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THE GROWTH OF SOLUTIONS TO HIGHER ORDER DIFFERENTIAL EQUATIONS WITH EXPONENTIAL POLYNOMIALS AS ITS COEFFICIENTS*

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Abstract By looking at the situation when the coefficients $P_j(z)$ $(j = 1, 2, \cdots, n-1)$ (or most of them) are exponential polynomials, we investigate the fact that all nontrivial solutions to higher order differential equations $f^{(n)} + P_{n-1}(z)f^{(n-1)} + \cdots + P_0(z)f = 0$ are of infinite order. An exponential polynomial coefficient plays a key role in these results.

Key words differential equations; entire solution; exponential polynomial; growth

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1 Introduction

This paper is devoted to considering the growth of solutions to higher order linear differential equations

$$f^{(n)} + P_{n-1}(z)f^{(n-1)} + \cdots + P_0(z)f = 0, \tag{1.1}$$

where $P_0(z) \not\equiv 0$ and $P_j(z)$ $(j = 1, 2, \cdots, n-1)$ are entire functions. Due to the classical result by Wittich [20], all solutions to (1.1) are entire functions with finite order if and only if all coefficients are polynomials. If $\max\{\rho(P_j), j = 1, 2, \cdots, n-1\} < \rho(P_0)$, then every nontrivial solution to (1.1) is of infinite order. In this paper, we are concentrating on looking at the situation when the coefficients (or most of them) of (1.1) are exponential polynomials.

Throughout this paper, an important and basic tool in our discussion is Nevanlinna theory, see e.g. [3, 9, 22, 23]. For a meromorphic function $f(z)$, we denote by $T(r, f)$ and $N(r, f)$, the characteristic function and the counting function of $f(z)$, respectively. In particular, we define the order and the lower order of a meromorphic function $f(z)$ by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r, f)}{\log r} \text{ and } \mu(f) = \liminf_{r\to\infty} \frac{\log^+ T(r, f)}{\log r},$$

while

$$\overline{\text{logdens}}(F) = \limsup_{r\to\infty} \frac{m_l(F \cap [1, r])}{\log r} \text{ and } \underline{\text{logdens}}(F) = \liminf_{r\to\infty} \frac{m_l(F \cap [1, r])}{\log r}$$

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stand for the upper and lower logarithmic densities of $F \subset [1,\infty)$, respectively.

We now recall the exponential polynomial of order n, which is defined by

$$g(z) = p_1(z)\mathrm{e}^{q_1(z)} + \cdots + p_l(z)\mathrm{e}^{q_l(z)},$$

where p_j and q_j are polynomials in z with degree $\deg(g) = \max_{1\le j\le l}\{\deg(q_j)\} = n$. As described in [12], $g(z)$ can be written in the normalized form

$$g(z) = H_0(z) + H_1(z)\mathrm{e}^{\omega_1 z^n} + \cdots + H_m(z)\mathrm{e}^{\omega_m z^n}, \tag{1.2}$$

where $m \le l$, H_j are either exponential polynomials of order $\le n-1$ or polynomials in z, and ω_j are pairwise different nonzero complex constants referred to as the leading coefficients of g.

We second recall that the Phragmén-Lindelöf indicator function of an entire function g of finite order $\rho(g) = \rho > 0$ is

$$h_g(\theta) = \limsup_{r\to\infty} r^{-\rho}\log|g(r\mathrm{e}^{\mathrm{i}\theta})|, \quad \theta \in [-\pi,\pi).$$

For an exponential polynomial (1.2), we can get that $h_g(\theta) = \max_{1\le j\le m}\{\Re(\omega_j \mathrm{e}^{in\theta})\}$, and so it follows from the proof of ([11], Satz 4) (see also [12], p. 462) that

$$h_g(\theta) = \lim_{r\to\infty} r^{-n}\log|g(r\mathrm{e}^{\mathrm{i}\theta})| \tag{1.3}$$

with finitely many possible exceptional values θ on $[-\pi,\pi)$. Hence an exponential polynomial is of completely regular growth (c.r.g.) on every ray with at most finitely many exceptions. By ([14],Theorem 1.3.4) or ([15], p. 140), it follows that exceptional rays are not possible, and hence g is of completely regular growth.

Considering infinite order solutions to the second order differential equation

$$f'' + A(z)f' + B(z)f = 0, \tag{1.4}$$

with entire coefficients $A(z)$ and $B(z)$, a natural goal that arises is to find the assumptions on the coefficients $A(z)$ and $B(z)$. It is known that (i) if either $\rho(A) < \rho(B)$ or $A(z)$ is a polynomial and $B(z)$ is transcendental, or (ii) if either $\rho(B) < \rho(A) \le 1/2$ or $A(z)$ is transcendental with $\rho(A) = 0$ and $B(z)$ is a polynomial, then every solution $f \not\equiv 0$ of (1.4) is of infinite order [4, 7, 13, 21]. Kwon improved Theorem 4 in [4], in which the angular sector $\theta_1 \le \arg z \le \theta_2$ is replaced by a smaller set E, and obtained

Theorem 1.1 ([8, Theorem 3]) Let E be a set of complex numbers satisfying $\overline{\mathrm{dens}}\{|z| : z \in E\} > 0$, and let $A(z)$ and $B(z)$ be entire functions which satisfy

$$|A(z)| \le \exp\{o(1)|z|^\beta\}$$

and

$$|B(z)| \ge \exp\{(1+o(1))\alpha|z|^\beta\},$$

respectively, as $z \to \infty$ in E. Then every solution $f \not\equiv 0$ of (1.4) is of infinite order, and hyper order $\rho_2(f) \ge \beta$.

If $\rho(A) = \rho(B)$ with different types, Wang and Chen [17] obtained that every nontrivial entire solution of (1.4) satisfies $\mu(f) = \infty$. However, if $\rho(A) = \rho(B)$, the conclusions are usually false. For example, $f(z) = \exp\{P(z)\}$ satisfies the differential equation

$$f'' + A(z)f' + (-P'' - (P')^2 - A(z)P')f = 0,$$

where $A(z)$ is a transcendental entire function and $P(z)$ is a nonconstant polynomial. Thus, for this situation, we can recall Theorem 2.1 from Heittokangas et al.[6].

Theorem 1.2 ([6, Theorem 2.1]) Let $A(z)$ and $B(z)$ be entire such that $\rho(A) = \rho(B) \in (0, \infty)$, and assume that both are of finite type. If (1.4) possesses a solution $f \not\equiv 0$ of finite order, then

$$h_B(\theta) \le \max\{0, h_A(\theta)\}, \quad \theta \in [-\pi, \pi).$$

In particular, if there exists an $\theta_0 \in [-\pi, \pi)$ such that $\max\{0, h_A(\theta_0)\} < h_B(\theta_0)$, then all solutions of (1.4) are of infinite order.

Heittokangas et al. [6, 19] studied the entire solutions of differential equation (1.1) with exponential polynomial coefficients and obtained some profound results. In this paper, we are considering the infinite order solutions of (1.1) with exponential polynomials as its coefficients and obtain

Theorem 1.3 Let $P_j(z)$ $(j = 0, \cdots, n-1)$ be entire such that $\rho(P_j) = \rho \in (0, \infty)$, and assume that all of them are of finite type. If (1.1) possesses a solution $f \not\equiv 0$ of finite order , then

$$h_{P_0}(\theta) \le \max\{0, h_{P_1}(\theta), h_{P_2}(\theta), \cdots, h_{P_{n-1}}(\theta)\}, \quad \theta \in [-\pi, \pi). \tag{1.5}$$

In particular, if there exists a $\theta_0 \in [-\pi, \pi)$ such that $\max\{0, h_{P_1}(\theta_0), h_{P_2}(\theta_0), \cdots, h_{P_{n-1}}(\theta_0)\} < h_{P_0}(\theta_0)$, then all solutions of (1.1) are of infinite order.

Remark 1.4 Inequality (1.5) cannot be replaced by

$$h_{P_0}(\theta) \le \max\{h_{P_1}(\theta), h_{P_2}(\theta), \cdots, h_{P_{n-1}}(\theta)\}, \quad \theta \in [-\pi, \pi).$$

For example, when $P_0(z) = 2,\ P_1(z) = 1 - \mathrm{e}^z$ and $P_2(z) = \mathrm{e}^z$, we have that $h_{P_0}(\theta) = 0$ and $h_{P_1}(\theta) = h_{P_2}(\theta) = \cos\theta$, though $f(z) = \mathrm{e}^{-z} - 1$ solves the differential equation

$$f^{'''} + P_2(z)f'' + P_1(z)f' + P_0 f = 0.$$

Obviously, $h_{P_0}(\theta) \le \max\{h_{P_1}(\theta), h_{P_2}(\theta)\}$ when $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, but $\max\{h_{P_1}(\theta), h_{P_2}(\theta)\} < h_{P_0}(\theta)$ when $\theta \in [-\pi, \pi) \backslash [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Theorem 1.5 Let $P_1(z)$ be an entire function of completely regular growth, and let $P_j (j \neq 1)$ be any entire function such that $\rho(P_1) > \max\{\rho(P_j) : j \neq 1\}$. Define $E = \{\theta \in [-\pi, \pi) : h_{P_1}(\theta) \le 0\}$. Then every solution $f \not\equiv 0$ of (1.1) satisfies

$$\rho(f) \ge \max\left\{\rho(P_1), \left(21\sqrt{m(E)}\right)^{-1} - 1\right\},$$

while $\rho(f) = \infty$ if $m(E) = 0$.

Definition 1.6 ([6, 11]) The convex hull $\mathrm{co}(W)$ of a finite set $W \subset \mathbb{C}$ is the intersection of finitely many closed half-planes each containing W. Hence $\mathrm{co}(W)$ is either a compact polygon or a line segment. We denote the perimeter of $\mathrm{co}(W)$ by $C(\mathrm{co}(W))$. If $\mathrm{co}(W)$ is a line segment, then $C(\mathrm{co}(W))$ equals twice the length of this line segment. Related to the leading coefficients in (1.2), we denote that $W = \{\overline{w_1}, \cdots, \overline{w_m}\}$ and $W_0 = W \bigcup \{0\}$.

Definition 1.7 ([10]) Suppose that $f \not\equiv 0$ is a solution of differential equation (1.1). If f satisfies

$$\limsup_{r\to\infty} \frac{\log T(r, f)}{r^n} = 0,$$

then we say that the differential equation (1.1) has a nontrivial n-subnormal solution.

Theorem 1.8 Let $P_j(z)(j=0,\cdots,n-1)$ be entire functions, $\alpha>1$ be a constant, and let $f_1,f_2,\cdots,f_n$ be linearly independent solutions of (1.1). Then there exists a constant $r_0=r_0(\alpha)>0$ such that for all $r\geq r_0$, we have

$$T(r,P_{n-1})\leq \max\{\log T(\alpha r,f_1),\log T(\alpha r,f_2),\cdots,\log T(\alpha r,f_n)\}+O(\log r). \tag{1.6}$$

If $P_{n-1}(z)$ satisfies

$$\limsup_{r\to\infty} r^{-1}T(r,P_{n-1})>0, \tag{1.7}$$

then at least one of $f_1,f_2,\cdots,f_n$ cannot be a 1−subnormal solution. In particular, if $P_{n-1}(z)$ is an exponential polynomial

$$P_{n-1}(z)=H_0(z)+H_1(z)\mathrm{e}^{w_1z^n}+\cdots+H_m(z)\mathrm{e}^{w_mz^n},$$

with $\rho(P_{n-1})=n\geq 1$, then (1.7) holds and (1.6) can be replaced by

$$(C(\mathrm{co}(W_0))+o(1))\frac{r^n}{2\pi}\leq \max\{\log T(r,f_1),\log T(r,f_2),\cdots,\log T(r,f_n)\}, \tag{1.8}$$

where $W_0=\{0,\overline{w_0},\cdots,\overline{w_m}\}$.

Theorem 1.9 Let $A(z)$ and $B(z)$ be two exponential polynomials with degree n, and let $E=\{\theta\in[-\pi,\pi):h_A(\theta)=h_B(\theta)\}$ with mes$E=0$. Then

(i) if $h_A(\theta)<h_B(\theta)$, $\theta\in[-\pi,\pi)\backslash E$, every transcendental solution f of (1.4) satisties $\rho(f)=\infty$ and $\rho_2(f)=n$;

(ii) if $h_A(\theta)>h_B(\theta)$, $\theta\in[-\pi,\pi)\backslash E$, every finite order solution f of (1.4) satisties $\rho(f)=n$.

Example 1.10 When $A(z)=\mathrm{e}^{2z}$ and $B(z)=-(\mathrm{e}^{3z}+\mathrm{e}^{2z}+\mathrm{e}^{z})$, we have $0<h_A(\theta)=2\cos\theta<h_B(\theta)=3\cos\theta$ when $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$ and $h_A(\theta)=2\cos\theta<h_B(\theta)=\cos\theta<0$ when $\theta\in[-\pi,\pi)\backslash[-\frac{\pi}{2},\frac{\pi}{2}]$. Hence, $h_A(\theta)<h_B(\theta)$, $\theta\in[-\pi,\pi)\backslash E$. Obviously, $f(z)=\mathrm{e}^{\mathrm{e}^z}$ solves the differential equation (1.4), which satisfies $\rho(f)=\infty$ and $\rho_2(f)=1=\deg(A)=\deg(B)$.

Example 1.11 When $A(z)=-(\mathrm{e}^{3z}+\mathrm{e}^{2z}+\mathrm{e}^{z})$ and $B(z)=\mathrm{e}^{2z}$, we have $h_A(\theta)=3\cos\theta>h_B(\theta)=2\cos\theta>0$ when $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$. $0>h_A(\theta)=\cos\theta>h_B(\theta)=2\cos\theta$ when $\theta\in[-\pi,\pi)\backslash[-\frac{\pi}{2},\frac{\pi}{2}]$. Hence, $h_A(\theta)>h_B(\theta)$, $\theta\in[-\pi,\pi)\backslash E$. Obviously, $f(z)=\mathrm{e}^{-z}+1$ solves the differential equation (1.4), which satisfies $\rho(f)=1=\deg(A)=\deg(B)$.

2 Lemmas

In order to prove our Theorems, we need the following lemmas.

Lemma 2.1 Let $P_j(z)$ $(j=0,\cdots,n-1)$ be entire functions such that $\rho(P_1)>\max\{\rho(P_j):j\neq 1\}$. Then every solution $f\not\equiv 0$ of (1.1) satisfies $\rho(f)\geq\rho(P_1)$.

Proof Write (1.1) as

$$|P_1(z)|\leq\left|\frac{f^{(n)}}{f'}\right|+\left|P_{n-1}(z)\frac{f^{(n-1)}}{f'}\right|+\cdots+\left|P_2(z)\frac{f''}{f'}\right|+\left|P_0(z)\frac{f}{f'}\right|.$$

By the lemma of logarithmic derivative, we have

$$T(r,P_1)=m(r,P_1)\leq\sum_{j=0,j\neq 1}^{n-1}m(r,P_j)+m\left(r,\frac{f}{f'}\right)+O(\log rT(r,f))$$

$$\leq \sum_{j=0,j\neq 1}^{n-1} T(r,P_j) + 3T(r,f) + O(\log rT(r,f)).$$

The assertion then follows by the assumption $\rho(P_1) > \max\{\rho(P_j) : j \neq 1\}$. □

Lemma 2.2 ([16, Theorem 1]) Suppose that F_j $(1 < j < L)$ are entire functions with order not exceeding $\rho < \infty$. Suppose that c_j $(1 < j < L)$ are complex numbers lying in a sector with the vertex at the origin and an angle opening 2γ for some γ in $[0, \frac{\pi}{2})$. For $\beta \in (0,1)$ and $r > 0$, let

$$U_r = \left\{\theta \in [0,2\pi] : \left|\sum_{j=1}^{L} c_j r\mathrm{e}^{\mathrm{i}\theta} \frac{F_j'(r\mathrm{e}^{\mathrm{i}\theta})}{F_j(r\mathrm{e}^{\mathrm{i}\theta})}\right| \geq \beta \left|\sum_{j=1}^{L} c_j n(r,0,F_j)\right|\right\}.$$

Then, for $M > 3L$, there exists a set $E = E_M \subset [1,\infty)$ with lower logarithmic density of at least $1 - 3L/M$ such that

$$m(U_r) > \left(\frac{(1-\beta)\cos\gamma}{7M(\rho+1)}\right)^2, \quad r \in E.$$

Lemma 2.3 ([5]) Let (f, H) be a given pair where f has finite order ρ, and let $\varepsilon > 0$ be a given constant. Then there exists a set $E \subset (1,\infty)$ that has finite logarithmic measure, such that, for all z satisfying $|z| \notin E \bigcup [0,1]$ and for all $(k,j) \in H$, we have

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho-1+\varepsilon)}.$$

Lemma 2.4 ([9, Proposition 1.4.8]) Let $f_1, \cdots, f_n$ be linearly independent meromorphic solutions of

$$f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_0(z) f = 0,$$

with meromorphic coefficients. Then the Wronskian determinant $W(f_1, \cdots, f_n)$ satisfies the differential equation $W' + a_{n-1}(z)W = 0$. In partical, if $a_{n-1}(z)$ is an entire function, then for some $c \in \mathbb{C}$, $W(f_1, \cdots, f_n) = c \exp\varphi$, where φ is a primitive function of $-a_{n-1}(z)$.

Lemma 2.5 ([11]) Let g be given by (1.2). Then

$$T(r,g) = C(\mathrm{co}(W_0))\frac{r^n}{2\pi} + o(r^n). \tag{2.1}$$

If $H_0(z) \neq 0$, then

$$m\left(r, \frac{1}{g}\right) = o(r^n), \tag{2.2}$$

while if $H_0(z) \equiv 0$, then

$$N\left(r, \frac{1}{g}\right) = C(\mathrm{co}(W))\frac{r^n}{2\pi} + o(r^n). \tag{2.3}$$

Lemma 2.6 ([5, Theorem 2]) Let $f(z)$ be a transcendental meromorphic function and $\alpha > 1$. Then, for $\forall\varepsilon > 0$,

(i) $\exists B > 0$, and a set H_1 with finite logarithmic measure, when $|z| = r \notin H_1$, we have

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq B\left[\frac{T(\alpha r, f)}{r}(\log r)^{\alpha} \log T(\alpha r, f)\right]^{j-i} \quad (0 \leq i < j);$$

(ii) there exists a set $H_2 \subset [0, 2\pi)$ with zero linear measure and constant $B > 0$ such that, when $\theta \in [0, 2\pi)\backslash H_2$, there exists $R_0 = R_0(\theta) > 1$, and when $\arg z = \theta$, $|z| = r > R_0$, we have

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \le B[T(\alpha r, f)\log T(\alpha r, f)]^{j-i} \quad (0 \le i < j).$$

Lemma 2.7 ([1]) Let $A_0, A_1, \cdots, A_{k-1}$ be entire functions of finite order. If $f(z)$ is a solution of equation

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0 f = 0,$$

then $\rho_2(f) \le \max\{\rho(A_j) : j = 0, 1, \cdots, k-1\}$.

Lemma 2.8 ([18, Lemma 2.5]) Let f be an entire function and suppose that

$$G(z) := \frac{\log^+ |f^{(k)}(z)|}{|z|^\rho}$$

is unbounded on some ray $\arg z = \theta$ with constant $\rho > 0$. Then there exists an infinite sequence of points $z_n = r_n e^{i\theta}$ $(n = 1, 2, \cdots)$, where $r_n \to \infty$ such that $G(z_n) \to \infty$ and

$$\left|\frac{f^{(j)}(z_n)}{f^{(k)}(z_n)}\right| \le \frac{1}{(k-j)!}(1 + o(1))r_n^{k-j}, \quad j = 0, 1, \cdots, k-1$$

as $n \to \infty$.

Lemma 2.9 ([18, Lemma 2.6]) Let f be an entire function with $\rho(f) = \rho < \infty$. Suppose that there exists a set $E \subset [-\pi, \pi)$ which has linear measure zero such that $\log^+ |f(re^{i\theta})| \le Mr^\sigma$ for any ray $\arg z = \theta \in [-\pi, \pi)\backslash E$, where M is a positive constant depending on θ, while σ is a positive constant independent of θ. Then $\rho(f) < \sigma$.

3 Proofs of Theorems

Proof of Theorem 1.3 Write (1.1) as

$$-P_0(z) = \frac{f^{(n)}}{f} + P_{n-1}(z)\frac{f^{(n-1)}}{f} + \cdots + P_1(z)\frac{f'}{f},$$

and then use the logarithmic derivative estimate to see that (1.5) holds for almost every θ. Since $P_j(z) (j = 0, \cdots, n-1)$ are of finite type, the indicator functions are continuous, and so (1.5) holds for every θ. The remaining assertion is a trivial consequence of (1.5). □

Proof of Theorem 1.5 Lemma 2.1 shows that $\rho(f) \ge \rho(P_1)$. Suppose, on the contrary, that (1.1) has a solution $f \not\equiv 0$ with $\rho(f) < \left(21\sqrt{m(E)}\right)^{-1} - 1$. Then f is of finite order of growth. We now split our proof into two cases.

Case 1 $m(E) > 0$. Then there exists sufficiently small $\varepsilon > 0$ such that

$$\rho(f) < \frac{1-\varepsilon}{7(3+\varepsilon)\sqrt{m(E)}} - 1 < \left(21\sqrt{m(E)}\right)^{-1} - 1. \tag{3.1}$$

We assert that f has infinitely many zeros. Otherwise, $f = Pe^Q$ with polynomials $P \ne 0$ and Q. Substituting f into (1.1), we conclude that

$$A_n(P, Q) + A_{n-1}(P, Q)P_{n-1} + \cdots + A_1(P, Q)P_1 + PP_0 = 0,$$

where $A_j(P,Q)$ $(j=1,2,\cdots,n)$ are polynomials of P,Q and their derivatives, with degrees $j+1$. Since $\rho(P_1)>\max\{\rho(P_j):j\neq 1\}$, we have $A_1(P,Q)=P'+PQ'=0$, and so $P=c\mathrm{e}^{-Q}$ for some constant c, which is a contradiction.

Set $L=1$, $\gamma=0$, $\beta=\varepsilon$, $c_1=1$ and $M=3+\varepsilon$ in Lemma 2.2. Then we have

$$U_r=\left\{\theta\in[0,2\pi]:r\left|\frac{f'(r\mathrm{e}^{\mathrm{i}\theta})}{f(r\mathrm{e}^{\mathrm{i}\theta})}\right|\geq\varepsilon n\left(r,\frac{1}{f}\right)\right\}, \tag{3.2}$$

and there exists a set $F_1\subset[1,\infty)$ with $\underline{\log\mathrm{dens}}(F_1)\geq\frac{\varepsilon}{3+\varepsilon}$ such that

$$m(U_r)>\left(\frac{1-\varepsilon}{7(3+\varepsilon)(\rho(f)+1)}\right)^2,\quad r\in F_1. \tag{3.3}$$

Hence (3.1) and (3.3) yield that $m(U_r)>m(E)$ and $m(U_r\backslash E)>0$.

Since $P_1(z)$ is of completely regular growth, it follows by [14, Theorem 1.2.1] that

$$\log|P_1(z)|=r^{\rho(P_1)}h_{P_1}(\theta)+o(r^{\rho(P_1)}) \tag{3.4}$$

for $z=r\mathrm{e}^{\mathrm{i}\theta}$ outside of a possible C_0-set $D\subset\mathbb{C}$, which can be covered by a system of Euclidean discs $D(a_n,r_n)$ such that

$$\lim_{r\to\infty}r^{-1}\sum_{|a_n|\leq r}r_n=0. \tag{3.5}$$

Let F_2 be the projection of D onto the non-negative real axis. Then F_2 is covered by the intervals $(|a_n|-r_n,|a_n|+r_n)$ of length $2r_n$. Consequently, $\overline{\log\mathrm{dens}}(F_2)\leq\mathrm{dens}(F_2)=0$ by (3.5).

We note from Lemma 2.3 that there exists a set F_3 with $\overline{\log\mathrm{des}}(F_3)=0$ such that

$$\left|\frac{f^{(j)}(z)}{f(z)}\right|\leq|z|^{j(\rho(f)-1+\varepsilon)},|z|\notin F_3\bigcup[0,1],\quad j=1,2,\cdots,n. \tag{3.6}$$

Define $F=F_1\backslash(F_2\bigcup F_3)$. Then $\overline{\log\mathrm{dens}}(F)\geq\frac{\varepsilon}{3+\varepsilon}$. Rewrite (1.1) as

$$-P_1(z)\frac{f'}{f}=\frac{f^{(n)}}{f}+P_{n-1}(z)\frac{f^{(n-1)}}{f}+\cdots+P_2(z)\frac{f''}{f}+P_0(z). \tag{3.7}$$

Thus, for $z=r\mathrm{e}^{\mathrm{i}\theta}$ such that $r\in F$ and $\theta\in U_r\backslash E$, we have, from (3.2), (3.4), (3.6) and (3.7), that

$$\begin{aligned}&\varepsilon r^{-1}n\left(r,\frac{1}{f}\right)\exp(r^{\rho(P_1)}h_{P_1}(\theta)+o(r^{\rho(P_1)}))\\ \leq&[M(r,P_{n-1})+M(r,P_{n-2})+\cdots+M(r,P_2)+1]r^{n(\rho(f)-1+\varepsilon)}+M(r,P_0),\end{aligned}$$

where $h_{P_1}(\theta)>0$. This contradicts the fact that $\rho(P_1)>\max\{\rho(P_j):j\neq 1\}$.

Case 2 $m(E)=0$. Set $L=1$, $\gamma=0$, $\beta=\frac{1}{2}$, $c_1=1$ and $M=4$ in Lemma 2.2. Then we have

$$U_r=\left\{\theta\in[0,2\pi]:2r\left|\frac{f'(r\mathrm{e}^{\mathrm{i}\theta})}{f(r\mathrm{e}^{\mathrm{i}\theta})}\right|\geq\varepsilon n\left(r,\frac{1}{f}\right)\right\},$$

and there exists a set $F_4\subset[1,\infty)$ with $\underline{\log\mathrm{dens}}(F_4)\geq\frac{1}{4}$ such that

$$m(U_r)>\left(\frac{1}{56(\rho(f)+1)}\right)^2,\quad r\in F_4.$$

It is clear that $m(U_r)>0$ and $m(U_r\backslash E)>0$. The remainder of the proof follows Case 1. □

Proof of Theorem 1.8 If $f_1, \cdots, f_n$ are linearly independent solutions of (1.1), then we have, from Lemma 2.4, that the Wronskian determinant

$$W(f_1, \cdots, f_n) = C \exp\left\{-\int^z P_{n-1}(t)\mathrm{d}t\right\}$$

for some constant $C \neq 0$. By using the logarithmic derivative estimate [2, Corollary 3.2.3], we have, for $\beta = \sqrt{\alpha}$, that

$$T(r, P_{n-1}) = T\left(r, \frac{W'}{W}\right) = O(\log T(\beta r, W)) + O(\log r). \tag{3.8}$$

Set $E = f_1 f_2 \cdots f_n$. Then

$$\frac{W}{E} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \frac{f_1'}{f_1} & \frac{f_2'}{f_2} & \cdots & \frac{f_n'}{f_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{f_1^{(n-1)}}{f_1} & \frac{f_2^{(n-1)}}{f_2} & \cdots & \frac{f_n^{(n-1)}}{f_n} \end{vmatrix},$$

and so

$$T(r, W) = m(r, W) \leq \sum_{j=1}^{n}(1 + o(1))T(r, f_j) \tag{3.9}$$

for all r outside of a set $E \subset [0, \infty)$ of finite linear measure. Set $\sigma = \int_E \mathrm{d}r$, and $r_0 = (\sigma + 1)/(\beta - 1)$. By checking the proof of Lemma 1.1.1 in [9], we have

$$T(r, W) \leq 2n \max\{T(\beta r, f_1), T(\beta r, f_2), \cdots, T(\beta r, f_n)\} \tag{3.10}$$

for all $r \geq r_0$. Since $\beta^2 = \alpha$, we have proved (1.6) from (3.8) and (3.10).

We now assert that at least one of $f_1, f_2, \cdots, f_n$ cannot be a 1-subnormal solution of (1.1). On the contrary, suppose that $f_1, f_2, \cdots, f_n$ are all 1-subnormal solutions of (1.1). By (1.7), there exists a sequence $\{r_n\}$ of positive real numbers tending to infinity such that

$$\lim_{n\to\infty} r_n^{-1} T(r_n, P_{n-1}) > 0.$$

Now, substituting $r = r_n$ into (1.6) and dividing (1.6) by αr_n, we arrive at a contradiction, as $n \to \infty$.

Finally, suppose that $P_{n-1}(z)$ is an exponential polynomial. We deduce from Lemma 2.5 that

$$T(r, P_{n-1}) = (C(\mathrm{co}(W_0)) + o(1))\frac{r^n}{2\pi}. \tag{3.11}$$

(1.6) and (3.11) give that

$$(C(\mathrm{co}(W_0)) + o(1))\frac{r^n}{2\pi\alpha^n} \leq \max\{\log T(r, f_1), \log T(r, f_2), \cdots, \log T(r, f_n)\} \tag{3.12}$$

for all $r \geq \alpha r_0$. By choosing $\beta = \beta(r) = 1 + (\sigma + 1)/r$ for $r \geq 1$, we derive from Lemma 1.1.1 in [9] that

$$\alpha^{-n} = \beta^{-2n} = 1 + o(1), \quad r \to \infty,$$

and hence (3.12) yields (1.8). □

Proof of Theorem 1.9 (i) Suppose, on the contrary, that $\rho(f) < \infty$. We deduce from Lemma 2.3 that there exists a set $F_5 \subset [1,\infty)$ of finite logarithmic measure and a positive constant M such that for all z satisfying $|z| = r \notin F_5 \cup [0,1]$,

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq r^M. \tag{3.13}$$

Since $h_A(\theta) < h_B(\theta)$, there must exist an θ_0 such that $h_A(\theta_0) < h_B(\theta_0)$. Thus, we obtain from (1.3) that

$$|A(re^{i\theta_0})| \leq \exp\{(h_A(\theta_0)+\varepsilon)r^n\} \tag{3.14}$$

and

$$|B(re^{i\theta_0})| \geq \exp\{(h_B(\theta_0)-\varepsilon)r^n\} \tag{3.15}$$

for all ε $\left(0 < \varepsilon < \frac{h_B(\theta_0)-h_A(\theta_0)}{2}\right)$, and for all sufficiently large r. Therefore, we obtain from (1.4) and (3.13)–(3.15) that

$$\exp\{(h_B(\theta_0)-\varepsilon)r^n\} \leq |B(re^{i\theta_0})| \leq \left|\frac{f''(z)}{f(z)}\right| + |A(z)|\left|\frac{f'(z)}{f(z)}\right| \leq 2r^M \exp\{(h_A(\theta_0)+\varepsilon)r^n\},$$

which is a contradiction, and so $\rho(f) = \infty$.

We further obtain from Lemma 2.6 that there exists a set F_6 of finite logarithmic measure and a constant $B > 0$ such that, for all z satisfying $|z| = r \notin F_6$,

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq BT(2r,f)^{2k}, \quad 0 \leq i < j \leq k. \tag{3.16}$$

Thus, for all z satisfying $|z| = r \notin F_6$, we deduce, from (1.4), (3.14)–(3.15), that

$$\exp\{(h_B(\theta_0)-h_A(\theta_0)-2\varepsilon)r^n\} \leq 2BT(2r,f)^4,$$

and so $\rho_2(f) \geq n$. On the other hand, Lemma 2.7 shows that $\rho_2(f) \leq \max\{\rho(A),\rho(B)\} = n$. We then easily obtain that $\rho_2(f) = n$.

(ii) We first affirm that $G(z) = \frac{\log^+|f'(re^{i\theta})|}{r^n}$ is bounded on ray $\arg z = \theta \in [-\pi,\pi)\backslash E$. On the contrary, if $G(z)$ is unbounded on ray $\arg z = \theta \in [-\pi,\pi)\backslash E$, then we obtain from Lemma 2.8 that there exists a positive constant K and an infinite sequence of points $z_n = r_n e^{i\theta}$ $(n = 1,2,\cdots)$, where $r_n \to \infty$ such that $G(z_n) \to \infty$ and

$$\left|\frac{f(z_n)}{f'(z_n)}\right| \leq Kr_n$$

as $n \to \infty$.

Thus, for all $0 < \varepsilon < \frac{h_A(\theta)-h_B(\theta)}{2}$, we obtain from Lemma 2.3 and (1.4) that

$$\begin{aligned}\exp\{(h_A(\theta)-\varepsilon)r_n^n\} \leq |A(r_ne^{i\theta})| &\leq \left|\frac{f''(r_ne^{i\theta})}{f'(r_ne^{i\theta})}\right| + \left|B(r_ne^{i\theta})\frac{f(r_ne^{i\theta})}{f'(r_ne^{i\theta})}\right| \\ &\leq Kr_n^M \exp\{(h_B(\theta)+\varepsilon)r_n^n\},\end{aligned}$$

which is a contradiction. Thus, for some positive constant M,

$$\frac{\log^+|f'(re^{i\theta})|}{r^n} \leq M,$$

when $\theta \in [-\pi,\pi)\backslash E$. We then obtain from Lemma 2.9 that $\rho(f) = \rho(f') \leq n$.

If $\rho(f) < n$, we then deduce a contradiction. By the Wiman-Valiron theory, there exists a set $F_6 \subset (1, \infty)$ with finite logarithmic messure such that, for all z satisfying $|z| = r \notin F_6 \bigcup [0, 1]$, and $|f(z)| = M(r, f)$,

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{v(r)}{z}\right)^j (1 + o(1)), \quad j = 1, 2. \tag{3.17}$$

Since $\rho(f) < n$, we choose a sequence $\{z_t : z_t = r_t e^{i\theta_t}\}$ such that $|f(z_t)| = M(r_t, f)$, $\theta_t \in [-\pi, \pi)\backslash E$ and $\lim\limits_{t\to\infty} \theta_t = \theta_0 \in [-\pi, \pi)\backslash E$ with $r_t \notin F_6 \bigcup [0, 1]$ as $r_t \to \infty$. Then $\{z_t\}$ satisfies (3.17) and $v(r_t) < r_t^n$.

Since θ_0 satisfies $h_A(\theta_0) > h_B(\theta_0)$ and the continuity of $h_A(\theta)$ and $h_B(\theta)$, we have

$$\lim_{t\to\infty} [h_A(\theta_t) - h_B(\theta_t)] > 0.$$

Hence there exists $N > 0$ such that $h_A(\theta_t) - h_B(\theta_t) > \frac{1}{2}(h_A(\theta_0) - h_B(\theta_0)) > 0$ for $t > N$.

Since $A(z)$ and $B(z)$ are exponential polynomials with degree n, we obtain from (1.3), (1.4) and (3.17) that, for any given ε $(0 < \varepsilon < \frac{1}{4}(h_A(\theta_t) - h_B(\theta_t)))$,

$$\begin{aligned} &\exp\{h_A(\theta_t)(1 - \varepsilon) r_t^n\} \left(\frac{v(r_t)}{r_t}\right) (1 + o(1)) \\ &\le \left(\frac{v(r_t)}{r_t}\right)^2 (1 + o(1)) + \exp\{h_B(\theta_t)(1 + \varepsilon) r_t^n\}. \end{aligned} \tag{3.18}$$

Therefore, we obtain from (3.18) that, for some positive constant M,

$$r_t \exp\{(h_A(\theta_t) - h_B(\theta_t) - 2\varepsilon) r_t^n\} \le M v(r_t) < M r_t^n,$$

which is a contradiction. Thus, $\rho(f) = n$. □

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