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Characteristic estimation of differential polynomials

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Abstract

In this paper, we give the characteristic estimation of a meromorphic function f with the differential polynomials $f^l(f^{(k)})^n$ and obtain that

$$T(r,f) \le M\overline{N}\left(r, \frac{1}{f^l(f^{(k)})^n - a}\right) + S(r,f)$$

holds for $M = \min\{\frac{1}{l-2}, 6\}$, integers $l(\ge 2)$, $n(\ge 1)$, $k(\ge 1)$, and a non-zero constant a. This quantitative estimate is an interesting and complete extension of earlier results. The value distribution of a differential monomial of meromorphic functions is also investigated.

MSC: 30D35; 30D05

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1 Introduction and main results

We assume that the reader is familiar with the fundamentals of Nevanlinna's value distribution theory of meromorphic functions (see e.g. [4, 10, 16]). Let f be a transcendental meromorphic function in the complex plane $\mathbb{C}$. We denote by $S(r,f)$ any quantity satisfying $S(r,f) = o(T(r,f))$, as $r \to \infty$, possibly outside of an exceptional set of finite logarithmic measure. A meromorphic function α defined in $\mathbb{C}$ is called a small function of f if $T(r,\alpha) = S(r,f)$.

We also introduce some other symbols (see [15]). Let $a \in \mathbb{C} \cup \{\infty\}$, k be a positive integer. Let $N_{k)}(r, \frac{1}{f-a})$ denote the counting function of those a-points of f (counting multiplicity) whose multiplicities are not greater than k, and let $\overline{N}_{k)}(r, \frac{1}{f-a})$ denote the corresponding reduced counting function. Similarly, let $N_{(k}(r, \frac{1}{f-a})$ denote the counting function of those a-points of f (counting multiplicity) whose multiplicities are not less than k, and let $\overline{N}_{(k}(r, \frac{1}{f-a})$ denote the corresponding reduced counting function. And let $N_k(r, \frac{1}{f-a})$ denote the counting function of those a-points of f with multiplicity k.

Hayman [5] proved the following well-known theorem.

Theorem 1.1 ([5, Theorem 9]) *Let f be a transcendental meromorphic function in the complex plane, and let l be a positive integer. If $l \geq 3$, then $f^l f'$ assumes every finite nonzero value infinitely often.*

Hayman also conjectured that Theorem 1.1 remained valid for $l \geq 1$. Mues [12] proved that $f^2 f' - 1$ has infinitely many zeros. Later on, many researchers investigated the zeros of differential monomial $f^l (f^{(k)})^n - a$ for positive integers l, n, k and a non-zero complex number a, and obtained some qualitative results, see e.g. [2, 3, 11, 13, 14], and some quantitative results, see e.g. [1, 6–9, 17].

Zhang [17] proved that the inequality $T(r,f) < 6N(r, \frac{1}{f^2 f' - 1}) + S(r,f)$ holds. Huang and Gu [6] extended the inequality and proved the following.

Theorem 1.2 ([6, Theorem 1]) *Let f be a transcendental meromorphic function in the complex plane, and let k be a positive integer. Then*

$$T(r,f) < 6N\left(r, \frac{1}{f^2 f^{(k)} - 1}\right) + S(r,f). \tag{1}$$

Karmakar and Sahoo further [8] proved the following.

Theorem 1.3 ([8, Theorem 1.1]) *Let f be a transcendental meromorphic function and $l(\geq 2)$, $k(\geq 1)$ be any integers, then*

$$T(r,f) < \frac{6}{2l-3}\overline{N}\left(r, \frac{1}{f^l f^{(k)} - 1}\right) + S(r,f). \tag{2}$$

Lahiri and Dewan [9] obtained the following estimate.

Theorem 1.4 ([9, Theorem 3.2]) *Let f be a transcendental meromorphic function, $\alpha(\not\equiv 0, \infty)$ be a small function of f. If $\psi = \alpha f^l (f^{(k)})^n$, where $l(\geq 0)$, $n(\geq 1)$, $k(\geq 1)$ are integers, then for any small function $a(\not\equiv 0, \infty)$ of ψ,*

$$\begin{aligned}(n+l)T(r,f) \leq{}& \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right) + nN_{(k)}\left(r, \frac{1}{f}\right) \\ &+ \overline{N}\left(r, \frac{1}{\psi - a}\right) + S(r,f),\end{aligned} \tag{3}$$

where $N_{(k)}(r, \frac{1}{f})$ denotes the counting function of zeros of f, a zero with multiplicity q is counted q times if $q \leq k$ and is counted k times if $q > k$.

Remark 1.1 Estimate (3) implies that, for $l \geq 3$, $n \geq 1$, $k \geq 1$,

$$T(r,f) \leq \frac{1}{l-2}\overline{N}\left(r, \frac{1}{f^l (f^{(k)})^n - a}\right) + S(r,f). \tag{4}$$

Jiang and Huang [7] proved the following.

Theorem 1.5 ([7, Theorem 1]) *Let f be a transcendental meromorphic function in the complex plane, $l(\geq 2)$, $n(\geq 2)$, $k(\geq 2)$ be integers, and a be a non-zero constant. Then*

$$T(r,f) \leq \frac{1}{l-1} N\left(r, \frac{1}{f^l(f^{(k)})^n - a}\right) + S^*(r,f), \tag{5}$$

where $S^*(r,f)$ denotes the quantity satisfying $S^*(r,f) = o(T(r,f))$ for all r outside a possible exceptional set E of logarithmic density 0.

We note that Theorem 1.3 does not hold for $n \geq 2$, Theorem 1.4 is invalid for $l = 2$, and Theorem 1.5 remains invalid for $l = 2, n = 1, k = 1$. Thus, by using a method different from the previous proofs, we continue to consider the characteristic estimate of more general forms $f^l(f^{(k)})^n - a$ for a non-zero constant a, integers $l \geq 2$, $n \geq 1$, and $k \geq 1$, and obtain its quantitative result as follows.

Theorem 1.6 *Let f be a transcendental meromorphic function with finite order in the complex plane, $l(\geq 2)$, $n(\geq 1)$, $k(\geq 1)$ be integers, and a be a non-zero constant. Then*

$$T(r,f) \leq M\overline{N}\left(r, \frac{1}{f^l(f^{(k)})^n - a}\right) + S(r,f) \tag{6}$$

for $M = \min\{\frac{1}{l-2}, 6\}$.

The quantity

$$\Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, \frac{1}{f-a})}{T(r,f)}$$

is called the deficiency of f at the point a. It is obvious that $0 \leq \Theta(a,f) \leq 1$. Thus, we present the value distribution of a differential monomial $f^l(f^{(k)})^n$.

Corollary 1.1 *Let f be a transcendental meromorphic function with finite order in the complex plane, $l(\geq 2)$, $n(\geq 1)$, $k(\geq 1)$ be integers, and a be a non-zero constant. Then*

$$\Theta\big(a, f^l\big(f^{(k)}\big)^n\big) \leq 1 - \frac{1}{M(nk+n+l)}$$

for $M = \min\{\frac{1}{l-2}, 6\}$.

2 Some lemmas

We now prepare some lemmas.

Lemma 2.1 *Let f be a transcendental meromorphic function with finite order. Then $f^l(f^{(k)})^n$ is not identically constant, where $l(\geq 2)$, $n(\geq 1)$, $k(\geq 1)$ are integers.*

Proof Contradicting to our assumption, we suppose that $f^l(f^{(k)})^n \equiv C$. Clearly, $C \neq 0$. Then $\frac{1}{f^{n+l}} = \frac{1}{C}(\frac{f^{(k)}}{f})^n$, and

$$(n+l)T(r,f) = m\left(r, \frac{1}{f^{n+l}}\right) + N\left(r, \frac{1}{f^{n+l}}\right) + O(1)$$

$$= m\left(r, \frac{1}{C}\left(\frac{f^{(k)}}{f}\right)^n\right) + O(1)$$
$$= nm\left(r, \frac{f^{(k)}}{f}\right) + O(1) = S(r,f),$$

a contradiction. □

Lemma 2.2 *Let f be a transcendental meromorphic solution with finite order. Suppose that $g(z) = f^2(f^{(k)})^n - a$, where $n(\geq 1)$, $k(\geq 1)$ are integers and a is a non-zero constant. Then*

$$(n+2)T(r,f) \leq \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) + nN_{k)}\left(r, \frac{1}{f}\right)$$
$$+ nk\overline{N}_{(k+1}\left(r, \frac{1}{f}\right) - N_0\left(r, \frac{1}{g'}\right) + S(r,f) \qquad (7)$$

and

$$\left[N(r,f) - \overline{N}(r,f)\right] + m(r,f) + (n+1)m\left(r, \frac{1}{f}\right) + N_0\left(r, \frac{1}{g'}\right)$$
$$\leq \overline{N}\left(r, \frac{1}{g}\right) + S(r,f), \qquad (8)$$

where $N_0(r, \frac{1}{g'})$ denotes the counting function of those zeros of g', but not zeros of f or g.

Proof It follows from Lemma 2.1 that g is not identically constant. Thus

$$\frac{a}{f^{n+2}} = \left(\frac{f^{(k)}}{f}\right)^n - \frac{g'}{f^{n+2}}\frac{g}{g'}.$$

We conclude from the lemma of the logarithmic derivative that

$$(n+2)m\left(r, \frac{1}{f}\right) \leq m\left(r, \left(\frac{f^{(k)}}{f}\right)^n\right) + m\left(r, \frac{g'}{f^{n+2}}\right) + m\left(r, \frac{g}{g'}\right) + O(1)$$
$$= T\left(r, \frac{g'}{g}\right) - N\left(r, \frac{g}{g'}\right) + S(r,f)$$
$$= \overline{N}(r,g) + N\left(r, \frac{1}{g}\right) - N\left(r, \frac{1}{g'}\right) + S(r,f)$$

and

$$(n+2)T(r,f) \leq (n+2)N\left(r, \frac{1}{f}\right) + \overline{N}(r,f) + N\left(r, \frac{1}{g}\right)$$
$$- N\left(r, \frac{1}{g'}\right) + S(r,f). \qquad (9)$$

Denote

$$N\left(r, \frac{1}{g'}\right) = N_{000}\left(r, \frac{1}{g'}\right) + N_{00}\left(r, \frac{1}{g'}\right) + N_0\left(r, \frac{1}{g'}\right), \qquad (10)$$

where $N_{000}(r, \frac{1}{g'})$ denotes the counting function of those zeros of g' which are from the zeros of g, $N_{00}(r, \frac{1}{g'})$ denotes the counting function of those zeros of g' which are from the zeros of f, $N_0(r, \frac{1}{g'})$ denotes the counting function of those zeros of g' which are not zeros of f or g. So, we have

$$N\left(r, \frac{1}{g}\right) - N_{000}\left(r, \frac{1}{g'}\right) = \overline{N}\left(r, \frac{1}{g}\right). \tag{11}$$

Let z_0 be a zero of f with multiplicity q. If $q \le k$, then z_0 is a zero of g' with multiplicity at least $2q - 1$. If $q \ge k + 1$, then z_0 is a zero of g' with multiplicity $(n + 2)q - (nk + 1)$. Thus, by simple calculation, we have

$$\begin{aligned} N_{00}\left(r, \frac{1}{g'}\right) &\ge 2N_{k)}\left(r, \frac{1}{f}\right) - \overline{N}_{k)}\left(r, \frac{1}{f}\right) \\ &\quad + (n + 2)N_{(k+1}\left(r, \frac{1}{f}\right) - (nk + 1)\overline{N}_{(k+1}\left(r, \frac{1}{f}\right) \\ &= 2N\left(r, \frac{1}{f}\right) - \overline{N}\left(r, \frac{1}{f}\right) + nN_{(k+1}\left(r, \frac{1}{f}\right) - nk\overline{N}_{(k+1}\left(r, \frac{1}{f}\right) \end{aligned}$$

and

$$\begin{aligned} &(n + 2)N\left(r, \frac{1}{f}\right) - N_{00}\left(r, \frac{1}{g'}\right) \\ &\quad \le nN\left(r, \frac{1}{f}\right) - nN_{(k+1}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f}\right) + nk\overline{N}_{(k+1}\left(r, \frac{1}{f}\right) \\ &\quad = nN_{k)}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f}\right) + nk\overline{N}_{(k+1}\left(r, \frac{1}{f}\right). \end{aligned} \tag{12}$$

Then we deduce from (9)–(12) that

$$\begin{aligned} (n + 2)T(r,f) &\le \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{f}\right) \\ &\quad + nN_{k)}\left(r, \frac{1}{f}\right) + nk\overline{N}_{(k+1}\left(r, \frac{1}{f}\right) - N_0\left(r, \frac{1}{g'}\right) + S(r,f), \end{aligned}$$

and so inequality (7) is proved.

We further get from (7) that

$$\begin{aligned} &N(r,f) + m(r,f) + (n + 1)N\left(r, \frac{1}{f}\right) + (n + 1)m\left(r, \frac{1}{f}\right) + O(1) \\ &\quad = (n + 2)T(r,f) \\ &\quad \le \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) + nN_{k)}\left(r, \frac{1}{f}\right) + nk\overline{N}_{(k+1}\left(r, \frac{1}{f}\right) \\ &\qquad - N_0\left(r, \frac{1}{g'}\right) + S(r,f) \\ &\quad \le \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) + nN_{k)}\left(r, \frac{1}{f}\right) + nN_{(k+1}\left(r, \frac{1}{f}\right) \end{aligned}$$

$$
\begin{aligned}
&- N_0\left(r, \frac{1}{g'}\right) + S(r,f) \\
&\leq \overline{N}(r,f) + (n+1)N\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) - N_0\left(r, \frac{1}{g'}\right) + S(r,f),
\end{aligned}
$$

that is,

$$
\begin{aligned}
&\left[N(r,f) - \overline{N}(r,f)\right] + m(r,f) + (n+1)m\left(r, \frac{1}{f}\right) + N_0\left(r, \frac{1}{g'}\right) \\
&\leq \overline{N}\left(r, \frac{1}{g}\right) + S(r,f),
\end{aligned}
$$

and so inequality (8) is proved. □

Lemma 2.3 *Let f be a transcendental meromorphic function with finite order. Suppose that*

$$
g(z) = f^2\left(f^{(k)}\right)^n - a,\ h(z) = \frac{g'(z)}{f(z)} = 2f'\left(f^{(k)}\right)^n + nf\left(f^{(k)}\right)^{n-1}f^{(k+1)},
$$

where $n(\geq 1)$, $k(\geq 1)$ are integers and a is a non-zero constant.

$$
\begin{aligned}
F(z) = {}& a_1\left(\frac{g'(z)}{g(z)}\right)^2 + a_2\left(\frac{g'(z)}{g(z)}\right)' + a_3\left(\frac{h'(z)}{h(z)}\right)' + a_4\left(\frac{h'(z)}{h(z)}\right)^2 \\
&+ a_5\left(\frac{g'(z)}{g(z)}\frac{h'(z)}{h(z)}\right),
\end{aligned}
\tag{13}
$$

where a_i's are defined by

$$
\begin{cases}
a_1 = -(2n^4 + 4n^3 + 2n^2 + 3n + 2); \\
a_2 = -2(n+1)(n^3 + n^2 + n + 2); \\
a_3 = 2n^2(n+1)^2; \\
a_4 = -2n^2(n+1)^2; \\
a_5 = 4(n+1)(n^3 + n^2 + 1),
\end{cases}
\tag{14}
$$

when $k = 1$, and are defined by

$$
\begin{cases}
a_1 = 2(nk+n)^2 - \frac{(3nk+3n+4)[(nk+n)^2 - 6(nk+n) - 24]}{nk+n+2}; \\
a_2 = -(nk+n+4)[(nk+n)^2 - 6(nk+n) - 24]; \\
a_3 = 2(nk+n)(nk+n+2)(nk+n+4); \\
a_4 = -4(nk+n)(nk+n+2); \\
a_5 = 4[(nk+n)^2 - 6(nk+n) - 24],
\end{cases}
\tag{15}
$$

when $k \geq 2$. Then $F(z) \not\equiv 0$.

Proof We use a similar method of Huang-Gu [6, Lemma 3]. Suppose that $F(z) \equiv 0$, we claim that

(i) $g(z) \not\equiv 0$;
(ii) $h(z) \not\equiv 0$;
(iii) all zeros of $f(z)$ are simple.

Suppose that z_1 is a zero of $g(z)$ with multiplicity $l(\geq 1)$. Then $f(z_1) \neq 0, \infty$, and z_1 is a zero of $h(z)$ with multiplicity $l-1$ since $g' = fh$. Using the Laurent series of $F(z)$ at the point z_1, we can calculate that the coefficient $A(l)$ of $(z-z_1)^{-2}$ is

$$A(l) = (a_1 + a_4 + a_5)l^2 - (a_2 + a_3 + 2a_4 + a_5)l + (a_3 + a_4).$$

Using (14) for $k = 1$, we have

$$A(l) = (n+2)l^2 + 2n(n+1)l > 0.$$

This shows that z_1 is a pole of $F(z)$, which contradicts $F(z) \equiv 0$. Hence $g(z) \not\equiv 0$ when $k = 1$. Using (15) for $k \geq 2$, we have

$$\begin{aligned} A(l) = {}& -\frac{(nk+n+4)^2(nk+n+6)}{nk+n+2} l^2 - (nk+n)(nk+n+4)(nk+n+6)l \\ & + 2(nk+n)(nk+n+2)^2. \end{aligned}$$

Clearly, $A(l) \neq 0$ for all positive integers l. This shows that z_1 is again a pole of $F(z)$, which contradicts $F(z) \equiv 0$. Hence $g(z) \not\equiv 0$ when $k \geq 2$.

Suppose that z_2 is a zero of $h(z)$ with multiplicity $l(\geq 1)$. By (i) we have $g(z_2) \neq 0, \infty$. Using the Laurent series of $F(z)$ at the point z_2, we can get the coefficient $B(l)$ of $(z-z_1)^{-2}$ is

$$B(l) = a_4 l^2 - a_3 l.$$

Using (14) for $k = 1$, we get

$$B(l) = -2n^2(n+1)^2\big(l^2 + l\big) < 0,$$

and so, the point z_2 is again a pole of $F(z)$, which contradicts $F(z) \equiv 0$.

Using (15) for $k \geq 2$, we get

$$B(l) = -4(nk+n)(nk+n+2)l^2 - 2(nk+n)(nk+n+2)(nk+n+4)l < 0,$$

and so, the point z_2 is a pole of $F(z)$, which contradicts $F(z) \equiv 0$. Hence conclusion (ii) $h(z) \not\equiv 0$ holds when $k \geq 1$.

Noting that $h(z) = \frac{g'(z)}{f(z)} = 2f'(f^{(k)})^n + nf(f^{(k)})^{n-1}f^{(k+1)}$ and (ii) $h(z) \not\equiv 0$, we can obtain (iii).

Setting $\phi(z) := \frac{h(z)}{g(z)}$, we conclude that $\phi(z)$ is an entire function, all zeros of $\phi(z)$ can occur only at multiple poles of $f(z)$ and the following expressions hold:

$$\frac{g'}{g} = \frac{fh}{g} = f\phi, \qquad \frac{h'}{h} = \frac{g'}{g} + \frac{\phi'}{\phi} = f\phi + \frac{\phi'}{\phi}.$$

First, we consider the case $k \geq 2$. Substituting the above two equalities into (13) yields

$$\begin{aligned}(a_1+a_4+a_5)f^2\phi^2+(a_2+a_3+2a_4+a_5)f\phi' \\ +(a_2+a_3)f'\phi+\left[a_3\left(\frac{\phi'}{\phi}\right)'+a_4\left(\frac{\phi'}{\phi}\right)^2\right]\equiv 0.\end{aligned} \tag{16}$$

Applying (15), we have $a_2+a_3=(nk+n+4)^2(nk+n+6)\neq 0$. And so $\phi\not\equiv 0$, otherwise $\frac{g'}{g}=f\phi\equiv 0$, that is, $g\equiv C$, which contradicts Lemma 2.1. Thus, it follows from (16) that

$$f'=\frac{1}{\phi}\alpha_{11}(z)+f\alpha_{12}(z)+f^2\phi\alpha_{13}(z), \tag{17}$$

where $\alpha_{1i}(z)$ ($i=1,2,3$) are differential polynomials of $\frac{\phi'}{\phi}$.

Differentiating both sides of (17) gives

$$\begin{aligned}f''=-\frac{1}{\phi}\frac{\phi'}{\phi}\alpha_{11}(z)+\frac{1}{\phi}\alpha_{11}'(z)+f'\alpha_{12}(z)+f\alpha_{12}'(z) \\ +2ff'\phi\alpha_{13}(z)+f^2\phi\left[\frac{\phi'}{\phi}\alpha_{13}(z)+\alpha_{13}'(z)\right].\end{aligned}$$

Applying the above equality to (17), we have

$$f''=\frac{1}{\phi}\alpha_{21}(z)+f\alpha_{22}(z)+f^2\phi\alpha_{23}(z)+f^3\phi^2\alpha_{24}(z),$$

where $\alpha_{2i}(z)$ ($i=1,2,3,4$) are differential polynomials of $\frac{\phi'}{\phi}$. Continuing the above process, we get

$$f^{(k)}=\frac{1}{\phi}\alpha_{k1}(z)+f\alpha_{k2}(z)+f^2\phi\alpha_{k3}(z)+\cdots+f^{k+1}\phi^k\alpha_{kk+2}(z), \tag{18}$$

where $\alpha_{ki}(z)$ ($i=1,2,\ldots,k+2$) are differential polynomials of $\frac{\phi'}{\phi}$.

Suppose that z_3 is a simple zero of $f(z)$. Together with (17), (18) and noting that $\phi(z_3)\neq 0,\infty$, we have

$$f'(z_3)=\frac{1}{\phi(z_3)}\alpha_{11}(z_3),\qquad f^{(k)}(z_3)=\frac{1}{\phi(z_3)}\alpha_{k1}(z_3).$$

Substituting the above two equalities into the expressions of $g(z)$ and $h(z)$ yields

$$g(z_3)=-a, h(z_3)=2f'(z_3)\big(f^{(k)}(z_3)\big)^n=\frac{2}{\phi^{n+1}(z_3)}\alpha_{11}(z_3)\alpha_{k1}^n(z_3).$$

Combining the above two equalities and the expression of $\phi(z):=\frac{h(z)}{g(z)}$, we get

$$a\phi^{n+2}(z_3)=-2\alpha_{11}(z_3)\alpha_{k1}^n(z_3). \tag{19}$$

Set $U(z):=a\phi^{n+2}(z)+2\alpha_{11}(z)\alpha_{k1}^n(z)$. We consider the following two cases.

Case 1. $U(z) \not\equiv 0$. It follows from (19) and (*iii*) that

$$N\left(r,\frac{1}{f}\right) = \overline{N}\left(r,\frac{1}{f}\right) \le N\left(r,\frac{1}{U}\right) \le T(r,U) + O(1) \\ \le O\bigl\{T(r,\phi)\bigr\} + O(1). \tag{20}$$

$$T(r,\phi) = m(r,\phi) = m\left(r,\frac{h}{g}\right) = m\left(r,\frac{g'}{g}\frac{1}{f}\right) \le m\left(r,\frac{1}{f}\right) + S(r,f). \tag{21}$$

Using (8) and noting that $N(r,\frac{1}{g}) = 0$, we have

$$m\left(r,\frac{1}{f}\right) = S(r,f). \tag{22}$$

It follows from (20)–(22) that

$$N\left(r,\frac{1}{f}\right) = S(r,f). \tag{23}$$

Applying (22) and (23), we get

$$T(r,f) = T\left(r,\frac{1}{f}\right) + O(1) = m\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f}\right) + O(1) = S(r,f),$$

a contradiction.

Case 2. $U(z) \equiv 0$. By the expression of $U(z)$, and noting that $\alpha_{11}(z), \alpha_{k1}(z)$ are differential polynomials of $\frac{\phi'}{\phi}$, we conclude that

$$T(r,\phi) = m(r,\phi) = S(r,\phi) \tag{24}$$

and

$$a\phi^{n+2}(z) \equiv -2\alpha_{11}(z)\alpha_{k1}^{n}(z). \tag{25}$$

Using (24), we conclude that $\phi(z)$ is a polynomial or a constant. If $\phi(z)$ is a polynomial, the left-hand side of (25) is a polynomial, and the right-hand side of (25) is a constant or rational function, a contradiction. So, $\phi(z)$ is a constant. If $\phi(z) \equiv 0$, then $\frac{g'}{g} = f\phi \equiv 0$, that is, g is a constant, a contradiction. If $\phi(z) \equiv C (C \neq 0)$, then we substitute this equality into (16) and get

$$(a_1 + a_4 + a_5)C^2 f^2 + (a_2 + a_3)Cf' \equiv 0.$$

Using (25) for $k \ge 2$, we have $a_1 + a_4 + a_5 = -\frac{(nk+n+4)^2(nk+n+6)}{nk+n+2} \neq 0$ and $a_2 + a_3 = (nk + n + 4)^2(nk + n + 6) \neq 0$. Thus $(\frac{1}{f})' = -\frac{f'}{f^2} \equiv -\frac{C}{nk+n+2} \neq 0$, and f is a rational function, a contradiction.

We now consider the case $k = 1$. Similar to the proof of the case $k \ge 2$, we obtain a contradiction. □

Lemma 2.4 *Let $f(z)$, $g(z)$, $h(z)$, and $F(z)$ be stated as in Lemma 2.3. Then all simple poles of $f(z)$ are zeros of $F(z)$.*

Proof Suppose that z_0 is a simple pole of $f(z)$, then

$$f(z)=\frac{A}{z-z_0}\bigl\{1+b(z-z_0)+c(z-z_0)^2+O\bigl((z-z_0)^3\bigr)\bigr\},$$

where $A\neq 0$, b,c are constants. We consider the following two cases.

Case 1. $k=1$. We have

$$\begin{aligned}
g(z)&=f^2\bigl(f'(z)\bigr)^n-a\\
&=\frac{(-1)^nA^{n+2}}{(z-z_0)^{2n+2}}\bigl\{1+2b(z-z_0)+\bigl[b^2-(n-2)c\bigr](z-z_0)^2+O\bigl((z-z_0)^3\bigr)\bigr\},\\
h(z)&=\frac{g'(z)}{f(z)}\\
&=\frac{(-1)^{n+1}2A^{n+1}}{(z-z_0)^{2n+2}}\\
&\quad\times\bigl\{(n+1)+nb(z-z_0)-\bigl(n^2-n+1\bigr)c(z-z_0)^2+O\bigl((z-z_0)^3\bigr)\bigr\}.
\end{aligned}$$

By using the above two equalities, we have

$$\begin{aligned}
\frac{g'(z)}{g(z)}&=\frac{-2}{z-z_0}\bigl\{(n+1)-b(z-z_0)+\bigl[b^2+(n-2)c\bigr](z-z_0)^2+O\bigl((z-z_0)^3\bigr)\bigr\},\\
\frac{h'(z)}{h(z)}&=\frac{-1}{z-z_0}\biggl\{2(n+1)-\frac{n}{n+1}b(z-z_0)\\
&\quad+\frac{n^2b^2+2(n+1)(n^2-n+1)c}{(n+1)^2}(z-z_0)^2+O\bigl((z-z_0)^3\bigr)\biggr\},\\
\biggl(\frac{g'(z)}{g(z)}\biggr)^2&=\frac{4}{(z-z_0)^2}\bigl\{(n+1)^2-2(n+1)b(z-z_0)\\
&\quad+\bigl[(2n+3)b^2+2(n+1)(n-2)c\bigr](z-z_0)^2+O\bigl((z-z_0)^3\bigr)\bigr\},\\
\biggl(\frac{g'(z)}{g(z)}\biggr)'&=\frac{2}{(z-z_0)^2}\bigl\{(n+1)-\bigl[b^2+(n-2)c\bigr](z-z_0)^2+O\bigl((z-z_0)^3\bigr)\bigr\},\\
\biggl(\frac{h'(z)}{h(z)}\biggr)'&=\frac{1}{(z-z_0)^2}\biggl\{2(n+1)-\frac{n^2b^2+2(n+1)(n^2-n+1)c}{(n+1)^2}(z-z_0)^2\\
&\quad+O\bigl((z-z_0)^3\bigr)\biggr\},\\
\biggl(\frac{h'(z)}{h(z)}\biggr)^2&=\frac{1}{(z-z_0)^2}\biggl\{4(n+1)^2-4nb(z-z_0)\\
&\quad+\frac{(4n+5)n^2b^2+8(n+1)^2(n^2-n+1)c}{(n+1)^2}(z-z_0)^2+O\bigl((z-z_0)^3\bigr)\biggr\},\\
\frac{g'(z)}{g(z)}\frac{h'(z)}{h(z)}&=\frac{2}{(z-z_0)^2}\bigl\{2(n+1)^2-(3n+2)b(z-z_0)
\end{aligned}$$

$$+ \left[(3n+2)b^2 + 2\left(2n^2 - 2n - 1\right)c\right](z-z_0)^2 + O\left((z-z_0)^3\right)\right\}.$$

By substituting the above equalities into (13) and performing some easy calculations, we have $F(z) = O((z-z_0))$, consequently, and so z_0 is a zero of $F(z)$.

Case 2. $k \geq 2$. We have

$$\begin{aligned} g(z) &= f^2\left(f^{(k)}\right)^n - a \\ &= \frac{(-1)^{nk}(k!)^n A^{n+2}}{(z-z_0)^{nk+n+2}}\left\{1 + 2b(z-z_0) + \left(b^2+2c\right)(z-z_0)^2 + O\left((z-z_0)^3\right)\right\} \end{aligned}$$

and

$$\begin{aligned} h(z) = \frac{g'(z)}{f(z)} = \frac{(-1)^{nk+1}(k!)^n A^{n+1}}{(z-z_0)^{nk+n+2}}&\left\{(nk+n+2)\right. \\ &\left. + (nk+n)b(z-z_0) + (nk+n-2)c(z-z_0)^2 + O\left((z-z_0)^3\right)\right\}. \end{aligned}$$

Using the above two equalities, we get

$$\begin{aligned} \frac{g'(z)}{g(z)} &= \frac{-1}{z-z_0}\left\{(nk+n+2) - 2b(z-z_0) + 2\left(b^2-2c\right)(z-z_0)^2 + O\left((z-z_0)^3\right)\right\}, \\ \frac{h'(z)}{h(z)} &= \frac{-1}{z-z_0}\left\{(nk+n+2) - \frac{nk+n}{nk+n+2}b(z-z_0)\right. \\ &\quad \left. + \frac{1}{nk+n+2}\left[\frac{(nk+n)^2b^2}{nk+n+2} - 2(nk+n-2)c\right](z-z_0)^2 + O\left((z-z_0)^3\right)\right\}, \\ \left(\frac{g'(z)}{g(z)}\right)^2 &= \frac{1}{(z-z_0)^2}\left\{(nk+n+2)^2 - 4(nk+n+2)b(z-z_0)\right. \\ &\quad \left. + \left[4(nk+n+3)b^2 - 8(nk+n+2)c\right](z-z_0)^2 + O\left((z-z_0)^3\right)\right\}, \\ \left(\frac{g'(z)}{g(z)}\right)' &= \frac{1}{(z-z_0)^2}\left\{(nk+n+2) - 2\left(b^2-2c\right)(z-z_0)^2 + O\left((z-z_0)^3\right)\right\}, \\ \left(\frac{h'(z)}{h(z)}\right)' &= \frac{1}{(z-z_0)^2}\left\{(nk+n+2)\right. \\ &\quad \left. - \frac{1}{nk+n+2}\left[\frac{(nk+n)^2b^2}{nk+n+2} - 2(nk+n-2)c\right](z-z_0)^2 + O\left((z-z_0)^3\right)\right\}, \\ \left(\frac{h'(z)}{h(z)}\right)^2 &= \frac{1}{(z-z_0)^2}\left\{(nk+n+2)^2 - 2(nk+n)b(z-z_0)\right. \\ &\quad \left. + \left[\frac{(2nk+2n+5)(nk+n)^2b^2}{(nk+n+2)^2} - 4(nk+n-2)c\right](z-z_0)^2 + O\left((z-z_0)^3\right)\right\}, \\ \frac{g'(z)}{g(z)}\frac{h'(z)}{h(z)} &= \frac{1}{(z-z_0)^2}\left\{(nk+n+2)^2 - (3nk+3n+4)b(z-z_0)\right. \\ &\quad \left. + \left[(3nk+3n+4)b^2 - 2(3nk+3n+2)c\right](z-z_0)^2 + O\left((z-z_0)^3\right)\right\}. \end{aligned}$$

By substituting the above equalities into (13) and performing some easy calculations, we again get $F(z) = O((z-z_0))$. It also shows that z_0 is a zero of $F(z)$. □

Definition 2.1 ([3]) Let f be a nonconstant meromorphic function in the complex plane and k be a positive integer. We call $M[f]=f^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}$ a differential monomial in f, where $n_0,n_1,\ldots,n_k$ are nonnegative integers, and $\gamma_M:=n_0+n_1+\cdots+n_k$ its degree. Further, let $M_j[f]$ denote differential monomials in f of degree γ_{M_j} for $j=1,2,\ldots,k$, and let α_j be meromorphic functions satisfying $T(r,\alpha_j)=S(r,f)$ for $j=1,2,\ldots,k$, then $P[f]=\alpha_1M_1[f]+\alpha_2M_2[f]+\cdots+\alpha_kM_k[f]$ is called a differential polynomial in f of degree $\gamma_P:=\max_{1\le j\le k}\gamma_{M_j}$. If the coefficients α_j only satisfy $m(r,\alpha_j)=S(r,f)$, then we call the function $P[f]$ a quasi-differential polynomial in f.

Lemma 2.5 ([3]) *Let f be a nonconstant meromorphic function and $Q^*[f]$, $Q[f]$ be quasi-differential polynomials in f with $Q[f]\not\equiv 0$. Let n be a positive integer and $f^nQ^*[f]=Q[f]$. If $\gamma_Q\le n$, then $m(r,Q^*[f])=S(r,f)$, where γ_Q is the degree of $Q[f]$.*

3 Proofs of theorems

In this section, we mainly give complete proofs for our main results.

Proof of Theorem 1.6. In what follows, we consider two cases.

Case 1. When $l\ge 3$, $n\ge 1$, $k\ge 1$, by inequality (4), we have

$$T(r,f)\le\frac{1}{l-2}\overline{N}\left(r,\frac{1}{f^l(f^{(k)})^n-a}\right)+S(r,f).$$

Case 2. When $l=2$, $n\ge 1$, $k\ge 1$, we consider two subcases.

Subcase 2.1. First we suppose that $k\ge 2$. From Lemma 2.3 and Lemma 2.4, we see immediately that $F\not\equiv 0$ and simple poles of $f(z)$ are the zeros of $F(z)$. We also conclude that the poles of $F(z)$ are with multiplicities two at most, which come from the multiple poles of $f(z)$, or from the zeros of $g(z)$, or from the zeros of $h(z)$.

Set $\beta=2f'(f^{(k)})^n+nf(f^{(k)})^{n-1}f^{(k+1)}-f(f^{(k)})^n\frac{g'}{g}$. Then $f\beta=-a\frac{g'}{g}$ and $h=-\frac{1}{a}\beta g$. We now consider the poles of β^2F. We note that the multiple poles of f with multiplicity $q(\ge 2)$ are the zeros of β with multiplicity $q-1$, and the zeros of h are either the zeros of g or the zeros of β. Thus,

$$N\bigl(r,\beta^2F\bigr)\le 4\overline{N}\left(r,\frac{1}{g}\right),$$

since the poles of β^2F come only from the zeros of g, and the multiplicity of poles of β^2F is 4 at most.

Noting that $m(r,F)=S(r,f)$ and $m(r,\beta^2)=S(r,f)$ from Lemma 2.5, we have $m(r,\beta^2F)=S(r,f)$. Therefore,

$$T\bigl(r,\beta^2F\bigr)\le 4\overline{N}\left(r,\frac{1}{g}\right).$$

Since the simple poles of f are the zeros of β^2F, hence

$$N_1(r,f)\le N\left(\frac{1}{\beta^2F}\right)\le T\bigl(r,\beta^2F\bigr)\le 4\overline{N}\left(r,\frac{1}{g}\right). \qquad (26)$$

It follows from (7) and (26) that

$$\begin{aligned}
&2(n+2)T(r,f)+N_1(r,f)\\
&\quad\le 2\overline{N}(r,f)+2\overline{N}\left(r,\frac{1}{f}\right)+6\overline{N}\left(r,\frac{1}{g}\right)+2nN_{k)}\left(r,\frac{1}{f}\right)\\
&\qquad+2nk\overline{N}_{(k+1}\left(r,\frac{1}{f}\right)-2N_0\left(r,\frac{1}{g'}\right)+S(r,f)
\end{aligned}$$

i.e.

$$\begin{aligned}
&2(n+2)T(r,f)+N_1(r,f)-2\overline{N}\left(r,\frac{1}{f}\right)\\
&\qquad-2nN_{k)}\left(r,\frac{1}{f}\right)-2nk\overline{N}_{(k+1}\left(r,\frac{1}{f}\right)\\
&\quad\le 2\overline{N}(r,f)+6\overline{N}\left(r,\frac{1}{g}\right)-2N_0\left(r,\frac{1}{g'}\right)+S(r,f),
\end{aligned}$$

which leads to

$$\begin{aligned}
&T(r,f)+m(r,f)+2(n+1)m\left(r,\frac{1}{f}\right)+\left[N(r,f)+N_1(r,f)-2\overline{N}(r,f)\right]\\
&\qquad+2\left[N\left(r,\frac{1}{f}\right)-\overline{N}\left(r,\frac{1}{f}\right)\right]\\
&\qquad+2n\left[N\left(r,\frac{1}{f}\right)-N_{k)}\left(r,\frac{1}{f}\right)-k\overline{N}_{(k+1}\left(r,\frac{1}{f}\right)\right]\\
&\quad\le 6\overline{N}\left(r,\frac{1}{g}\right)-2N_0\left(r,\frac{1}{g'}\right)+S(r,f).
\end{aligned}\tag{27}$$

We note that

$$\begin{aligned}
&N(r,f)+N_1(r,f)-2\overline{N}(r,f)\\
&\quad=N_1(r,f)+N_{(2}(r,f)+N_1(r,f)-2\left[N_1(r,f)+\overline{N}_{(2}(r,f)\right]\\
&\quad=N_{(2}(r,f)-2\overline{N}_{(2}(r,f)\ge 0
\end{aligned}$$

and

$$\begin{aligned}
&N\left(r,\frac{1}{f}\right)-N_{k)}\left(r,\frac{1}{f}\right)-k\overline{N}_{(k+1}\left(r,\frac{1}{f}\right)\\
&\quad=N_{(k+1}\left(r,\frac{1}{f}\right)-k\overline{N}_{(k+1}\left(r,\frac{1}{f}\right)\\
&\quad\ge N_{(k+1}\left(r,\frac{1}{f}\right)-\frac{k}{k+1}N_{(k+1}\left(r,\frac{1}{f}\right)\ge 0.
\end{aligned}$$

By combining the above two inequalities and (27), we have

$$T(r,f)\le 6\overline{N}\left(r,\frac{1}{g}\right)+S(r,f).\tag{28}$$

Subcase 2.2. Suppose that $k = 1$. Set $\beta = 2(f')^{n+1} + nf(f')^{n-1}f'' - f(f')^n\frac{g'}{g}$. Then $f\beta = -a\frac{g'}{g}$ and $h = -\frac{1}{a}\beta g$. We again consider the poles of $\beta^2 F$.

Arguing similarly as in Subcase 2.1, we have

$$T(r,\beta^2 F) \le 4\overline{N}\left(r,\frac{1}{g}\right),$$

and (26) is still valid.

It follows from (7) and (26) that

$$\begin{aligned}
&2(n+2)T(r,f) + N_1(r,f) \\
&\quad \le 2\overline{N}(r,f) + 2\overline{N}\left(r,\frac{1}{f}\right) + 6\overline{N}\left(r,\frac{1}{g}\right) + 2nN_1\left(r,\frac{1}{f}\right) \\
&\qquad + 2n\overline{N}_{(2}\left(r,\frac{1}{f}\right) - 2N_0\left(r,\frac{1}{g'}\right) + S(r,f) \\
&\quad = 2\overline{N}(r,f) + 2(n+1)\overline{N}\left(r,\frac{1}{f}\right) + 6\overline{N}\left(r,\frac{1}{g}\right) - 2N_0\left(r,\frac{1}{g'}\right) + S(r,f),
\end{aligned}$$

which yields

$$\begin{aligned}
&T(r,f) + m(r,f) + 2(n+1)m\left(r,\frac{1}{f}\right) + \left[N(r,f) + N_1(r,f) - 2\overline{N}(r,f)\right] \\
&\quad + 2(n+1)\left[N\left(r,\frac{1}{f}\right) - \overline{N}\left(r,\frac{1}{f}\right)\right] \le 6\overline{N}\left(r,\frac{1}{g}\right) - 2N_0\left(r,\frac{1}{g'}\right) + S(r,f).
\end{aligned}$$

Noting that $N(r,f) + N_1(r,f) - 2\overline{N}(r,f) \ge 0$ and $N(r,\frac{1}{f}) - \overline{N}(r,\frac{1}{f}) \ge 0$, we have

$$T(r,f) \le 6\overline{N}\left(r,\frac{1}{g}\right) + S(r,f). \tag{29}$$

Thus, from the above two cases, we have

$$T(r,f) \le M\overline{N}\left(r,\frac{1}{f^l(f^{(k)})^n - a}\right) + S(r,f)$$

for $M = \min\{\frac{1}{l-2}, 6\}$ and positive integers $l(\ge 2)$, $n(\ge 1)$, $k(\ge 1)$. □

Proof of Corollary 1.1 Set $\psi := f^l(f^{(k)})^n$, where $l(\ge 2)$, $n(\ge 1)$, $k(\ge 1)$ are integers. It follows from Lemma 2.1 that $\psi \not\equiv 0$. By using Theorem 1.6, we have

$$T(r,f) \le M\overline{N}\left(r,\frac{1}{\psi - a}\right) + S(r,f), \tag{30}$$

where $M = \min\{\frac{1}{l-2}, 6\}$.

Applying the lemma of the logarithmic derivative, we get

$$\begin{aligned}
T(r,\psi) &= T\left(r,f^l\left(f^{(k)}\right)^n\right) \\
&\le lT(r,f) + n\left[m\left(r,\frac{f^{(k)}}{f}\right) + m(r,f) + N(r,f) + k\overline{N}(r,f)\right]
\end{aligned}$$

$$\leq (nk+n+l)T(r,f)+S(r,f) \tag{31}$$

and

$$\begin{aligned}(n+l)T(r,f) &= T\left(r,\frac{1}{f^{n+l}}\right)+O(1) = T\left(r,\left(\frac{f^{(k)}}{f}\right)^n\frac{1}{\psi}\right)+O(1)\\ &\leq N\left(r,\left(\frac{f^{(k)}}{f}\right)^n\right)+T(r,\psi)+S(r,f)\\ &= nk\left[\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)\right]+T(r,\psi)+S(r,f)\\ &\leq (2nk+1)T(r,\psi)+S(r,f).\end{aligned} \tag{32}$$

It follows from (31) and (32) that

$$S(r,f) = S(r,\psi). \tag{33}$$

Combining (30), (31), and (33), we have

$$T(r,\psi) \leq M(nk+n+l)\overline{N}\left(r,\frac{1}{\psi-a}\right)+S(r,\psi), \tag{34}$$

where $M = \min\{\frac{1}{l-2},6\}$. By the definition of the deficiency $\Theta(a,\psi)$ and (33), we have, for $M = \min\{\frac{1}{l-2},6\}$,

$$\begin{aligned}\Theta(a,\psi) &= 1-\limsup_{r\to\infty}\frac{\overline{N}(r,\frac{1}{\psi-a})}{T(r,\psi)}\\ &\leq 1-\limsup_{r\to\infty}\frac{\frac{1}{M(nk+n+l)}T(r,\psi)-S(r,\psi)}{T(r,\psi)}\\ &= 1-\frac{1}{M(nk+n+l)}.\end{aligned}$$

□

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Authors' contributions
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