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GROWTH OF SOLUTIONS TO HIGHER ORDER DIFFERENTIAL EQUATIONS WITH MITTAG-LEFFLER COEFFICIENTS

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Abstract. The classical problem of finding conditions on the entire coefficients A_j $(j = 0, 1, \cdots, k-1)$ ensuring that all nontrivial solutions to higher order differential equations $f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0$ are of infinite lower order is being discussed in this paper. In particular, we assume that the coefficients (or most of them) are Mittag-Leffler functions.

1. Introduction and main results

This paper is devoted to considering the growth of solutions to higher order linear differential equations

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = 0, \tag{1.1}$$

where $A_0(z) \not\equiv 0$ and $A_j(z)$ $(j = 1, 2, \cdots, k-1)$ are entire functions. As is well known, all solutions to (1.1) are entire functions. Due to the classical result by Wittich [14], all solutions to (1.1) are of finite order if and only if all coefficients are polynomials. As for number of linearly independent solutions of infinite order (in which case at least one of the coefficients is transcendental), see Frei [3] for the classical result and [8] for more detailed recent investigations. Trivially, if $\max\{\rho(A_j), j = 1, 2, \cdots, k-1\} < \rho(A_0)$, then every nontrivial solution to (1.1)

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is of infinite order. In this paper, we are concentrating to looking at the situation when the coefficients (or most of them) of (1.1) are Mittag-Leffler functions.

Throughout this paper, we use the key results and notations of the Nevanlinna theory of meromorphic functions, see e.g. [5, 16, 17]. In particular, we need to apply the notions of order ρ, lower order μ and hyper-order ρ_2 frequently. Moreover, we need to apply various notions of densities and measures. As a suitable reference for them, the reader may look at [7]. Also, we need to make use of the Wiman-Valiron theory, see e.g. [9, 12].

We next recall Mittag-Leffler function, see [5], p. 83–86:

$$E_\rho(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma\left(1+\frac{k}{\rho}\right)},\ 0<\rho<\infty. \tag{1.2}$$

Using Stirling's formula, it is easy to verify that this power series has an infinite radius of convergence. For several specific values of ρ, Mittag–Leffler function reduces back to well-known elementary functions such as to the exponential function e^z for $\rho = 1$ and to $\cos\sqrt{z}$ for $\rho = 1/2$. Furthermore, by use of the Hankel integral representation for $\Gamma-$function, Mittag–Leffler function $E_\rho(z)$ has the uniform asymptotic behavior

$$E_\rho(z) = \begin{cases} \rho\exp(z^\rho) + O(|z|^{-1}), & |\arg z| \le \frac{\pi}{2\rho} \\ O(|z|^{-1}), & \frac{\pi}{2\rho} < |\arg z| \le \pi. \end{cases}$$

Recalling the characteristic function

$$T(r, E_\rho) = \begin{cases} \frac{1}{\pi\rho} r^\rho + o(r^\rho), & \frac{1}{2} \le \rho < \infty \\ \frac{\sin\pi\rho}{\pi\rho} r^\rho + o(r^\rho), & 0<\rho<\frac{1}{2}, \end{cases}$$

we immediately conclude that Mittag–Leffler function $E_\rho(z)$ is a transcendental entire function of regular growth of order $\rho(E_\rho) = \mu(E_\rho)$.

Considering infinite order solutions to

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_0(z) f = 0 \tag{1.3}$$

with entire coefficients, a natural topic might be looking the oscillation and growth of such solutions. In particular, analysis of the hyper-order of these solutions would shed some light on the situation. As a simple example, we may recall [2], Theorem 4, due to Chen and Yan

Theorem 1.1. [2, Theorem 4] *Let $A_0(z), A_1(z), \cdots, A_{k-1}(z)$ be entire functions such that*

$$\max\{\rho(A_j), j = 1, 2, \cdots, k-1\} < \rho(A_0) < \infty.$$

Then every nontrivial entire solution f to (1.3) is of infinite order and satisfies $\rho_2(f) = \rho(A_0)$.

In this paper, we consider the hyper-order of transcendental entire solutions to (1.3) with Mittag-Leffler coefficients. We may list our main results as follows.

Theorem 1.2. *Suppose that $A_j(z) = E_{\rho_j}(z)$ with $\rho_j > \frac{1}{2}$ $(j = 0, 1, \cdots, k-1)$. If $\max\{\rho_j \ (j = 1, \cdots, k-1)\} = \rho < \rho_0$, then every nontrivial entire solution f to (1.3) satisfies $\mu(f) = \rho(f) = \infty$ and $\rho_2(f) = \rho_0$.*

Definition 1. An exponential polynomial of order n is an entire function of the form

$$g(z) = P_1(z)e^{Q_1(z)} + \cdots + P_l(z)e^{Q_l(z)},$$

where P_j and Q_j are polynomials in z with $\max\limits_{1\le j\le l}\{\deg(Q_j)\} = n$.

The Phragmén-Lindelöf indicator function of an entire g of finite order $\rho = \rho(g) > 0$ is

$$h_g(\theta) = \limsup_{r\to\infty} \frac{\log|g(re^{i\theta})|}{r^\rho}, \ \theta \in [-\pi, \pi).$$

For example, if $g(z) = \exp(wz^n)$, where $w \in \mathbb{C}\backslash\{0\}$ and n is a positive integer, then $h_g(\theta) = \Re(we^{in\theta})$. If g is of finite type, then h_g is continuous.

If $A_s(z)$ $(s \in \{0, 1, \cdots, k-1\})$ is a dominant coefficient, we further obtain

Theorem 1.3. *Let $A_s(z)$ be an exponential polynomial satisfying $h_{A_s}(\theta) > 0$ for all $\theta \in [-\pi, \pi)$. Suppose that $A_j(z) = E_{\rho_j}(z)$ with $\rho_j > \frac{1}{2}$ $(j \neq s, j = 0, 1, \cdots, k-1)$. If $\rho_s > \rho = \max\{\rho_j, \ j \neq s, j = 0, 1, \cdots, k-1\}$. Then every nontrivial entire solutions to (1.3) satisfies $\mu(f) = \rho(f) = \infty$ and $\rho_2(f) = \rho_s$.*

Theorem 1.4. *Suppose that the coefficients $A_j(z) = E_{\rho_j}(z)$ are of order $\rho_j > \frac{1}{2}$ $(j = 1, 2, 3, \cdots, k-1)$ with $\rho_1 \le \min\{\rho_j, \ (j = 2, 3, \cdots, k-1\}$ and $A_0(z)$ is an entire function with $0 < \mu(A_0) < 1$, and that $A_1(z)$ has a finite deficient value a. Then every nontrivial solution f to (1.3) satisfies $\mu(f) = \rho(f) = \infty$ and $\rho_2(f) \ge \mu(A_0)$.*

Theorem 1.5. *Suppose that $A_j(z) = E_{\rho_j}(z)$ are of order $\rho_j > \frac{1}{2}$ $(j = 1, \cdots, k-1)$ and that $\frac{1}{2} \le \mu(A_0) < \rho_1 \le \min\{\rho_j, \ (j = 2, \cdots, k-1)\}$. Then every nontrivial solution f to (1.3) satisfies $\rho(f) = \infty$ and $\rho_2(f) \ge \mu(A_0)$.*

2. Auxiliary results

We first recall the familiar growth property of complex exponential function $e^{Q(z)}$, where $Q(z) = a_n z^n + \cdots + a_1 z + a_0$ is a polynomial of degree n, see e.g. [13]. The complex plane divides into $2n$ equal open angles by the rays

$$\arg z = -\frac{\arg a_n}{n} + (2j-1)\frac{\pi}{2n},\ (j = 0, 1, \cdots, 2n-1).$$

In each of these sectors, $e^{Q(z)}$ either (1) blows up exponentially; or (2) decays to zero exponentially.

To consider entire functions that have a somewhat similar behavior as $e^{Q(z)}$, we define entire functions of exponential growth type, see [11], as follows.

Definition 2. [11, Definition 2] A transcendental entire function $A(z)$ is of exponential growth type, denoted as $A(z) \in \mathcal{A}$, provided that $\rho(A) = \mu(A) < \infty$, and for two positive constants c, d and for a real-valued function $\delta_A(\theta)$ defined on $[0, 2\pi)$, continuous outside an exceptional set F of finitely many points, it holds that for any given $\theta \in [0, 2\pi)\backslash F$, there are a constant τ, and positive constants $R = R(\theta)$ and $M = M(\theta)$ such that when $|z| = r > R$,

$$\begin{array}{lll} (\mathcal{A}_1) & |A(re^{i\theta})| \geq \exp\{c\delta_A(\theta)r^d\} & if \quad \delta_A(\theta) > 0, \\ (\mathcal{A}_2) & |A(re^{i\theta})| \leq Mr^{\tau} & if \quad \delta_A(\theta) < 0, \end{array}$$

where $\tau < 2(\rho(A) - 1)$.

Remark. Since Mittag-Leffler function $E_\rho(z)$ is of regular growth, we have $A(z) := E_\rho(z) \in \mathcal{A}$ for $\rho > \frac{1}{2}$. Moreover, $\delta_A(\theta) = -1$ for $\theta \in (\frac{\pi}{2\rho}, 2\pi - \frac{\pi}{2\rho})$, and $\delta_A(\theta) = \cos(\rho\theta)$ for $\theta \in [-\frac{\pi}{2\rho}, \frac{\pi}{2\rho})$, see [11].

We next recall four results that all are from the seminal paper [6].

Lemma 2.1. [6, Corollary 1] *Let f be a meromorphic function of finite order $\rho(f)$ and let $\varepsilon > 0$ be a given constant. Then there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi_0 \in [0, 2\pi)\backslash E$, then there exists a constant $R_0 = R_0(\psi_0) > 1$ such that for all z satisfying $\arg z = \psi_0$ and $|z| \geq R_0$, and for all integers $j > i \geq 0$, we have*

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| < |z|^{(j-i)(\rho(f)-1+\varepsilon)}.$$

Lemma 2.2. [6, Corollary 2] *Let f be a transcendental meromorphic function with $\rho(f) < \infty$. Let $H = \{(k_1, j_1), (k_2, j_2), \cdots, (k_q, j_q)\}$ be a finite set of distinct pairs of integers that satisfy $k_i > j_i \geq 0$ for $i = 1, 2, \cdots, q$, and let $\varepsilon > 0$ be a given constant. Then there exists a set $E \subset (1, \infty)$ that has finite logarithmic* *measure, such that for all z satisfying $|z| \notin E \cup [0,1]$ and for all $(k,j) \in H$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho(f)-1+\varepsilon)}.$$

Lemma 2.3. [6, Theorem 3] *Let f be a nontrivial entire function, and let $\alpha > 1$ be a given real constant. Let j and i be two integers such that $j > i \geq 0$. Then there exists a set $F \subset [0,\infty)$ having finite logarithmic measure and a constant $B > 0$ depending on α, j, i only, such that for all z satisfying $|z| = r \notin F \cup [0,1]$, we have*

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq B\left[\frac{T(\alpha r, f)}{r}\log^{\alpha} r \log T(\alpha r, f)\right]^{j-i}.$$

Lemma 2.4. [6, Theorem 4] *Let f be a nontrivial entire function, and let $\alpha > 1$ and $\varepsilon > 0$ be given constants. Then there exist a constant $c > 0$ and a set $F \subset [0,\infty)$ having finite linear measure such that for all z satisfying $|z| = r \notin F$, we have*

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq c\left[T(\alpha r, f) r^{\varepsilon}\log T(\alpha r, f)\right]^{j},\ \ (j = 1, 2, \cdots, k).$$

For the convenience of the reader, we recall the next three lemmata.

Lemma 2.5. [1, Theorem] *Let f be a transcendental meromorphic function with $0 \leq \mu(f) < 1$. Then, for every $\alpha \in (\mu(f), 1)$, the set $E := \{r \in [0,\infty) : m(r) > M(r)\cos\pi\alpha\}$ satisfies $\overline{\log dens E} \geq 1 - \frac{\mu(f)}{\alpha}$, where $m(r) := \inf\limits_{|z|=r} \log|f(z)|$, and $M(r) := \sup\limits_{|z|=r} \log|f(z)|$.*

Lemma 2.6. [4, Lemma 1] *Let f be a meromorphic function of finite order $\rho(f)$. For any given $\varepsilon > 0$ and $0 < \sigma < \frac{1}{2}$, there is a constant $K(\rho(f), \varepsilon)$ and a set $E(\varepsilon) \subset [0,\infty)$ of lower logarithmic density of $\underline{\log dens E(\varepsilon)} \geq 1 - \varepsilon$, such that, for $r \in E(\varepsilon)$ and each interval J of length σ, we have*

$$r\int_J \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| d\theta < K(\rho, \varepsilon)\left(\sigma \log\frac{1}{\sigma}\right) T(r, f).$$

Lemma 2.7. [15, p. 180] *Let f be an entire function of lower order $\mu(f) \in [\frac{1}{2}, \infty)$. Then there exists a sector $S(\alpha, \beta) = \{z : \alpha < \arg z < \beta\}$ with $\beta - \alpha \geq \frac{\pi}{\mu(f)}$ such that*

$$\limsup_{r\to\infty} \frac{\log\log|f(re^{i\theta})|}{\log r} \geq \mu(f)$$

holds for all the rays $\arg z = \theta \in (\alpha, \beta)$, where $0 \leq \alpha < \beta \leq 2\pi$.

It remains to complete this section of auxiliary results by two simple lemmata, whose proofs we shortly add here.

Lemma 2.8. *Suppose that* $A_j(z) = E_{\rho_j}(z)$ *with* $\rho_j > \frac{1}{2}$ $(j = 0, 1, \cdots, k-1)$. *If there exists* $s \in \{0, 1, \cdots, k-1\}$ *such that* $\rho_s > \rho = \max\{\rho_j, j \neq s,\ j = 0, 1, \cdots, k-1\}$. *Then every nontrivial solution to (1.3) satisfies* $\rho(f) \geq \mu(f) \geq \rho_s$.

Proof. We obtain from (1.3) that

$$-A_s = \frac{f^{(k)}}{f^{(s)}} + \cdots + A_{s+1}\frac{f^{(s+1)}}{f^{(s)}} + A_{s-1}\frac{f^{(s-1)}}{f^{(s)}} + \cdots + A_1\frac{f'}{f^{(s)}} + A_0\frac{f}{f^{(s)}}. \tag{2.1}$$

By elementary Nevanlinna theory, we have

$$\begin{aligned} T(r, A_s) = m(r, A_s) \quad &\leq \quad \sum_{j\neq s, j=0}^{k-1} m(r, A_j) + \sum_{j=s+1}^{k-1} m\left(r, \tfrac{f^{(j)}}{f^{(s)}}\right) \\ &+ \quad \sum_{j=0}^{s-1} m\left(r, \tfrac{f^{(j)}}{f^{(s)}}\right) + O(1) \end{aligned} \tag{2.2}$$

as $r \to \infty$. Since

$$m(r, f^{(j)}/f^{(s)}) \leq T(r, f^{(j)}) + T(r, f^{(s)}) + O(1) \leq BT(r, f)$$

as $r \to \infty$, it follows from (2.2) that

$$T(r, A_s) - \sum_{j\neq s, j=0}^{k-1} T(r, A_j) - O(\log rT(r, f)) \leq BT(r, f) \tag{2.3}$$

as $r \to \infty$. The claim now follows from (2.3) since $A_j(z) = E_{\rho_j}$ $(j = 1, 2, \cdots, k-1)$ are of regular growth and $\rho_s > \rho = \max\{\rho_j,\ j \neq s, j = 0, 1, \cdots, k-1\}$. $\square$

Lemma 2.9. *Suppose that* $A_j(z) = E_{\rho_j}(z)$ *with* $\rho_j > \frac{1}{2}$ $(j = 0, 1, \cdots, k-1)$. *If there exists* $s \in \{0, 1, \cdots, k-1\}$ *such that* $\rho_s > \rho = \max\{\rho_j, j \neq s, j = 0, 1, \cdots, k-1\}$. *Then every infinite order solution to (1.3) satisfies* $\rho_2(f) \leq \rho_s$.

Proof. We obtain from (1.3) that

$$\begin{aligned} \left|\frac{f^{(k)}}{f}\right| \quad &\leq \quad |A_{k-1}|\left|\frac{f^{(k-1)}}{f}\right| + \cdots + |A_{s+1}|\left|\frac{f^{(s+1)}}{f}\right| \\ &+ \quad |A_s|\left|\frac{f^{(s)}}{f}\right| + |A_{s-1}|\left|\frac{f^{(s-1)}}{f}\right| + \cdots + |A_1|\left|\frac{f'}{f}\right| + |A_0|. \end{aligned} \tag{2.4}$$

By the Wiman-Valiron theory, we obtain for the central index $\nu(r)$ of f that

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu(r)}{z}\right)^j (1 + o(1)),\ j = 1, 2, \cdots, k, \tag{2.5}$$

where z satisfies $|f(z)| = M(r, f)$ and $|z| = r \notin F_0 \cup [0, 1]$, where $F_0 \subset (1, \infty)$ has finite logarithmic measure.

As stated above, $\rho(A_j) = \rho(E_j(z)) = \rho_j$. Thus, for any $0 < \varepsilon < \rho_s - \rho$,

$$\begin{aligned} &|A_s(z)| \le \exp(r^{\rho_s+\varepsilon}), \\ &|A_j(z)| \le \exp(r^{\rho+\varepsilon}) \ (j = 0, 1, \cdots, k-1, j \ne s). \end{aligned} \tag{2.6}$$

Thus, it easily follows from (2.4)-(2.6) that, for all z satisfying $|f(z)| = M(r, f)$, $|z| = r \notin F_0 \cup [0, 1]$,

$$\nu(r)^k |1 + o(1)| \le kr^k \exp(r^{\rho_s+\varepsilon}) \nu(r)^{k-1} |1 + o(1)|.$$

This immediately results in $\rho_2(f) \le \rho_s$. □

3. Proof of Theorem 1.2

Suppose that f is a solution to (1.3) and $\max\{\rho_j \ (j = 1, \cdots, k-1)\} = \rho < \rho_0$, then it follows that

$$|A_0| \le \left|\frac{f^{(k)}}{f}\right| + |A_{k-1}| \left|\frac{f^{(k-1)}}{f}\right| + \cdots + |A_1| \left|\frac{f'}{f}\right|. \tag{3.1}$$

By Lemma 2.4, it is immediate to find a set $F_1 \subset [0, \infty)$ of finite linear measure such that for all z satisfying $|z| = r \notin F_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \le r[T(2r, f)]^k, \ (j = 1, 2, \cdots, k). \tag{3.2}$$

Since $A_0(z) = E_{\rho_0}(z)$, we deduce from Definition 2 that for all $\theta \in E_0 = [-\frac{\pi}{2\rho_0}, \frac{\pi}{2\rho_0}]$, $\delta_{A_0}(\theta) > 0$. Furthermore, there exist constants $0 < \alpha < 1$ and $R_0 = R_0(\theta) > 0$ such that when $|z| = r > R_0$,

$$|A_0(re^{i\theta})| \ge \exp\{\alpha \delta_{A_0}(\theta) r^{\rho_0}\}. \tag{3.3}$$

Since $\max\{\rho_j \ (j = 1, \cdots, k-1)\} = \rho < \rho_0$ and every $A_j(z)$ is of regular growth, we have

$$\lim_{r\to\infty} \frac{\log |A_j(re^{i\theta})|}{r^{\rho_0}} = 0, \ (j = 1, 2, \cdots, k-1),$$

and so

$$|A_j(re^{i\theta})| \le \exp\{o(1) r^{\rho_0}\} \ (j = 1, 2, \cdots, k-1). \tag{3.4}$$

Thus, it follows from (3.1)-(3.4) that there exists a set $F = [0, \infty) \setminus (F_1 \cup [0, R_0])$ of positive upper density $\overline{dens}F > 0$ such that

$$\exp\{(1 + o(1)) \alpha \delta_{A_0}(\theta) r^{\rho_0}\} \le [T(2r, f)]^k$$

as $|z| = r \to \infty$ in F. Therefore, $\mu(f) = \rho(f) = \infty$ and

$$\rho_2(f) = \limsup_{r\to\infty} \frac{\log^+ \log^+ T(r,f)}{\log r} \geq \rho_0.$$

By Lemma 2.9, $\rho_2(f) = \rho_0$.

Remark. Observe that proving $\rho_2(f) \geq \rho_0$ may also immediately be seen as follows. Indeed, suppose that $\rho_2(f) = \tau < \rho_0$. By [10, Lemma 1.3],

$$m\left(r, \frac{f^{(j)}}{f}\right) = O(r^{\tau+\varepsilon})$$

for $j = 1, \ldots, k$. Since $T(r, A_j) = m(r, A_j) = O(r^{\rho+\varepsilon})$ for $j = 1, \ldots, k-1$, we observe by (3.1) that

$$T(r, A_0) = m(r, A_0) = O(r^{\tau+\varepsilon}) + O(r^{\rho+\varepsilon}).$$

Hence $\rho(A_0) \leq \max\{\tau, \rho\} < \rho_0$, a contradiction.

4. Proof of Theorem 1.3

By Lemma 2.8, all solutions to (1.3) are transcendental. Write now (1.3) in the form

$$\begin{aligned} -A_s &= \frac{f^{(k)}}{f^{(s)}} + \cdots + A_{s+1}\frac{f^{(s+1)}}{f^{(s)}} \\ &+ \frac{f}{f^{(s)}}\left(A_{s-1}\frac{f^{(s-1)}}{f} + \cdots + A_1\frac{f'}{f} + A_0\right). \end{aligned} \tag{4.1}$$

By Lemma 2.3, there exists a set $F_2 \subset [0,\infty)$ of finite logarithmic measure and a constant $B > 0$, such that for all z satisfying $|z| = r \notin F_2 \cup [0,1]$, we have

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq BT(2r, f)^{2k}, 0 \leq i < j \leq k. \tag{4.2}$$

By the Wiman-Valiron theory,

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu(r)}{z}\right)^j (1 + o(1)),\ j = 1, 2, \cdots, k, \tag{4.3}$$

where $|f(z)| = M(r, f)$ and $|z| = r \notin F_3 \cup [0,1]$, where $F_3 \subset (1,\infty)$ is of finite logarithmic measure. Since f is transcendental, $\nu(r) \to \infty$ as $r \to \infty$. Hence, for all z such that $|z| = r \notin F_3 \cup [0,1]$ and $|f(z)| = M(r, f)$, (4.3) implies that

$$\frac{f(z)}{f^{(j)}(z)} \leq 2r^j,\ j = 1, 2, \cdots, k. \tag{4.4}$$

Note $F_4=\{\theta \in [0, 2\pi) : |f(re^{i\theta})| = M(r, f)$ when $r \notin F_3 \cup [0, 1]\}$. Since $A_s(z)$ is an exponential polynomials, it follows from the proof of [19, Satz 4], [20, p. 462], and [18, Theorem 1.3.4] that

$$\lim_{r\to\infty} \frac{\log |A_s(re^{i\theta})|}{r^{\rho_s}} = h_{A_s}(\theta)$$

for all $\theta \in [-\pi, \pi)$. Since $h_{A_s}(\theta)$ is continuous in $[-\pi, \pi)$, there exists θ_1 such that $h_{A_s}(\theta) \geq h_{A_s}(\theta_1)$ for all $\theta \in [-\pi, \pi)$. Hence, for all sufficiently small ε, there exists $R_0 = R_0(\theta) > 0$ such that when $|z| = r > R_0$ and $\theta \in F_4$

$$|A_s(re^{i\theta})| \geq \exp\{(1 - \varepsilon)h_{A_s}(\theta_1)r^{\rho_s}\}. \tag{4.5}$$

Since $\max\{\rho_j\ (j = 1, \cdots, k - 1), j \neq s\} = \rho < \rho_s$ and $A_j(z)$ is of regular growth, we have

$$\lim_{r\to\infty} \frac{\log |A_j(re^{i\theta})|}{r^{\rho_s}} = 0,\ \ (j = 1, 2, \cdots, k - 1, j \neq s).$$

Therefore,

$$|A_j(re^{i\theta})| \leq \exp\{o(1)r^{\rho_s}\}\ (j = 1, 2, \cdots, k - 1, j \neq s). \tag{4.6}$$

Note the set $F := [1, \infty)\backslash(F_2 \cup F_3 \cup [0, R_0])$ is of infinite logarithmic measure. It follows from (4.1), (4.2), (4.4)-(4.6) that, for all z satisfying $|z| = r \in F$ and all $\arg z \in F_4$,

$$\exp\{(1 + o(1))h_{A_s}(\theta_1)r^{\rho_s} \leq [T(2r, f)]^{2k}$$

as $|z| = r \to \infty$ in F. Therefore, $\mu(f) = \rho(f) = \infty$ and

$$\rho_2(f) = \limsup_{r\to\infty} \frac{\log^+ \log^+ T(r, f)}{\log r} \geq \rho_s.$$

By Lemma 2.9, we have $\rho_2(f) = \rho_s$.

5. Proof of Theorem 1.4

Since A_0 is an entire function with $0 < \mu(A_0) < 1$, we deduce from Lemma 2.5 that, for every $\alpha \in (\mu(A_0), 1)$, $E := \{r \in [0, \infty) : m(r) > M(r) \cos \pi\alpha\}$ satisfies $\overline{\log dens}E \geq 1 - \frac{\mu(A_0)}{\alpha}$. Thus, there exists a positive constant r_0 such that for all sufficiently small $\varepsilon > 0$ and all $r \in E\backslash[0, r_0]$,

$$|A_0(z)| > \exp\{r^{\mu(A_0)-\varepsilon}\}. \tag{5.1}$$

Since $A_1(z)$ has a finite deficient value a, say $\delta(A_1, a) = 2\delta > 0$, there exists a constant r_1 such that $m(r, \frac{1}{A_1-a}) > \delta T(r, A_1)$ for all $r > r_1 > r_0$. Therefore, for all $r > r_1$, there exists $z_r = re^{i\theta_r}$ such that

$$\log |A_1(z_r) - a| \leq -\delta T(r, A_1). \tag{5.2}$$

Set $0 < \varepsilon_0 < 1 - \frac{\mu(A_0)}{\alpha}$. Applying Lemma 2.6 to $A_1(z) - a$, we now choose σ_0 such that $\sigma_0 < \min\{\frac{1}{2}, 2\pi - \frac{\pi}{\rho}\}$ and $K(\rho(A_1), \varepsilon_0)\sigma_0 \log \frac{1}{\sigma_0} < \frac{\delta}{4}$. Then, for an interval $J \subset (\frac{\pi}{2\rho_1}, 2\pi - \frac{\pi}{2\rho_1})$ of length of σ_0, and for all $r > r_1$ in a set $E(\varepsilon_0)$ of lower logarithmic density $\underline{\log dens} E(\varepsilon_0) \geq 1 - \varepsilon_0$, we have

$$r \int_J \left| \frac{A_1'(re^{i\theta})}{A_1(re^{i\theta}) - a} \right| d\theta < \frac{\delta}{2} T(r, A_1). \tag{5.3}$$

If $\theta_r \notin J$, then we get $\log |A_1(z_r) - a| \to \infty$, which contradicts (5.2), hence $\theta_r \in J$. Let $|z_r| = r \in E(\varepsilon_0) \backslash [0, r_1]$. We conclude from (5.2) and (5.3) that, for all $\theta \in J_0 = [\theta_r - \frac{\sigma_0}{2}, \theta_r + \frac{\sigma_0}{2}] \cap J$,

$$\begin{aligned}
\log |A_1(re^{i\theta}) - a| &= \log |A_1(re^{i\theta_r}) - a| + \int_{\theta_r}^{\theta} \frac{d(\log |A_1(re^{i\varphi}) - a|)}{d\varphi} d\varphi \\
&\leq -\delta T(r, A_1) + r \int_{\theta_r}^{\theta} \left| \frac{A_1'(re^{i\varphi})}{A_1(re^{i\varphi}) - a} \right| d\varphi \\
&\leq -\frac{\delta}{2} T(r, A_1) < 0,
\end{aligned}$$

and so

$$|A_1(re^{i\theta}) - a| \leq 1 \tag{5.4}$$

for $r \in E(\varepsilon_0) \backslash [0, r_1]$ and $\theta \in J_0$.

Since for $A_j(z) = E_{\rho_j}(z)$, $\rho_j > \frac{1}{2}$ $(j = 2, 3, \cdots, k-1)$, we have $\delta_{\rho_j}(\theta) < 0$ for $\theta \in J_0 \subset (\frac{\pi}{2\rho_1}, 2\pi - \frac{\pi}{2\rho_1})$. Hence, there exists $M > 0$ and $r_2 > 0$ such that, for all z satisfying $|z| = r > r_2$ and all $\theta = \arg z \in J_0$,

$$|A_j(re^{i\theta})| \leq M r^{-1}, \ (j = 2, 3, \cdots, k-1). \tag{5.5}$$

Set $F := E(\varepsilon_0) \bigcap E$. Then

$$\overline{\log dens} F + \underline{\log dens}(E(\varepsilon_0) \backslash E) \geq \underline{\log dens} E(\varepsilon_0),$$

and so

$$\overline{\log dens} F \geq 1 - \varepsilon_0 - \underline{\log dens} E^c.$$

Since $0 < \varepsilon_0 < 1 - \frac{\mu(A_0)}{\alpha}$, $\overline{\log dens} E + \underline{\log dens} E^c = 1$ and $\overline{\log dens} E \geq 1 - \frac{\mu(A_0)}{\alpha}$, we have

$$\overline{\log dens} F \geq 1 - \varepsilon_0 - \frac{\mu(A_0)}{\alpha} > 0.$$

Assume now, contrary to the claim, that there exists a solution f of finite order to (1.3). Then by Lemma 2.2, there is a set $F_6 \subset (1, \infty)$ of finite logarithmic measure such that for all z satisfying $|z| \notin F_6 \cup [0,1]$,

$$\left|\frac{f^{(j)}}{f}\right| \leq |z|^{j(\rho(f)-1+\varepsilon)},\ j=1,2,\cdots,k. \tag{5.6}$$

By (1.3),

$$|A_0(z)| \leq \left|\frac{f^{(k)}(z)}{f(z)}\right| + \cdots + |A_2(z)|\left|\frac{f''(z)}{f(z)}\right| + (|A_1(z)-a|+|a|)\left|\frac{f'(z)}{f(z)}\right|. \tag{5.7}$$

Thus, for $r \in F\backslash(F_6 \cup [0,1] \cup [0,r_1] \cup [0,r_2])$ and $\theta \in J_0$, we obtain from (5.1) and (5.4)-(5.7) that

$$\exp\{r^{\mu(A_0)-\varepsilon}\} \leq [(k-1)Mr^{-1}+1+|a|]r^{k(\rho(f)-1+\varepsilon)}$$

as $r \to \infty$, which is impossible. Therefore $\rho(f)=\infty$.

On the other hand, for $r \in F\backslash(F_1 \cup [0,r_1] \cup [0,r_2])$ and $\theta \in J_0$, we obtain from (3.2), (5.1), (5.4), (5.5) and (5.7) that

$$\exp\{r^{\mu(A_0)-\varepsilon}\} \leq k(1+|a|)T(2r,f)$$

as $r \to \infty$. This obviously results in $\mu(f)=\infty$ and $\rho_2(f) \geq \mu(A_0)$.

6. Proof of Theorem 1.5

Since $A_j(z)=E_{\rho_j}(z) \in \mathcal{A}$, $\rho_j > \frac{1}{2}$ $(j=1,\cdots,k-1)$ and $\rho_1 \leq \min\{\rho_j\ (j=2,\cdots,k-1)\}$, we obtain, see Definition 2, that $\delta_{A_j}(\theta)<0,\ j=1,2,\cdots,k-1$ when $\theta \in J_1=(\frac{\pi}{2\rho_1}, 2\pi-\frac{\pi}{2\rho_1})$. Hence, there exists $M>0$ and $r_1>0$ such that, for all z satisfying $|z|=r>r_1$ and all $\theta=\arg z \in J_1$,

$$|A_j(re^{i\theta})| \leq Mr^{-1},\ (j=1,2,\cdots,k-1). \tag{6.1}$$

Denote now by $\mathfrak{m}(E)$ the linear measure of a set $E \subset \mathbb{R}$. We easily see that

$$\begin{aligned}\zeta(A_1) &:= \frac{1}{2\pi}\mathfrak{m}\left(\left\{\theta \in [0,2\pi) : \limsup_{r\to\infty}\frac{\log^+|A_1(re^{i\theta})|}{\log r} < \infty\right\}\right)\\ &= \frac{1}{2\pi}\mathfrak{m}\left(\frac{\pi}{2\rho_1}, 2\pi-\frac{\pi}{2\rho_1})\right) = 1-\frac{1}{2\rho_1}. \qquad (6.2)\end{aligned}$$

Assume now, on the contrary, that there exists a solution f with $\rho(f)<\infty$, we obtain from Lemma 2.1 that there exists a set $E \subset [0,2\pi)$ that has linear measure zero, such that if $\psi_0 \in [0,2\pi)\backslash E$, then there exists a constant $r_2=r_2(\psi_0)>1$ such that for all z satisfying $\arg z=\psi_0$ and $|z| \geq r_2$, and for all integers $j>i\geq 0$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| < |z|^{j(\rho(f)-1+\varepsilon)},\ j=1,2,\cdots,k. \tag{6.3}$$

Lemma 2.7 tells that there exists a sector $S(\alpha,\beta)=\{z:\alpha<\arg z<\beta\}$ with

$$\beta-\alpha\geq\frac{\pi}{\mu(A_0)}>\frac{\pi}{\rho_1}=2\pi(1-\zeta(A_1))$$

such that

$$\limsup_{r\to\infty}\frac{\log\log|A_0(re^{i\theta})|}{\log r}\geq\mu(A_0) \tag{6.4}$$

holds for all the rays $\arg z=\theta\in(\alpha,\beta)$. Thus for any given $\theta\in(\alpha,\beta)$, there exists one sequence $\{r_n\}(r_n\to\infty$ as $n\to\infty)$ such that

$$|A_0(r_ne^{i\theta})|\geq\exp\{r_n^{\mu(A_0)-\varepsilon}\}$$

for sufficiently large n. We obtain from (3.1),(6.1), (6.3) and (6.4) that

$$\exp\{r_n^{\mu(A_0)-\varepsilon}\}\leq kMr_n^{-1}r_n^{k(\rho(f)-1+\varepsilon)}$$

as $n\to\infty$, which is impossible. Then $\rho(f)=\infty$.

We also have, when $\theta\in E_0$, from (3.1), (3.2), (6.1), (6.3) and (6.4) that

$$\exp\{r_n^{\mu(A_0)-\varepsilon}\}\leq kT(2r_n,f)$$

as $r\to\infty$. This results in $\rho_2(f)\geq\mu(A_0)$.

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