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On Certain Type of Difference Polynomials of Meromorphic Functions

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Abstract—In this paper, the authors investigate zeros of difference polynomials of the form $f(z)^n H(z, f) - s(z)$, where $f(z)$ is a meromorphic function, $H(z, f)$ is a difference polynomial of $f(z)$ and $s(z)$ is a small function. The authors first obtain some inequalities for the relationship of the zero counting function of $f(z)^n H(z, f) - s(z)$ and the characteristic function and pole counting function of $f(z)$. Based on the above inequalities, the authors then establish some difference analogues of a classical result of Hayman for meromorphic functions.

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1. INTRODUCTION

Let $f(z)$ be a meromorphic function in the complex plane $\mathbb{C}$. We assume that the reader is familiar with the basic notions of Nevanlinna's theory (see [1–2]). We use $\sigma(f)$ to denote the order of growth of $f(z)$, $\sigma_2(f)$ to denote the hyper order of $f(z)$, and $\delta(\infty, f)$ to denote the Nevanlinna deficiency of $f(z)$.

Many authors have been interested in the value distribution of differential polynomials of meromorphic functions and obtained fruitful results. In particular, Hayman proved the following results.

Theorem A. (see [3]) If $f(z)$ is a transcendental entire function and $n \geqslant 2$, then $f'(z)f(z)^n$ assumes all finite values except possibly zero infinitely often.

Theorem B. (see [3]) If $f(z)$ is a transcendental meromorphic function and $n \geqslant 3$, then $f'(z)f(z)^n$ assumes all finite values except possibly zero infinitely often.

The difference analogues of Nevanlinna value distribution theory have been established in [4–8]. Using these theories, many authors considered the value distribution of difference polynomials of entire functions. In particular, the following result can be viewed as a difference analogue of Theorem A.

Theorem C. (see [9–11]) Let $f(z)$ be a transcendental entire function of finite order, and c be a non-zero complex constant. Then for $n \geqslant 2$, $f(z)^n f(z + c)$ assumes every non-zero value $a \in \mathbb{C}$ infinitely often.

For meromorphic functions, it is easy to see that a direct difference analogue of Theorem B cannot hold. Indeed, take $f(z) = \tan z$. Then

$$f(z)^3 f\left(z + \frac{\pi}{2}\right) = -\tan^2 z$$

never takes the value 1.

A natural question is: What can be said about the conclusion of Theorem B if $f'(z)$ of Theorem B is replaced by $f(z+\eta)(\eta \in \mathbb{C}/\{0\})$? For this question, the following results are obtained in [12–13].

Theorem D. (see [12]) Let $f(z)$ be a transcendental meromorphic function such that its order $\sigma(f) < \infty$, let η be a non-zero complex number, and let $n \geqslant 1$ be an integer. Suppose that $P(z) \not\equiv 0$ is a polynomial. Then

$$\begin{aligned}&\overline{N}\left(r, \frac{1}{f(z)^n f(z+\eta) - P(z)}\right)\\ &\geqslant nT(r, f(z)) + m(r, f(z)) - 2\overline{N}(r, f(z)) - 2\overline{N}\left(r, \frac{1}{f(z)}\right) - N\left(r, \frac{1}{f(z)}\right) + o\left(\frac{T(r, f(z))}{r^{1-\varepsilon}}\right) + O(1),\end{aligned}$$

as $r \notin E$ and $r \to \infty$, where E denotes a set of finite logarithmic measure.

Theorem E. (see [12–13]) Let $f(z)$ be a transcendental meromorphic function such that its order $\sigma(f) < \infty$, let η be a non-zero complex number, and let $n \geqslant 6$ be an integer. Suppose that $P(z) \not\equiv 0$ is a polynomial. Then $f(z)^n f(z+\eta) - P(z)$ has infinitely many zeros.

We pose three questions related to Theorems D and E.

Question 1. What happens if $f(z+\eta)$ is generalized to difference polynomials?

Question 2. Is it possible to reduce the condition "$n \geqslant 6$" in Theorem E?

Question 3. Applying Theorem D, we cannot get Theorem C. So Theorem D is not a direct improvement of Theorem C to the case of meromorphic functions. Is it possible to obtain such a direct improvement?

2. RESULTS

To formulate our results, we introduce some notations.

The difference polynomial $H(z, f)$ of a meromorphic function $f(z)$ is defined by

$$H(z, f) = \sum_{\lambda \in J} a_\lambda(z) \prod_{j=1}^{\tau_\lambda} f(z + \delta_{\lambda,j})^{\mu_{\lambda,j}}, \tag{2.1}$$

where J is an index set, $\delta_{\lambda,j}$ are complex constants, $\mu_{\lambda,j}$ are non-negative integers, and the coefficients $a_\lambda(z)(\not\equiv 0)$ are small meromorphic functions of $f(z)$.

The degree of the monomial $a_\lambda(z) \prod_{j=1}^{\tau_\lambda} f(z+\delta_{\lambda,j})^{\mu_{\lambda,j}}$ is defined by

$$d_\lambda = \sum_{j=1}^{\tau_\lambda} \mu_{\lambda,j}. \tag{2.2}$$

The degree of $H(z, f)$ is defined by

$$d_H = \deg_f H(z, f) = \max_{\lambda \in J} d_\lambda. \tag{2.3}$$

Let the different $\delta_{\lambda,j}$ in $H(z,f)$ be $\delta_1, \cdots, \delta_m$, and let

$$\chi = \begin{cases} 1, \text{ if } \delta_s = 0 \text{ for some } s \in \{1, \cdots, m\}, \\ 0, \text{ if } \delta_t \neq 0 \text{ for all } t = 1, \cdots, m. \end{cases} \tag{2.4}$$

In this paper, we consider Questions 1–3 in the introduction section and obtain some results using different methods than [12–13]. Among our results, Theorem 2.1 and Corollary 2.1 answer Questions 1 and 3, and Corollary 2.2, Theorems 2.2, 2.3 and their corollaries offer partial results concerning Question 2.

Theorem 2.1. Let $f(z)$ be a transcendental meromorphic function satisfying $\sigma_2(f) < 1$, let $H(z,f)(\not\equiv 0)$ be a difference polynomial in $f(z)$ of the form (2.1) with $m \geqslant 1$ different $\delta_{\lambda,j}$, let d_H and χ be defined by (2.3) and (2.4) respectively, and let $n > md_H$ be an integer. If $s(z) \not\equiv 0$ is a small meromorphic function of $f(z)$, then

$$\begin{aligned} &2\overline{N}\Big(r, \frac{1}{f(z)^n H(z,f) - s(z)}\Big) \\ &\geqslant (n-1)T(r,f(z)) - (m-\chi)d_H N(r,f(z)) - (2m+1-2\chi)\overline{N}(r,f(z)) + S(r,f). \end{aligned}$$

For a difference monomial

$$F(z,f) = f(z+c_1)^{i_1} f(z+c_2)^{i_2} \cdots f(z+c_m)^{i_m}, \tag{2.5}$$

where $m \geqslant 1$ is an integer, $i_1, i_2, \cdots, i_m$ are positive integers, and $c_1, c_2, \cdots, c_m$ are different non-zero complex constants, we obtain the following corollary.

Corollary 2.1. Let $f(z)$ be a transcendental meromorphic function satisfying $\sigma_2(f) < 1$, let $F(z,f)$ be a difference monomial in $f(z)$ of the form (2.5), let $\deg_f F(z,f) = d_F$, and let $n > d_F$ be an integer. If $s(z) \not\equiv 0$ is a small meromorphic function of $f(z)$, then

$$\begin{aligned} &2\overline{N}\Big(r, \frac{1}{f(z)^n F(z,f) - s(z)}\Big) \\ &\geqslant (n-1)T(r,f(z)) - d_F N(r,f(z)) - (2m+1)\overline{N}(r,f(z)) + S(r,f). \end{aligned}$$

Especially, if $F(z,f) = f(z+\eta)$ $(\eta \in \mathbb{C}/\{0\})$, then

$$2\overline{N}\Big(r, \frac{1}{f(z)^n f(z+\eta) - s(z)}\Big) \geqslant (n-1)T(r,f(z)) - N(r,f(z)) - 3\overline{N}(r,f(z)) + S(r,f).$$

Theorem 2.1 and Corollary 2.1 generalize Theorem D to difference polynomials and are direct improvements of Theorem C to meromorphic functions. Furthermore, using Corollary 2.1 we can get Corollary 2.2, which is a version to reduce the condition "$n \geqslant 6$" in Theorem E.

Corollary 2.2. Let $f(z)$ be a transcendental meromorphic function satisfying $\sigma_2(f) < 1$ and $\delta(\infty, f(z)) > \frac{1}{2}$, let η be a non-zero complex number, and let $n \geqslant 3$ be an integer. If $s(z) \not\equiv 0$ is a small meromorphic function of $f(z)$, then $f(z)^n f(z+\eta) - s(z)$ has infinitely many zeros.

For the difference monomial (2.5), if the poles and zeros of $f(z)$ satisfy some conditions, we can obtain a better estimate.

Theorem 2.2. Let $f(z)$ be a transcendental meromorphic function satisfying $\sigma_2(f) < 1$, let $F(z,f)$ be a difference monomial in $f(z)$ of the form (2.5), let $\deg_f F(z,f) = d_F$, and let $n > d_F$ be an integer.

Suppose that the poles z_i and zeros z_j of $f(z)$ satisfy $z_i - z_j \neq c_l$ $(l = 1, \cdots, m)$, except for finitely many exceptional poles and zeros. If $s(z) \not\equiv 0$ is a small meromorphic function of $f(z)$, then

$$2\overline{N}\Big(r, \frac{1}{f(z)^n F(z,f) - s(z)}\Big) \geqslant (n-1)T(r, f(z)) - (2m+1)\overline{N}(r, f(z)) + S(r,f).$$

Corollary 2.3. Let $f(z)$ be a transcendental meromorphic function satisfying $\sigma_2(f) < 1$, let η be a non-zero complex number, and let $n \geqslant 5$ be an integer. Suppose that the poles z_i and zeros z_j of $f(z)$ satisfy $z_i - z_j \neq c_l$ $(l = 1, \cdots, m)$, except for finitely many exceptional poles and zeros. If $s(z) \not\equiv 0$ is a small meromorphic function of $f(z)$, then $f(z)^n f(z+\eta) - s(z)$ has infinitely many zeros.

At last, we estimate the zeros of $f(z)^n H(z,f) - s(z)$ under the assumption that $f(z)$ has two Borel exceptional values.

Theorem 2.3. Let $f(z)$ be a finite order transcendental meromorphic function with two Borel exceptional values $a, b \in \mathbb{C} \cup \{\infty\}$, let $H(z,f)(\not\equiv 0)$ be a difference polynomial in $f(z)$ of the form (2.1) with $m \geqslant 1$ different $\delta_{\lambda,j}$, let d_λ and d_H be defined by (2.2) and (2.3) respectively, and let n be a positive integer. Suppose that $s(z) \not\equiv 0$ is a small meromorphic function of $f(z)$.

(i) If $a, b \in \mathbb{C}$, $a^n \sum\limits_{\lambda \in J} a_\lambda(z) a^{d_\lambda} - s(z) \not\equiv 0$, $b^n \sum\limits_{\lambda \in J} a_\lambda(z) b^{d_\lambda} - s(z) \not\equiv 0$ and $n > m d_H$, then

$$N\Big(r, \frac{1}{f(z)^n H(z,f) - s(z)}\Big) \geqslant (n - m d_H) T(r, f(z)) + S(r,f).$$

(ii) If $a \in \mathbb{C}$, $b = \infty$ and $a^n \sum\limits_{\lambda \in J} a_\lambda(z) a^{d_\lambda} - s(z) \not\equiv 0$, then

$$N\Big(r, \frac{1}{f(z)^n H(z,f) - s(z)}\Big) \geqslant n T(r, f(z)) + S(r,f).$$

From Theorem 2.3, we can easily get the following corollary, which reduces the condition "$n \geqslant 6$" to "$n \geqslant 2$" for meromorphic functions with two Borel exceptional values.

Corollary 2.4. Let $f(z)$ be a finite order transcendental meromorphic function with two Borel exceptional values $a, b \in \mathbb{C} \cup \{\infty\}$, let η be a non-zero complex number, and let $n \geqslant 2$ be an integer. Suppose that $s(z) \not\equiv 0$ is a small meromorphic function of $f(z)$, and that one of the following two conditions holds:

(i) $a, b \in \mathbb{C}$, $a^{n+1} - s(z) \not\equiv 0$ and $b^{n+1} - s(z) \not\equiv 0$;

(ii) $a \in \mathbb{C}$, $b = \infty$ and $a^{n+1} - s(z) \not\equiv 0$.

Then $f(z)^n f(z+\eta) - s(z)$ has infinitely many zeros.

3. PROOF OF THEOREM 2.1

We need the following lemmas.

Lemma 3.1. (see [7]) Let $f(z)$ be a non-constant meromorphic function and $c \in \mathbb{C}$. If $\sigma_2(f) < 1$ and $\varepsilon > 0$, then

$$m\Big(r, \frac{f(z+c)}{f(z)}\Big) = o\Big(\frac{T(r, f(z))}{r^{1-\sigma_2(f)-\varepsilon}}\Big)$$

for all r outside of a set E of finite logarithmic measure.

By [14, Lemma 1], [15, p. 66] and [7, Lemma 8.3], we immediately deduce the following lemma.

Lemma 3.2. Let $f(z)$ be a non-constant meromorphic function of $\sigma_2(f) < 1$, and let $c \neq 0$ be an arbitrary complex number. Then

$$T(r, f(z+c)) = T(r, f(z)) + S(r, f),$$
$$N(r, f(z+c)) = N(r, f(z)) + S(r, f),$$
$$\overline{N}(r, f(z+c)) = \overline{N}(r, f(z)) + S(r, f).$$

Applying logarithmic derivative lemma and Lemma 3.1 to Theorem 2.3 of [8], we get the following lemma.

Lemma 3.3. Let $f(z)$ be a transcendental meromorphic solution of hyper order $\sigma_2(f) < 1$ of a differential-difference equation of the form

$$U(z, f)P(z, f) = Q(z, f),$$

where $U(z, f)$ is a difference polynomial in $f(z)$ with small meromorphic coefficients, $P(z, f)$ and $Q(z, f)$ are differential-difference polynomials in $f(z)$ such that the proximity functions of the coefficients of $P(z, f)$ and $Q(z, f)$ are of the type $S(r, f)$. Assume that $\deg_f U(z, f) = n$, $\deg_f Q(z, f) \leqslant n$ and $U(z, f)$ contains just one term of maximal total degree in $f(z)$ and its shifts. Then

$$m(r, P(z, f)) = S(r, f).$$

Using a similar proof as in [16, Theorem 1.1] or [17, Lemma 2], we get the following lemma.

Lemma 3.4. Let $f(z)$ be a transcendental meromorphic function satisfying $\sigma_2(f) < 1$, let $H(z, f) (\not\equiv 0)$ be a difference polynomial in $f(z)$ of the form (2.1) with $m \geqslant 1$ different $\delta_{\lambda,j}$, let $F(z, f)$ be a difference monomial in $f(z)$ of the form (2.5), and let $\deg_f H(z, f) = d_H$ and $\deg_f F(z, f) = d_F$. Then

$$T(r, H(z, f)) \leqslant m d_H T(r, f(z)) + S(r, f), \tag{3.1}$$
$$T(r, F(z, f)) \leqslant d_F T(r, f(z)) + S(r, f). \tag{3.2}$$

From the proof of [17, Lemma 2], we get the following lemma.

Lemma 3.5. Let $f_1, f_2, \cdots, f_n$ be meromorphic functions. Then

$$N\Big(r, \sum_{\lambda \in I} f_1^{i_{\lambda,1}} f_2^{i_{\lambda,2}} \cdots f_n^{i_{\lambda,n}}\Big) \leqslant \sum_{i=1}^{n} \sigma N(r, f_i),$$

where $I = \{(i_{\lambda,1}, i_{\lambda,2}, \cdots, i_{\lambda,n})\}$ is an index set, and $\sigma = \max\limits_{\lambda \in I}\{i_{\lambda,1} + i_{\lambda,2} + \cdots + i_{\lambda,n}\}$.

Proof of Theorem 2.1. Set

$$\psi(z) = f(z)^n H(z, f) - s(z). \tag{3.3}$$

First observe that $\psi(z) \not\equiv 0$. Indeed, if $\psi(z) \equiv 0$, then

$$H(z, f) \equiv \frac{s(z)}{f(z)^n}. \tag{3.4}$$

Since $n > m d_H$, comparing the characteristic functions of both sides of (3.4) and using (3.1) of Lemma 3.4, we get a contradiction. So $\psi(z) \not\equiv 0$.

Differentiating both sides of (3.3), we obtain

$$\psi'(z) = n f(z)^{n-1} f'(z) H(z, f) + f(z)^n H'(z, f) - s'(z). \tag{3.5}$$

Since $\psi(z) \not\equiv 0$, multiplying both sides of (3.3) by $\frac{\psi'(z)}{\psi(z)}$, we get

$$\psi'(z) = \frac{\psi'(z)}{\psi(z)} f(z)^n H(z, f) - \frac{\psi'(z)}{\psi(z)} s(z). \tag{3.6}$$

Subtracting (3.5) from (3.6), we get

$$f(z)^{n-1} E(z) = s'(z) - \frac{\psi'(z)}{\psi(z)} s(z), \tag{3.7}$$

where

$$E(z) = nf'(z)H(z, f) - \frac{\psi'(z)}{\psi(z)} f(z)H(z, f) + f(z)H'(z, f). \tag{3.8}$$

We affirm that $E(z) \not\equiv 0$. Otherwise, since $s(z) \not\equiv 0$, it follows from (3.7) that

$$\frac{\psi'(z)}{\psi(z)} = \frac{s'(z)}{s(z)},$$

which gives $\psi(z) = C_1 s(z)$, where C_1 is a non-zero constant. Substituting $\psi(z) = C_1 s(z)$ into (3.3), we get

$$H(z, f) = \frac{(C_1 + 1)s(z)}{f(z)^n}. \tag{3.9}$$

Similarly as in (3.4), by (3.9) and (3.1), we get a contradiction. So $E(z) \not\equiv 0$.

By (3.1), we have $T(r, \psi(z)) \leqslant (n + md_H)T(r, f(z)) + S(r, f)$. So

$$m\Big(r, \frac{\psi'(z)}{\psi(z)}\Big) = S(r, \psi) = S(r, f). \tag{3.10}$$

Applying Lemma 3.3 to (3.7), we have

$$m(r, E(z)) = S(r, f). \tag{3.11}$$

Next we estimate $N(r, E(z))$. By (3.8), we see that the poles of $E(z)$ come from the poles of $f(z)$, the poles of $H(z, f)$, and the poles of $\frac{\psi'(z)}{\psi(z)}$. We denote by $N(r, |E(z) = f(z) = \infty)$ the counting function of those common poles of $E(z)$ and $f(z)$ in $|z| < r$, where each such point is counted according to its multiplicity in $E(z)$, denote by $N(r, |E(z) = H(z, f) = \infty, f(z) \neq \infty)$ the counting function of those common poles of $E(z)$ and $H(z, f)$ in $|z| < r$, where each such point is not a pole of $f(z)$, and each such point is counted according to its multiplicity in $E(z)$, and denote by $N(r, |E(z) = \frac{\psi'(z)}{\psi(z)} = \infty, f(z) \neq \infty, H(z, f) \neq \infty)$ the counting function of those common poles of $E(z)$ and $\frac{\psi'(z)}{\psi(z)}$ in $|z| < r$, where each such point is not a pole of $f(z)$ or a pole of $H(z, f)$, and each such point is counted according to its multiplicity in $E(z)$. Then

$$\begin{aligned} N(r, E(z)) = {} & N(r, |E(z) = f(z) = \infty) \\ & + N(r, |E(z) = H(z, f) = \infty, f(z) \neq \infty) \\ & + N\Big(r, |E(z) = \frac{\psi'(z)}{\psi(z)} = \infty, f(z) \neq \infty, H(z, f) \neq \infty\Big). \end{aligned} \tag{3.12}$$

Suppose that z_0 is a pole of $E(z)$ with order k.

If z_0 is a pole of $f(z)$ with order p, by (3.7), $n \geqslant 2$ and the fact that $\frac{\psi'(z)}{\psi(z)}$ has at most simple poles, we see that z_0 must be a pole of $s(z)$ with order q and $k + (n-1)p \leqslant q + 1$. We then deduce from $n \geqslant 2$ that $k \leqslant q$. So

$$N(r, |E(z) = f(z) = \infty) \leqslant N(r, s(z)) = S(r, f). \tag{3.13}$$

If z_0 is not a pole of $f(z)$ and z_0 is a pole of $H(z,f)$ with order l, then by (3.8) and the fact that $\frac{\psi'(z)}{\psi(z)}$ has at most simple poles, we see that $k \leqslant l+1$.

We denote by $N(r,|H(z,f)=\infty, f(z)\neq\infty)$ the counting function of those poles of $H(z,f)$ in $|z|<r$, where each such point is not a pole of $f(z)$, and each such point is counted according to its multiplicity in $H(z,f)$, and denote by $\overline{N}(r,|H(z,f)=\infty, f(z)\neq\infty)$ the counting function of those poles of $H(z,f)$ in $|z|<r$, where each such point is not a pole of $f(z)$, and each such point is counted one time. Then

$$\begin{aligned}&N(r,|E(z)=H(z,f)=\infty, f(z)\neq\infty)\\ \leqslant\ &N(r,|H(z,f)=\infty, f(z)\neq\infty)+\overline{N}(r,|H(z,f)=\infty, f(z)\neq\infty).\end{aligned}\tag{3.14}$$

We will prove that

$$N(r,|H(z,f)=\infty, f(z)\neq\infty)\leqslant (m-\chi)d_H N(r,f(z))+S(r,f).\tag{3.15}$$

Let the different $\delta_{\lambda,j}$ in $H(z,f)$ be $\delta_1,\cdots,\delta_m$. If $\delta_t\neq 0$ for all $t=1,\cdots,m$, then $f(z)$ is not contained in $H(z,f)$ and by (2.4) we have $\chi=0$. Since the coefficients of $H(z,f)$ are small functions of $f(z)$ and the degree of $H(z,f)$ is d_H, we deduce from Lemma 3.5 that

$$N(r,|H(z,f)=\infty, f(z)\neq\infty)=N(r,H(z,f))\leqslant\sum_{j=1}^{m}d_H N(r,f(z+\delta_j))+S(r,f).$$

So by Lemma 3.2 and $\chi=0$, we have

$$N(r,|H(z,f)=\infty, f(z)\neq\infty)\leqslant (m-\chi)d_H N(r,f(z))+S(r,f).$$

If $\delta_s=0$ for some $s\in\{1,\cdots,m\}$, then $f(z+\delta_s)=f(z)$ and by (2.4) we have $\chi=1$. Since the coefficients of $H(z,f)$ are small functions of $f(z)$ and the degree of $H(z,f)$ is d_H, we deduce from Lemma 3.5 that

$$\begin{aligned}&N(r,|H(z,f)=\infty, f(z)\neq\infty)\\ \leqslant\ &d_H N(r,f(z+\delta_1))+\cdots+d_H N(r,f(z+\delta_{s-1}))+d_H N(r,f(z+\delta_{s+1}))\\ &+\cdots+d_H N(r,f(z+\delta_m))+S(r,f)\\ =\ &\sum_{j=1}^{m}d_H N(r,f(z+\delta_j))-d_H N(r,f(z+\delta_s))+S(r,f).\end{aligned}$$

So by Lemma 3.2, $\chi=1$ and $f(z+\delta_s)=f(z)$, we have

$$N(r,|H(z,f)=\infty, f(z)\neq\infty)\leqslant (m-\chi)d_H N(r,f(z))+S(r,f).$$

Therefore, we proved that (3.15) holds. Similarly, we can prove that

$$\begin{aligned}&\overline{N}(r,|H(z,f)=\infty, f(z)\neq\infty)\\ \leqslant\ &\sum_{j=1}^{m}\overline{N}(r,f(z+\delta_j))-\chi\overline{N}(r,f(z))+S(r,f)\\ =\ &(m-\chi)\overline{N}(r,f(z))+S(r,f).\end{aligned}\tag{3.16}$$

If z_0 is not a pole of $f(z)$ and z_0 is not a pole of $H(z,f)$, then z_0 must be a pole of $\frac{\psi'(z)}{\psi(z)}$. Since $\frac{\psi'(z)}{\psi(z)}$ has at most simple poles, we deduce from (3.8) that $k=1$. The poles of $\frac{\psi'(z)}{\psi(z)}$ come from the poles of $\psi(z)$ and the zeros of $\psi(z)$. If z_0 is a pole of $\psi(z)$, then by (3.3), we see that z_0 must be a pole of $s(z)$. So

$$N\Big(r,|E(z)=\frac{\psi'(z)}{\psi(z)}=\infty, f(z)\neq\infty, H(z,f)\neq\infty\Big)$$

$$\leqslant \overline{N}(r,s(z)) + \overline{N}\Big(r,\frac{1}{\psi(z)}\Big)$$
$$= \overline{N}\Big(r,\frac{1}{\psi(z)}\Big) + S(r,f). \tag{3.17}$$

We deduce from (3.11)–(3.17) that

$$T(r,E(z)) \leqslant (m-\chi)d_H N(r,f(z)) + (m-\chi)\overline{N}(r,f(z)) + \overline{N}\Big(r,\frac{1}{\psi(z)}\Big) + S(r,f). \tag{3.18}$$

By (3.7) and (3.10), we get

$$\begin{aligned}(n-1)T(r,f(z)) &\leqslant T(r,E(z)) + T\Big(r,\frac{\psi'(z)}{\psi(z)}\Big) + S(r,f)\\ &= T(r,E(z)) + N\Big(r,\frac{\psi'(z)}{\psi(z)}\Big) + S(r,f)\\ &= T(r,E(z)) + \overline{N}\Big(r,\frac{1}{\psi(z)}\Big) + \overline{N}(r,\psi(z)) + S(r,f).\end{aligned} \tag{3.19}$$

Since $H(z,f)$ has m different $\delta_{\lambda,j}$ and χ is defined by (2.4), we deduce from (3.3) and Lemma 3.2 that

$$\begin{aligned}\overline{N}(r,\psi(z)) &\leqslant \overline{N}(r,f(z)) + \overline{N}(r,H(z,f)) - \chi\overline{N}(r,f(z)) + S(r,f)\\ &\leqslant (1+m-\chi)\overline{N}(r,f(z)) + S(r,f).\end{aligned} \tag{3.20}$$

We deduce from (3.18)–(3.20) that

$$\begin{aligned}&2\overline{N}\Big(r,\frac{1}{f(z)^n H(z,f)-s(z)}\Big)\\ &= 2\overline{N}\Big(r,\frac{1}{\psi(z)}\Big) \geqslant (n-1)T(r,f(z)) - (m-\chi)d_H N(r,f(z))\\ &- (2m+1-2\chi)\overline{N}(r,f(z)) + S(r,f).\end{aligned}$$

4. PROOF OF THEOREM 2.2

Set

$$\psi(z) = f(z)^n F(z,f) - s(z).$$

Since $n > d_F$, we deduce from (3.2) of Lemma 3.4 that $\psi(z) \not\equiv 0$. Since $F(z,f)$ is a special case of $H(z,f)$, we also have (3.5)–(3.12), where $H(z,f)$ is replaced by $F(z,f)$. Next we discuss each term in (3.12).

Suppose that z_0 is a pole of $E(z)$ with order k.

If z_0 is a pole of $f(z)$, as in (3.13) of Theorem 2.1, we get

$$N(r,|E(z)=f(z)=\infty) \leqslant N(r,s(z)) = S(r,f). \tag{4.1}$$

If z_0 is not a pole of $f(z)$ and z_0 is a pole of $F(z,f)$, then z_0 must be a pole of $f(z+c_t)$ for some $t \in \{1,\cdots,m\}$. So z_0+c_t is a pole of $f(z)$. Since the poles z_i and zeros z_j of $f(z)$ satisfy $z_i - z_j \neq c_l$ $(l=1,\cdots,m)$, except for finitely many exceptional poles and zeros, we will assume that z_0 is not a zero of $f(z)$. So, when estimating $N(r,|E(z)=F(z,f)=\infty, f(z)\neq\infty)$, we may have an error term of the type $O(\log r)$. Since $f(z_0)\neq 0,\infty$, we see that z_0 is a pole of $f(z)^{n-1}E(z)$ with order k. By (3.7), we see that z_0 is a pole of $\frac{\psi'(z)}{\psi(z)}$ with order 1 and $k=1$, or z_0 is a pole of $s(z)$ with order q and $k \leqslant q+1$. Therefore,

$$N(r, |E(z)=F(z,f)=\infty, f(z)\neq\infty) \leqslant \overline{N}(r, F(z,f)) + N(r, s(z)) + \overline{N}(r, s(z)) + O(\log r).$$

By Lemma 3.2, we have

$$\overline{N}(r, F(z,f)) \leqslant \sum_{j=1}^{m} \overline{N}(r, f(z+c_j)) = m\overline{N}(r, f(z)) + S(r,f). \tag{4.2}$$

If z_0 is not a pole of $f(z)$ and z_0 is not a pole of $F(z,f)$, then z_0 must be a pole of $\frac{\psi'(z)}{\psi(z)}$. As in (3.17) of Theorem 2.1, we get

$$N\Big(r, |E(z)=\frac{\psi'(z)}{\psi(z)}=\infty, f(z)\neq\infty, F(z,f)\neq\infty\Big) \leqslant \overline{N}\Big(r, \frac{1}{\psi(z)}\Big) + S(r,f). \tag{4.3}$$

By (3.11), (3.12) and (4.1)–(4.3), we get

$$T(r, E(z)) \leqslant m\overline{N}(r, f(z)) + \overline{N}\Big(r, \frac{1}{\psi(z)}\Big) + S(r,f). \tag{4.4}$$

By (3.7) and (3.10), we get

$$(n-1)T(r, f(z)) \leqslant T(r, E(z)) + \overline{N}\Big(r, \frac{1}{\psi(z)}\Big) + \overline{N}(r, \psi(z)) + S(r,f). \tag{4.5}$$

Since $F(z,f) = f(z+c_1)^{i_1}\cdots f(z+c_m)^{i_m}$, $\psi(z) = f(z)^n F(z,f) - s(z)$ and $c_1, \cdots, c_m$ are different non-zero complex constants, we deduce from Lemma 3.2 that

$$\begin{aligned}\overline{N}(r, \psi(z)) &\leqslant \overline{N}(r, f(z)) + \overline{N}(r, F(z,f)) + S(r,f) \\ &\leqslant (1+m)\overline{N}(r, f(z)) + S(r,f).\end{aligned} \tag{4.6}$$

We deduce from (4.4)–(4.6) that

$$\begin{aligned}&2\overline{N}\Big(r, \frac{1}{f(z)^n F(z,f) - s(z)}\Big) \\ &= 2\overline{N}\Big(r, \frac{1}{\psi(z)}\Big) \geqslant (n-1)T(r, f(z)) - (2m+1)\overline{N}(r, f(z)) + S(r,f).\end{aligned}$$

5. PROOF OF THEOREM 2.3

We need the following lemma.

Lemma 5.1.(see [18]) Suppose that h is a non-constant meromorphic function satisfying

$$\overline{N}(r,h) + \overline{N}(r,1/h) = S(r,h).$$

Let $f = a_0h^p + a_1h^{p-1} + \cdots + a_p$, and $g = b_0h^q + b_1h^{q-1} + \cdots + b_q$ be polynomials in h with coefficients $a_0, a_1, \cdots, a_p, b_0, b_1, \cdots, b_q$ being small functions of h and $a_0b_0a_p \not\equiv 0$. If $q \leqslant p$, then $m(r, g/f) = S(r,h)$.

Proof of Theorem 2.3. Set

$$\psi(z) = f(z)^n H(z,f) - s(z). \tag{5.1}$$

First we assume that the condition (i) in Theorem 2.3 holds. Let

$$g(z) = \frac{f(z)-a}{f(z)-b}.$$

Then $0, \infty$ are two Borel exceptional values of $g(z)$. By Hadamard factorization theorem, $g(z)$ takes the form

$$g(z)=w(z)\mathrm{e}^{h(z)},$$

where $w(z)$ is a meromorphic function such that $\sigma(w(z))<\sigma(g(z))$, and $h(z)$ is a polynomial such that $\sigma(g(z))=\deg h(z)\geqslant 1$. So

$$f(z)=\frac{bw(z)\mathrm{e}^{h(z)}-a}{w(z)\mathrm{e}^{h(z)}-1},\quad f(z)^n=\frac{b^n w(z)^n\mathrm{e}^{nh(z)}+\cdots+(-a)^n}{w(z)^n\mathrm{e}^{nh(z)}+\cdots+(-1)^n}. \tag{5.2}$$

Denoting

$$W_\lambda(z)=w(z+\delta_{\lambda,1})^{\mu_{\lambda,1}}\cdots w(z+\delta_{\lambda,\tau_\lambda})^{\mu_{\lambda,\tau_\lambda}}$$

and substituting (5.2) into $H(z,f)$, we get

$$\begin{aligned}H(z,f)&=\sum_{\lambda\in J}a_\lambda(z)\prod_{j=1}^{\tau_\lambda}\frac{b^{\mu_{\lambda,j}}w(z+\delta_{\lambda,j})^{\mu_{\lambda,j}}\mathrm{e}^{\mu_{\lambda,j}h(z+\delta_{\lambda,j})}+\cdots+(-a)^{\mu_{\lambda,j}}}{w(z+\delta_{\lambda,j})^{\mu_{\lambda,j}}\mathrm{e}^{\mu_{\lambda,j}h(z+\delta_{\lambda,j})}+\cdots+(-1)^{\mu_{\lambda,j}}}\\&=\sum_{\lambda\in J}a_\lambda(z)\frac{b^{\mu_{\lambda,1}+\cdots+\mu_{\lambda,\tau_\lambda}}W_\lambda(z)\mathrm{e}^{\mu_{\lambda,1}h(z+\delta_{\lambda,1})+\cdots+\mu_{\lambda,\tau_\lambda}h(z+\delta_{\lambda,\tau_\lambda})}+\cdots+(-a)^{\mu_{\lambda,1}+\cdots+\mu_{\lambda,\tau_\lambda}}}{W_\lambda(z)\mathrm{e}^{\mu_{\lambda,1}h(z+\delta_{\lambda,1})+\cdots+\mu_{\lambda,\tau_\lambda}h(z+\delta_{\lambda,\tau_\lambda})}+\cdots+(-1)^{\mu_{\lambda,1}+\cdots+\mu_{\lambda,\tau_\lambda}}}.\end{aligned}$$

Denoting

$$s_\lambda(z)=W_\lambda(z)\mathrm{e}^{\mu_{\lambda,1}(h(z+\delta_{\lambda,1})-h(z))}\cdots\mathrm{e}^{\mu_{\lambda,\tau_\lambda}(h(z+\delta_{\lambda,\tau_\lambda})-h(z))},$$

we have

$$\begin{aligned}H(z,f)&=\sum_{\lambda\in J}a_\lambda(z)\frac{b^{d_\lambda}s_\lambda(z)\mathrm{e}^{d_\lambda h(z)}+\cdots+(-a)^{d_\lambda}}{s_\lambda(z)\mathrm{e}^{d_\lambda h(z)}+\cdots+(-1)^{d_\lambda}}\\&=\frac{\Big(\sum\limits_{\lambda\in J}a_\lambda(z)b^{d_\lambda}\Big)\prod\limits_{\lambda\in J}s_\lambda(z)\mathrm{e}^{\sum_{\lambda\in J}d_\lambda h(z)}+\cdots+\Big(\sum\limits_{\lambda\in J}a_\lambda(z)a^{d_\lambda}\Big)(-1)^{\sum_{\lambda\in J}d_\lambda}}{\prod\limits_{\lambda\in J}s_\lambda(z)\mathrm{e}^{\sum_{\lambda\in J}d_\lambda h(z)}+\cdots+(-1)^{\sum_{\lambda\in J}d_\lambda}}.\end{aligned}\tag{5.3}$$

By (5.1)–(5.3) and denoting $S(z)=\prod\limits_{\lambda\in J}s_\lambda(z)$, $D=\sum\limits_{\lambda\in J}d_\lambda$, we get

$$\begin{aligned}\psi(z)&=\frac{b^n w(z)^n\Big(\sum\limits_{\lambda\in J}a_\lambda(z)b^{d_\lambda}\Big)S(z)\mathrm{e}^{(n+D)h(z)}+\cdots+\Big(\sum\limits_{\lambda\in J}a_\lambda(z)a^{d_\lambda}\Big)(-a)^n(-1)^D}{w(z)^nS(z)\mathrm{e}^{(n+D)h(z)}+\cdots+(-1)^{n+D}}-s(z)\\&=\frac{\Big(b^n\sum\limits_{\lambda\in J}a_\lambda(z)b^{d_\lambda}-s(z)\Big)w(z)^nS(z)\mathrm{e}^{(n+D)h(z)}+\cdots+(-1)^{n+D}\Big(a^n\sum\limits_{\lambda\in J}a_\lambda(z)a^{d_\lambda}-s(z)\Big)}{w(z)^nS(z)\mathrm{e}^{(n+D)h(z)}+\cdots+(-1)^{n+D}}.\end{aligned}\tag{5.4}$$

We see that $\psi(z)$ is a rational function in $\mathrm{e}^{h(z)}$ and the coefficients in (5.4) are all small functions of $\mathrm{e}^{h(z)}$. Since $a^n\sum\limits_{\lambda\in J}a_\lambda(z)a^{d_\lambda}-s(z)\not\equiv 0$ and $b^n\sum\limits_{\lambda\in J}a_\lambda(z)b^{d_\lambda}-s(z)\not\equiv 0$, by Lemma 5.1, we get

$$m\Big(r,\frac{1}{\psi(z)}\Big)=S(r,\mathrm{e}^{h(z)})=S(r,f).$$

Moreover, by (5.1) and Lemma 3.4, we have

$$nT(r,f(z))=T\Big(r,\frac{\psi(z)+s(z)}{H(z,f)}\Big)\leqslant T(r,\psi(z))+md_HT(r,f(z))+S(r,f).$$

So

$$N\Big(r, \frac{1}{f(z)^n H(z,f) - s(z)}\Big) = N\Big(r, \frac{1}{\psi(z)}\Big) \geqslant (n - md_H)T(r, f(z)) + S(r, f).$$

Now we assume that the condition (ii) in Theorem 2.3 holds. Then $f(z)$ takes the form

$$f(z) = w(z)e^{h(z)} + a, \tag{5.5}$$

where $w(z)$ is a meromorphic function such that $\sigma(w(z)) < \sigma(f(z))$, and $h(z)$ is a polynomial such that $\sigma(f(z)) = \deg h(z) \geqslant 1$. Substituting (5.5) into $f(z)^n$, we get

$$f(z)^n = w(z)^n e^{nh(z)} + \cdots + a^n. \tag{5.6}$$

Using the notations $W_\lambda(z)$ and $s_\lambda(z)$ as above and substituting (5.5) into $H(z,f)$, we get

$$\begin{aligned} H(z,f) &= \sum_{\lambda \in J} a_\lambda(z) \prod_{j=1}^{\tau_\lambda} (w(z+\delta_{\lambda,j})^{\mu_{\lambda,j}} e^{\mu_{\lambda,j} h(z+\delta_{\lambda,j})} + \cdots + a^{\mu_{\lambda,j}}) \\ &= \sum_{\lambda \in J} a_\lambda(z) (W_\lambda(z) e^{\mu_{\lambda,1} h(z+\delta_{\lambda,1}) + \cdots + \mu_{\lambda,\tau_\lambda} h(z+\delta_{\lambda,\tau_\lambda})} + \cdots + a^{\mu_{\lambda,1} + \cdots + \mu_{\lambda,\tau_\lambda}}) \\ &= \sum_{\lambda \in J} a_\lambda(z)(s_\lambda(z) e^{d_\lambda h(z)} + \cdots + a^{d_\lambda}). \end{aligned}$$

Since $H(z,f) \not\equiv 0$ and $d_H = \max\limits_{\lambda \in J} d_\lambda$, we see that $H(z,f)$ takes the form

$$H(z,f) = \sum_{\lambda \in J} a_\lambda(z) a^{d_\lambda} \not\equiv 0 \tag{5.7}$$

or

$$H(z,f) = l_q(z)e^{qh(z)} + \cdots + l_1(z)e^{h(z)} + \sum_{\lambda \in J} a_\lambda(z) a^{d_\lambda}, \quad 1 \leqslant q \leqslant d_H, \tag{5.8}$$

where $l_j(z) (j = 1, \cdots, q)$ are all small functions of $e^{h(z)}$ and $l_q(z) \not\equiv 0$.

If (5.7) holds, by (5.1), (5.6) and Lemma 5.1, we get

$$N\Big(r, \frac{1}{\psi(z)}\Big) = nT(r, f(z)) + S(r, f).$$

If (5.8) holds, by (5.1), (5.6) and Lemma 5.1, we get

$$N\Big(r, \frac{1}{\psi(z)}\Big) = (n+q)T(r, f(z)) + S(r, f), \quad 1 \leqslant q \leqslant d_H.$$

Therefore,

$$N\Big(r, \frac{1}{f(z)^n H(z,f) - s(z)}\Big) = N\Big(r, \frac{1}{\psi(z)}\Big) \geqslant nT(r, f(z)) + S(r, f).$$

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