the definition and properties of the Mellin transformation. The reader can refer to Sneddon (1972) for more details.

Definition 1 For a locally integrable function $f(x)$ in $(0, +\infty)$, the Mellin transform of $M[f](y)$ of $f(x)$ is defined by

$$M[f](y) = \int_0^\infty f(x) x^{y-1} dx, \quad y \in \mathbb{C},$$

and if this integral converges for $a_1 < \mathcal{R}(y) < a_2$, then the inverse Mellin transform is given by

$$f(x) = M^{-1}[M[f]](x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} M[f](y) x^{-y} dy.$$

Here, $\mathcal{R}(y)$ is the real part of complex number y.

Proposition 5 *Let* $f(x)$ *be a locally integrable function in* $(0, +\infty)$. *Suppose that the Mellin transform* $M[f](y)$ *of* $f(x)$ *exists for* $a_1 < \mathcal{R}(y) < a_2$. *Then, for any positive integer* n,

$$M\left[\left(x\frac{\partial}{\partial x}\right)^n f\right](y) = (-y)^n M[f](y).$$

Proposition 6 *For* $\alpha \in \mathbb{C}$ *with* $\mathcal{R}(\alpha) > 0$ *and* $b \in \mathbb{R}$, *the inverse Mellin transform of* $f(y) = e^{\alpha(y+b)^2}$ *is given by*

$$M^{-1}[f](x) = \frac{1}{2\sqrt{\pi\alpha}} x^b e^{-\frac{1}{4\alpha}(\log x)^2}.$$

Proof of Proposition 2

We consider the Mellin transform $M[P](t, y)$ of $P(t, x)$:

$$M[P](t, y) = \int_0^\infty P(t, x) x^{y-1} dx.$$

The inverse Mellin transform is given by

$$P(t, x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} M[P](t, y) x^{-y} dy.$$

Then, we transform the non-homogeneous PDE (6) into the following ordinary differential equation (ODE):

$$\begin{aligned} &\frac{dM[P]}{dt}(t, y) + \frac{1}{2}\theta^2 Q(y) M[P](t, y) = M[F](t, y), \\ &Q(y) = y^2 + (1 - k_2)y - k_1, \end{aligned} \tag{23}$$

where $k_1 = 2\beta/\theta^2$, $k_2 = 2(\beta - r)/\theta^2$.

Since $M[P](T, y) = 0$, the non-homogeneous ODE (23) yields

$$M[P](t, y) = -\int_t^T e^{\frac{1}{2}\theta^2 Q(y)(\eta - t)} M[F](\eta, y) d\eta$$

and

$$P(t, x) = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \int_t^T e^{\frac{1}{2}\theta^2 Q(y)(\eta - t)} M[F](\eta, y) x^{-y} d\eta dy.$$

Let us consider the following function

$$\mathcal{G}(t, x) \triangleq \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}\theta^2 Q(y) t} x^{-y} dy.$$

Then,

$$\mathcal{G}(t, x) = e^{-\frac{1}{2}\theta^2 \left\{ \left(\frac{1-k_2}{2} \right)^2 + k_1 \right\} t} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}\theta^2 \left(y + \frac{1-k_2}{2} \right)^2 t} x^{-y} dy.$$

By Proposition 6 in Appendix D, we can obtain

$$\mathcal{G}(t,x) = e^{-\frac{1}{2}\theta^2\left\{\left(\frac{1-k_2}{2}\right)^2+k_1\right\}t}\frac{x^{\frac{1-k_2}{2}}}{\theta\sqrt{2\pi t}}\exp\left\{-\frac{1}{2}\frac{(\log x)^2}{\theta^2 t}\right\}.$$

Since $e^{\frac{1}{2}\theta^2\mathcal{Q}(y)t}$ and $M[F](t,y)$ are the Mellin transforms of $\mathcal{G}(t,x)$ and $F(t,x)$, respectively, the Mellin convolution theorem in Proposition 1 implies

$$P(t,x) = -\int_t^T\int_0^\infty F(\eta,u)\mathcal{G}(\eta-t,\frac{x}{u})\frac{1}{u}du d\eta.$$

For any $\alpha \in \mathbb{R}$ and $b > 0$, by direct computation we can obtain

$$\int_b^\infty u^{-\alpha}\mathcal{G}(t,\frac{x}{u})\frac{1}{u}du = x^{-\alpha}e^{-\frac{1}{2}\theta^2\{k_1-(1-k_2)\alpha-\alpha^2\}t}\mathcal{N}\left(\frac{\log\frac{x}{b}-\theta^2\left(\frac{1-k_2}{2}+\alpha\right)t}{\theta\sqrt{t}}\right).$$

Thus we have

$$\begin{aligned}P(t,x) = \rho x\int_t^T e^{-r(\eta-t)}\mathcal{N}\left(d^+(\eta-t,\frac{x}{R_x(\eta)})\right)d\eta \\ - l\int_t^T e^{-\beta(\eta-t)}\mathcal{N}\left(d^-(\eta-t,\frac{x}{R_x(\eta)})\right)d\eta.\end{aligned}$$

□

Proof of Proposition 3

By Theorem 1 in Yang and Koo (2018) and the first-order condition, for $w > G(t)$, there exists a unique $x^* = x^*(t,w)$ such that

$$V(t,w) = \inf_{x>0}\left[\widehat{V}(t,x)+x(w-G(t))\right] = \widehat{V}(t,x^*(t,w))+x^*(t,w)(w-G(t)).$$

and

$$w - G(t) = -\partial_x\widehat{V}(t,x^*(t,w)), \tag{24}$$

that is,

$$x^*(t,w) = \mathcal{J}_{\widehat{V}}(t,-w+G(t)),$$

where $\mathcal{J}_{\widehat{V}}(t,\cdot)$ is the inverse function of $\partial_x\widehat{V}(t,\cdot)$.

Since $\widehat{V}(t,x) = \underline{\widetilde{V}}(t,x)+xG(t)+P(t,x)$ and $-w+G(t) = \partial_x\widehat{V}(t,x^*(t,w))$,

$$w(t,x^*) = -\partial_x\widehat{V}(t,x^*)+G(t) = -\partial_x\underline{\widetilde{V}}(t,x^*)-\partial_x P(t,x^*). \tag{25}$$

By the explicit form of $\underline{\widetilde{V}}(t,x)$ in (10),

$$\partial_x \underline{\widetilde{V}} = -x^{-\frac{1}{\gamma}} \left(A \cdot e^{-K(T_1-t)} + \frac{1-e^{-K(T_1-t)}}{K} \right). \tag{26}$$

From (7), (24), (25), and (26), we can see that $x^* = x^*(t,w)$ is a unique solution to the following integral equation:

$$\begin{aligned} w = {} & (x^*)^{-\frac{1}{\gamma}} \left(A \cdot e^{-K(T_1-t)} + \frac{1-e^{-K(T_1-t)}}{K} \right) \\ & + \frac{l}{\theta x^* \sqrt{2\pi}} \int_t^T \exp\left\{ -\beta(\eta - t) - \frac{d^-(\eta - t, \frac{x^*}{R_x(\eta)})^2}{2} \right\} \frac{1}{\sqrt{\eta - t}} d\eta \\ & - \frac{\rho}{\theta \sqrt{2\pi}} \int_t^T \exp\left\{ -r(\eta - t) - \frac{d^+(\eta - t, \frac{x^*}{R_x(\eta)})^2}{2} \right\} \frac{1}{\sqrt{\eta - t}} d\eta \\ & - \rho \int_t^T e^{-r(\eta - t)} \mathcal{N}\left(d^+\left(\eta - t, \frac{x^*}{R_x(\eta)} \right) \right) d\eta. \end{aligned}$$

Also, we obtain the value function V as follows:

$$\begin{aligned} V(t,w) &= \widehat{V}(t,x^*) - x^* \partial_x \widehat{V}(t,x^*) \\ &= \left(\underline{\widetilde{V}}(t,x^*) - x^* \partial_x \underline{\widetilde{V}}(t,x^*) \right) + \left(P(t,x^*) - x^* \partial_x P(t,x^*) \right) \\ &= \frac{1}{1-\gamma} (x^*)^{-\frac{1-\gamma}{\gamma}} \left(A \cdot e^{-K(T_1-t)} + \frac{1-e^{-K(T_1-t)}}{K} \right) \\ &\quad - \frac{\rho x^*}{\theta \sqrt{2\pi}} \int_t^T \exp\left\{ -r(\eta - t) - \frac{d^+(\eta - t, \frac{x^*}{R_x(\eta)})^2}{2} \right\} \frac{1}{\sqrt{\eta - t}} d\eta \\ &\quad + \frac{l}{\theta \sqrt{2\pi}} \int_t^T \exp\left\{ -\beta(\eta - t) - \frac{d^-(\eta - t, \frac{x^*}{R_x(\eta)})^2}{2} \right\} \frac{1}{\sqrt{\eta - t}} d\eta \\ &\quad - l \int_t^T e^{-\beta(\eta - t)} \mathcal{N}\left(d^-\left(\eta - t, \frac{x^*}{R_x(\eta)} \right) \right) d\eta. \end{aligned}$$

□

Proof of Theorem 1

From Theorem 1 in Yang and Koo (2018), the optimal consumption c_s^* is given by

$$c_s^* = \mathcal{J}_{U_1}(s, X_s^*) = (X_s^*)^{-\frac{1}{\gamma}} \quad \text{with} \quad X_s^* = x^*(t,w)e^{\beta(s-t)}H_s.$$

By Proposition 3 and time-consistency of Problem 1, we can easily derive

$$\begin{aligned} W_s^* = {} & (X_s^*)^{-\frac{1}{\gamma}}\left(A \cdot e^{-K(T_1-s)} + \frac{1-e^{-K(T_1-s)}}{K}\right) \\ & + \frac{l}{\theta X_s^* \sqrt{2\pi}} \int_s^T \exp\left\{-\beta(\eta - s) - \frac{d^-(\eta - s, \frac{X_s^*}{R_x(\eta)})^2}{2}\right\} \frac{1}{\sqrt{\eta - s}} d\eta \\ & - \frac{\rho}{\theta\sqrt{2\pi}} \int_s^T \exp\left\{-r(\eta - s) - \frac{d^+(\eta - s, \frac{X_s^*}{R_x(\eta)})^2}{2}\right\} \frac{1}{\sqrt{\eta - s}} d\eta \\ & - \rho \int_s^T e^{-r(\eta - s)} \mathcal{N}\left(d^+\left(\eta - s, \frac{X_s^*}{R_x(\eta)}\right)\right) d\eta. \end{aligned}$$

From (25) we know that

$$W_s^* = -\partial_x \underline{\widetilde{V}}(s, X_s^*) - \partial_x P(s, X_s^*),$$

and the dynamic of X_s^* is given by

$$dX_s^* = (\beta - r)X_s^* ds - \theta X_s^* dB_s.$$

Thus, by Itô's formula, the local martingale term[5] of dW_s^* is given by

$$\theta X_s^* \left(\partial_{xx} \underline{\widetilde{V}}(s, X_s^*) + \partial_{xx} P(s, X_s^*)\right) dB_s. \tag{27}$$

Comparing (27) with the wealth evolution Eq. (2) via stopping times, we deduce that the optimal portfolio process π_s^* is given by

$$\pi_s^* = \frac{\theta}{\sigma} X_s^* \left(\partial_{xx} \underline{\widetilde{V}}(s, X_s^*) + \partial_{xx} P(s, X_s^*)\right) = -\frac{\theta}{\sigma} X_s^* \cdot \partial_x W_s^*. \tag{28}$$

□

Proof of Proposition 4

It follows from (28) that the optimal portfolio for $w < R_w(t)$ $(x > R_x(t))$ is given by

[5] The term in (27) is indeed a martingale, which is implied in the proof of the verification theorem (Theorem 1 and Lemma 2) in Yang and Koo (2018).

$$\pi_t^* = \frac{\theta}{\sigma} x^* \Big(\partial_{xx} \underline{\widetilde{V}}(t, x^*) + \partial_{xx} P(t, x^*) \Big).$$

On the other hand, the optimal portfolio π_t^* for $w \geq R_w(t)$ $(x \leq R_x(t))$ is given by

$$\pi_t^* = \frac{\theta}{\sigma} x^* \partial_{xx} \underline{\widetilde{V}}(t, x^*).$$

Clearly,

$$\lim_{x \to R_x(t)^-} \pi_t^* = \frac{\theta}{\sigma} R_x(t) \partial_{xx} \underline{\widetilde{V}}(t, R_x(t)).$$

Note that for $(t, x) \in \mathbf{WR}_x$,

$$\begin{aligned} \partial_{xx} P(t, x) &= \frac{2}{\theta^2 x^2} (\partial_t P + \mathcal{L} P)(t, x) - \frac{2}{\theta^2 x^2} \big(\partial_t P + (\beta - r) x \partial_x P - \beta P \big)(t, x) \\ &= \frac{2}{\theta^2 x^2} (l - \rho x) - \frac{2}{\theta^2 x^2} \big(\partial_t P + (\beta - r) x \partial_x P - \beta P \big)(t, x). \end{aligned}$$

Since $P \in C^\infty(\{(t, x) \mid x \geq R_x(t), t \in [0, T]\})$ and $P(t, R_x(t)) = \partial_t P(t, R_x(t)) = \partial_x P(t, R_x(t)) = 0$ (see Lemma 4 in Yang and Koo 2018), we deduce that

$$\lim_{x \to R_x(t)^+} \partial_{xx} P(t, x) = \frac{2}{\theta^2 (R_x(t))^2} (l - \rho R_x(t)).$$

Since $R_x(t) < l/\rho$ for $(t, x) \in \mathbf{WR}_x$ (see Lemmas 5 and 6 in Yang and Koo 2018), we have

$$\begin{aligned} \lim_{x \to R_x(t)^+} \pi_t^* &= \frac{\theta}{\sigma} R_x(t) \partial_{xx} \underline{\widetilde{V}}(t, R_x(t)) + \frac{2}{\theta \sigma R_x(t)} (l - \rho R_x(t)) \\ &> \frac{\theta}{\sigma} R_x(t) \partial_{xx} \underline{\widetilde{V}}(t, R_x(t)) \\ &= \lim_{x \to R_x(t)^-} \pi_t^*. \end{aligned}$$

□

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