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# Zero distribution of some difference polynomials

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**Abstract**. In this paper, suppose that $a, c \in \mathbb{C} \setminus \{0\}$, $c_j \in \mathbb{C}(j = 1, 2, \cdots, n)$ are not all zeros and $n \geq 2$, and $f(z)$ is a finite order transcendental entire function with Borel finite exceptional value or with infinitely many multiple zeros, the zero distribution of difference polynomials of $f(z + c) - af^n(z)$ and $f(z)f(z + c_1) \cdots f(z + c_n)$ are investigated. A number of examples are also presented to show that our results are best possible in a certain sense.

## §1 Introduction

During the last decade, Nevanlinna theory for differences of meromorphic functions has been an interesting topic, see e.g. [3, 9]. To some extent, this is due to the extensive investigations related to discrete Painlevé equations, see e.g. [7].

In this paper, we assume that the reader is familiar with the basic notations of Nevanlinna value distribution, see e.g. [10, 16, 21]. In addition to the main theorems in Nevanlinna theory, we frequently need to apply the notion of exponent of convergence $\lambda(f)$ for zeros of $f$. Unless otherwise specified, a meromorphic function $\alpha$ is said to be small function, relative to a given meromorphic function $f$ of finite order $\rho$, if for any $\varepsilon > 0$, and for some $\lambda < \rho$, $T(r, \alpha) = O(r^{\lambda+\varepsilon}) + S(r, f)$ outside of a possible exceptional set of finite logarithmic measure.

A number of investigations during the last decade are prompting this paper. As to these developments we refer to [3], Chapter 4.1 and the references [1, 2, 4, 5, 13, 15, 17–19].

The original idea of the present paper is to offer some difference analogues for Picard's values of meromorphic functions and of their derivatives, which were obtained by Hayman as follows.

***Theorem 1.A*** [11] *Suppose that $f(z)$ is a transcendental entire function. Then*

*(1) for all $n \geq 3$, and $a \neq 0$, $\varphi(z) = f^{'}(z) - af(z)^n$ assumes every value $b \in \mathbb{C}$ infinitely often;*

*(2) for all $n \geq 2$, $\psi(z) = f^{'}(z)f(z)^n$ assumes every non-zero value $b \in \mathbb{C}$ infinitely often.*

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The main purpose of this paper is to establish some difference analogues of Theorem 1.A. In section 2, if $f(z)$ is a finite order transcendental entire function with a Borel exceptional value $d$, we show that: (1) $\lambda(\Phi(z)-b)=\sigma(f)$ for $\Phi(z)=f(z+c)-af(z)^n, n\geq 2, a\in\mathbb{C}\setminus\{0\}$ and $b(\neq d-ad^n)\in\mathbb{C}$. (2) $\lambda(\Psi(z)-b)=\sigma(f)$ for $\Psi(z)=f(z)f(z+c_1)\cdots f(z+c_n)$ and $b(\neq d^{n+1})\in\mathbb{C}$. In Section 3, we proceed to considering that $\Psi(z)=f(z)f(z+c_1)\cdots f(z+c_n)$ assumes every value $b\in\mathbb{C}$ infinitely often if $f(z)$ has infinitely many multiple zeros and $c_j\in\mathbb{C}\setminus\{0\},(j=1,2,\cdots,n)$ are complex constants. If a finite order entire function $f(z)$ has few zeros only, the claim that $\Psi(z)=f(z)f(z+c_1)\cdots f(z+c_n)$ assumes every value $b\in\mathbb{C}$ infinitely often will be treated in Section 4.

## §2 Difference polynomials of entire functions with Borel exceptional value

Zheng has recently presented a counterpart of Theorem 1.A (1), Theorem 4.2.1 in [22], as follows.

***Theorem 2.A*** [22] *Suppose that $f(z)$ is a transcendental entire function of finite order, and $a, c$ are non-zero constants. Then for any integer $n\geq 3$,*

$$\Phi(z)=f(z+c)-af(z)^n$$

*assumes every finite value $b\in\mathbb{C}$ infinitely often.*

Example 2.1 was listed to show that the assumption of $\sigma(f)<\infty$ in Theorem 2.A is best possible.

***Example 2.1*** *Assume that $f(z)=e^{e^z}$. Then $\sigma(f)=+\infty$ and*

$$\Phi_1(z)=f(z+\log 3)+f(z)^3=2e^{3e^z}$$

*is zero-free.*

Example 2.2 shows that the assumption that $n\geq 3$ can not be omitted.

***Example 2.2*** *Assume that $f(z)=e^z+d$. Then $\sigma(f)=1<\infty$. For $e^c\neq a$,*

$$\Phi_2(z)=f(z+c)-af(z)=(e^c-a)e^z+d(1-a)$$

*can not assume the value $d(1-a)$, and for $e^c=2ad$,*

$$\Phi_3(z)=f(z+c)-af(z)^2=-ae^{2z}+d(1-ad)$$

*can not assume the value $d(1-ad)$.*

When $n=1$ and $a=1$ in Theorem 1.A, Chen and Shon obtained the following difference version of Theorem 1.A (1), Theorem 3 in [5], if $f(z)$ is transcendental entire function with the exponent of convergence of zeros $\lambda(f)=\lambda<1$.

***Theorem 2.B*** [5] *Suppose that $c\in\mathbb{C}\setminus\{0\}$ and $f(z)$ is a transcendental entire function of order of growth $\sigma(f)=\sigma=1$, and has infinitely many zeros with the exponent of convergence of zeros $\lambda(f)=\lambda<1$. Then $g(z)=\Delta f(z)=f(z+c)-f(z)$ has infinitely many zeros and infinitely many fixed points.*

We notice that $d$ is the Borel exceptional value of $f(z)=e^z+d$ in Example 2.2. What can be said about the zeros of $f(z+c)-af(z)^n, n\in\mathbb{N}, a\in\mathbb{C}\setminus\{0\}$ if $f(z)$ has a Borel exceptional value $d$ ? Here, we obtain the following results.

***Theorem 2.1*** *Suppose that $f(z)$ is a finite order transcendental entire function with a Borel exceptional value $d$, and $c\in\mathbb{C}\setminus\{0\}$ is a complex constant. Then $\Phi(z)=f(z+c)-af(z)^n, n\geq$*

$2, a \in \mathbb{C} \setminus \{0\}$ *assumes every* $b(\neq d - ad^n) \in \mathbb{C}$ *infinitely often, and* $\lambda(\Phi(z) - b) = \sigma(f)$.

***Corollary 2.1*** *Suppose that* $f(z)$ *is a finite order transcendental entire function , and* $c \in \mathbb{C} \setminus \{0\}$ *is a complex constant. If* $f(z)$ *has a Borel exceptional value* 0*, then* $\Phi(z) = f(z+c) - af(z)^n, n \geq 2, a \in \mathbb{C} \setminus \{0\}$ *assumes every non-zero value* $b \in \mathbb{C}$ *infinitely often.*

**Remark 2.1** *Theorem 2.1 does not remain valid if* $n = 1$ *and* $e^c = a$*. For example, Let* $f(z) = e^z + d$*, then* $\Phi(z) = f(z+c) - af(z) \equiv 0$ *has no zero.*

The following simple result proved in [15], Theorem 2, which is a difference version of Theorem 1.A(2).

***Theorem 2.C*** [15] *Suppose that* $f(z)$ *is a finite order transcendental entire function, and* $c$ *is a non-zero complex constant. Then* $f(z)^n f(z+c), n \geq 2$ *assumes every non-zero value* $b \in \mathbb{C}$ *infinitely often.*

***Example 2.3*** *Assume that* $f(z) = e^z + 1$*. Then* $f(z)f(z+\pi i) - 1$ *is zero-free.*

Example 2.3 shows that Theorem 2.C is invalid if $n = 1$. Thus, Chen, et.al. completed the case $n = 1$ of Theorem 2.C and obtained Theorem 1.2 in [4].

***Theorem 2.D*** [4] *Suppose that* $f(z)$ *is a finite order transcendental entire function with Borel exceptional value* $d$*, and* $c \in \mathbb{C} \setminus \{0\}$ *is a complex constant. Set* $H(z) = f(z)f(z+c)$*. Then for every* $b(\neq d^2) \in \mathbb{C}$*,* $\lambda(H - b) = \sigma(f)$.

Now, we extend Theorem 2.D and obtain a more general version as follows.

***Theorem 2.2*** *Suppose that* $f(z)$ *is a finite order transcendental entire function with Borel exceptional value* $d$*, and* $c_j \in \mathbb{C} \setminus \{0\}, (j = 1, 2, \cdots, n)$ *are complex constants. Then* $\Psi(z) = f(z)f(z+c_1)\cdots f(z+c_n)$ *assumes every* $b(\neq d^{n+1}) \in \mathbb{C}$ *infinitely often, and* $\lambda(\Psi(z) - b) = \sigma(f)$.

The next example shows that the equality in Theorem 2.2 may appear indeed.

***Example 2.4*** *Assume that* $f(z) = e^z + 2$*. Then the value 2 is a Borel exceptional value of* $f(z)$*. For any given value* $b \neq 2^4 = 16$*, we have*

$$\Psi(z) - b = f(z)f(z+\pi i)f\left(z + \frac{\pi}{2}i\right)f\left(z + \frac{3\pi}{2}i\right) - b = -e^{4z} + 16 - b,$$

*which satisfies* $\lambda(\Psi(z) - b) = \sigma(f) = 1$.

***Corollary 2.2*** *Suppose that* $f(z)$ *is a finite order transcendental entire function , and* $c_j \in \mathbb{C} \setminus \{0\}, (j = 1, 2, \cdots, n)$ *are complex constants. If* $f(z)$ *has a Borel exceptional value* 0*, then* $\Psi(z) = f(z)f(z+c_1)\cdots f(z+c_n)$ *assumes every non-zero* $b \in \mathbb{C}$ *infinitely often.*

We now prepare some lemmas to prove Theorem 2.1 and Theorem 2.2. We firstly need a precise asymptotic relation between the shift $f(z+c)$ and finite order meromorphic function $f(z)$ due to Chiang and Feng, Theorem 2.1 in [6].

***Lemma 2.1*** [6] *Let* $f(z)$ *be a meromorphic function with order* $\rho = \rho(f) < \infty$ *and let* $c$ *be a fixed non-zero complex number, then for each* $\varepsilon > 0$*, we have*

$$T(r, f(z+c)) = T(r, f) + O(r^{\rho-1+\varepsilon}) + O(\log r).$$

The following Lemma 2.2, frequently applied below, is an estimation of growth of difference polynomials.

***Lemma 2.2*** *Suppose that* $f(z)$ *is a transcendental entire function with finite order* $\rho$*, and* $c \in \mathbb{C} \setminus \{0\}$ *is a complex constant. Set* $\Phi(z) = f(z+c) - af(z)^n, n \geq 2, a \in \mathbb{C} \setminus \{0\}$*. Then* $\sigma(\Phi) = \sigma(f) = \rho$.

**Proof.** Since $f(z)$ is a transcendental entire function with finite order $\rho$, it is obvious that

$\rho(f(z+c)) = \rho(f) = \rho$ by Lemma 2.1, and

$$\rho(\Phi) \le \rho(f) = \rho. \tag{1}$$

On the other hand, by Lemma 2.1, we obtain that

$$\begin{aligned} nT(r,f) + O(1) &= T(r, af(z)^n) = T(r, f(z+c) - \Phi(z)) \\ &\le T(r, f(z+c)) + T(r, \Phi(z)) + O(1) \\ &= T(r,f) + T(r,\Phi(z)) + O(r^{\rho-1+\varepsilon}) + O(\log r), \end{aligned}$$

which implies that

$$\rho = \rho(f) \le \rho(\Phi). \tag{2}$$

(1) and (2) show that Lemma 2.2 is arrived.

**Remark 2.2** *If $n = 1$, Lemma 2.2 is invalid. For example, $f(z) = e^z$, then $\Phi(z) = f(z+c) - af(z) \equiv 0$, provide that $e^c = a$.*

We need to recall the following lemma 2.3. The version below is a simple modification of a lemma due to Hiromi and Ozawa, Lemma 1 in [12], see also Theorem 1.51 in [21].

***Lemma 2.3*** [21] *Let $f_j(z)(j = 1, 2, \cdots, n)(n \ge 2)$ be meromorphic functions, $g_j(z)(j = 1, 2, \cdots, n)$ be entire functions, and satisfy*

*(1)* $\sum\limits_{j=1}^{n} f_j(z) e^{g_j(z)} = 0$;

*(2) when $1 \le j < k \le n, g_j(z) - g_k(z)$ is not a constant;*

*(3) when $1 \le j \le n, 1 \le h < k \le n$,*

$$T(r, f_j) = o\left\{T(r, e^{g_h - g_k})\right\} (r \to +\infty, r \notin E),$$

*where $E \subset (1, +\infty)$ is of finite linear measure or finite logarithmic measure.*

*Then $f_j(z) \equiv 0 (j = 1, 2, \cdots, n)$.*

We also need the proximity function and pointwise estimates of $f(z+\eta)/f(z)$, which is a discrete version of the classical logarithmic derivative estimates of $f(z)$, see Corollary 2.6 in [6].

***Lemma 2.4*** [6] *Let $\eta_1$ and $\eta_2$ be two complex numbers such that $\eta_1 \ne \eta_2$ and let $f(z)$ be a finite order meromorphic function. Let $\rho$ be the order of $f(z)$, then for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O(r^{\rho-1+\varepsilon}).$$

The following lemma, frequently applied below, is an estimation for growth for difference products.

***Lemma 2.5*** *Suppose that $f(z)$ is a transcendental entire function of finite order $\rho$, and $c_j \in \mathbb{C} \setminus \{0\} (j = 1, 2, \cdots, n)$ are complex constants. Set $\Psi(z) = f(z)f(z+c_1)\cdots f(z+c_n)$. Then $\sigma(\Psi) = \sigma(f) = \rho$.*

**Proof.** We can rearrange the expression of $\Psi(z)$ to obtain

$$\Psi(z) = f^{n+1}(z) \cdot \prod_{j=1}^{n} \frac{f(z+c_j)}{f(z)}. \tag{3}$$

It follows from Lemma 2.4 and (3) that, for each sufficiently small $\varepsilon > 0$,

$$m(r, \Psi) \le (n+1)m(r,f) + \sum_{j=1}^{n} m\left(r, \frac{f(z+c_j)}{f(z)}\right) = (n+1)m(r,f) + O(r^{\rho-1+\varepsilon}), \tag{4}$$

and

$$(n+1)m(r,f)=m(r,f^{n+1})\leq m(r,\Psi)+\sum_{j=1}^{n}m\left(r,\frac{f(z)}{f(z+c_j)}\right)=m(r,\Psi)+O(r^{\rho-1+\varepsilon}). \quad (5)$$

It follows from (4), (5) and $f(z)$ is transcendental entire function that $\sigma(\Psi)=\sigma(f)=\rho$.

In the following, we proceed to prove Theorem 2.1 and Theorem 2.2.

***Proof of Theorem 2.1*** Suppose now, contrary to the claim, that we have $\lambda(\Phi-b)<\sigma(f)<+\infty$ for $\Phi(z)=f(z+c)-af(z)^n$. It follows from Lemma 2.2 that $\sigma(\Phi-b)=\sigma(\Phi)=\sigma(f)<+\infty$, which implies that $\lambda(\Phi-b)<\sigma(\Phi-b)<+\infty$. This shows that there exists a positive integer $k$ such that $\sigma(\Phi-b)=\sigma(\Phi)=\sigma(f)=k$. Thus we can rewrite $\Phi(z)-b$ as the form

$$\Phi(z)-b=R(z)e^{\beta z^k}, \quad (6)$$

where $\beta$ is a non-zero constant and $R(z)$ is an entire function with

$$\sigma(R)\leq\max\{\lambda(\Phi-b),k-1\}.$$

Since $d$ is the Borel exceptional value of $f(z)$ and $\sigma(f)=k$, we can rewrite $f(z)$ as the form

$$f(z)=d+P(z)e^{\alpha z^k}, \quad (7)$$

where $P(z)$ is an entire function with order $\sigma(P)<\sigma(f)=k,\alpha$ is a non-zero constant.

From (7), we have

$$f(z+c)=d+P(z+c)Q(z)e^{\alpha z^k}, \quad (8)$$

where

$$Q(z)=\exp\left\{\alpha\sum_{j=1}^{k}\binom{k}{j}z^{k-j}c^j\right\},\quad \sigma(Q)=k-1.$$

It follows from (6)–(8) that

$$\begin{aligned}&-aP^n(z)e^{n\alpha z^k}-a\sum_{j=1}^{n-2}\left(\binom{n}{j}d^jP^{n-j}(z)e^{(n-j)\alpha z^k}\right)\\&+\left[P(z+c)Q(z)-and^{n-1}P(z)\right]e^{\alpha z^k}+d-ad^n-b=R(z)e^{\beta z^k}.\end{aligned} \quad (9)$$

Since $aP(z)^n\not\equiv 0$ and $R(z)\not\equiv 0$, by comparing the growths of both side of (9), we obtain $\beta=n\alpha$. So we can rewrite (9) as the form

$$\begin{aligned}&[R(z)+aP^n(z)]e^{n\alpha z^k}+a\sum_{j=1}^{n-2}\left(\binom{n}{j}d^jP^{n-j}(z)e^{(n-j)\alpha z^k}\right)\\&-\left[P(z+c)Q(z)-and^{n-1}P(z)\right]e^{\alpha z^k}+b-(d-ad^n)=0.\end{aligned} \quad (10)$$

It follows from Lemma 2.3 and (10) that $b=d-ad^n$. This contradicts to the assumption that $b\neq d-ad^n$. Hence we have $\lambda(\Phi-b)=\sigma(f)$.

***Proof of Theorem 2.2*** Suppose now, contrary to the claim, that we have $\lambda(\Phi-b)<\sigma(f)<+\infty$ for $\Psi(z)=f(z)f(z+c_1)\cdots f(z+c_n)$. It follows from Lemma 2.5 that $\sigma(\Psi-b)=\sigma(\Psi)=\sigma(f)<+\infty$. Thus, $\lambda(\Psi-b)<\sigma(\Psi-b)<+\infty$. This shows that there exists a positive integer $k$ such that $\sigma(f)=\sigma(\Psi)=\sigma(\Psi-b)=k$. Thus we can rewrite $\Psi(z)-b$ as the form

$$\Psi(z)-b=Q(z)e^{\beta z^k}, \quad (11)$$

where $\beta$ is a non-zero constant and $Q(z)$ is an entire function with

$$\sigma(Q) \le \max\{\lambda(\Psi - b), k-1\}.$$

On the other hand, since $d$ is the Borel exceptional value of $f(z)$ and $\sigma(f) = k$, we can rewrite $f(z)$ as the form

$$f(z) = d + P(z)e^{\alpha z^k}, \tag{12}$$

where $P(z)$ is an entire function with order $\sigma(P) < \sigma(f) = k, \alpha$ is a non-zero constant.

From (12), we have

$$f(z + c_j) = d + P(z + c_j)P_j(z)e^{\alpha z^k}, \tag{13}$$

where

$$P_j(z) = \exp\left\{\alpha\binom{k}{1}z^{k-1}c_j + \alpha\binom{k}{2}z^{k-2}c_j^2 + \cdots + \alpha c_j^k\right\}, \quad \sigma(P_j) = k-1, (j = 1, 2, \cdots, n).$$

It follows from (11)−(13) that

$$P(z)\left(\prod_{j=1}^{n} P(z + c_j)P_j(z)\right)e^{(n+1)\alpha z^k} + A_n(z)e^{n\alpha z^k} + \cdots + A_1(z)e^{\alpha z^k} + d^{n+1} - b = Q(z)e^{\beta z^k}, \tag{14}$$

where $A_j(z)(j = 1, 2, \cdots, n)$ are difference polynomials in $P(z)$, $P_j(z)$ and some of the shifts $P(z + c_1), P(z + c_2), \cdots, P(z + c_n)$. Thus, $\sigma(A_j) < \sigma(f) = k(j = 1, 2, \cdots, n)$ and

$$\sigma\left(P(z)\left(\prod_{j=1}^{n} P(z + c_j)P_j(z)\right)\right) < \sigma(f) = k.$$

Together with (12)−(14), we obtain, by using the method of induction,

$$P(z)\left(\prod_{j=1}^{n} P(z + c_j)P_j(z)\right) \not\equiv 0.$$

Thus, by comparing the growths of both side of (14), we obtain $\beta = (n+1)\alpha$. So we can rewrite (14) as the form

$$\left[P(z)\left(\prod_{j=1}^{n} P(z + c_j)P_j(z)\right) - Q(z)\right]e^{(n+1)\alpha z^k} + A_n(z)e^{n\alpha z^k} + \cdots + A_1(z)e^{\alpha z^k} + d^{n+1} - b = 0. \tag{15}$$

It follows from Lemma 2.3 and (15) that $b = d^{n+1}$. This contradicts to the assumption that $b \neq d^{n+1}$. Hence we have $\lambda(\Psi - b) = \sigma(f)$.

## §3 Difference polynomials of entire functions with multiple zeros

In this section, the starting point to us is the following Theorem 1.4 in [4], which is a difference version of Theorem 1.A(2) when $n = 1$.

***Theorem 3.A*** [4] *Suppose that $f(z)$ is a transcendental entire function of finite order, and $c \in \mathbb{C}\backslash\{0\}$ is a complex constant. If $f(z)$ has infinitely many multiple zeros, then $H(z) = f(z)f(z + c)$ takes every value $a \in \mathbb{C}$ infinitely often.*

We now extend Theorem 3.A to more general version and obtain

***Theorem 3.1*** *Suppose that $f(z)$ is a transcendental entire function of finite order, and $c_j \in \mathbb{C}\setminus\{0\}, (j=1,2,\cdots,n)$ are complex constants. If $f(z)$ has infinitely many multiple zeros, then $\Psi(z)=f(z)f(z+c_1)\cdots f(z+c_n)$ assumes every value $b\in\mathbb{C}$ infinitely often.*

The following example shows that the situation in Theorem 3.1 may appear indeed.

***Example 3.1*** *Assume that $f(z)=(e^z+1)^2$. Then $f(z)$ has infinitely many multiple zeros. For any given $b\in\mathbb{C}$, we have*

$$\Psi(z)-b=f(z)f(z+\pi i)f\left(z+\frac{\pi}{2}i\right)f\left(z+\frac{3\pi}{2}i\right)-a=e^{8z}-2e^{4z}+1-b,$$

*which has infinitely many zeros.*

We now proceed to prove Theorem 3.1.

***Proof of Theorem 3.1*** Suppose that $f(z)$ has infinitely many multiple zeros. If $b=0$, then assertion of Theorem 3.1 is arrived. Thus, we only prove the case $b\in\mathbb{C}\setminus\{0\}$. If $\Psi(z)$ assumes value $b$ only finitely often, then we can rewrite $\Psi(z)-b$ as the form

$$\Psi(z)-b=f(z)f(z+c_1)\cdots f(z+c_n)-b=q(z)e^{p(z)}, \tag{16}$$

where $q(z)$ is polynomial, and $p(z)$ is an entire function. It follows from Lemma 2.5 that $\sigma(e^{p(z)})=\sigma(\Psi(z)-b)=\sigma(f)$.

Differentiating (16) and eliminating $e^{p(z)}$, we obtain

$$f^{'}(z)\prod_{j=1}^{n}f(z+c_j)+f(z)\left[\sum_{k=1}^{n}f^{'}(z+c_k)\prod_{\substack{j\neq k\\ j=1}}^{n}f(z+c_j)\right]=\frac{q^*(z)}{q(z)}\left[f(z)\prod_{j=1}^{n}f(z+c_j)-b\right], \tag{17}$$

where $q^*(z)=q^{'}(z)+q(z)p^{'}(z)$. Obviously, $q^*(z)\not\equiv 0$. Indeed, if $q^*(z)\equiv 0$, i.e.,

$$\Psi^{'}(z)=f^{'}(z)\prod_{j=1}^{n}f(z+c_j)+f(z)\left[\sum_{k=1}^{n}f^{'}(z+c_k)\prod_{\substack{j\neq k\\ j=1}}^{n}f(z+c_j)\right]\equiv 0.$$

Thus, we see that there exists a constant $l$ such that

$$f(z)f(z+c_1)\cdots f(z+c_n)=l. \tag{18}$$

Since $f(z)$ is transcendental entire function, it follows from (18) that we have $l\neq 0$. Thus,

$$f^{n+1}(z)=l\cdot\prod_{j=1}^{n}\frac{f(z)}{f(z+c_j)}. \tag{19}$$

It follows from Lemma 2.4 and (19) that we obtain

$$(n+1)T(r,f)=(n+1)m(r,f)\leq\sum_{j=1}^{n}m\left(r,\frac{f(z)}{f(z+c_j)}\right)+S(r,f)=O(r^{\rho-1+\varepsilon})+S(r,f).$$

This is a contradiction. Hence $q^*(z)\not\equiv 0$.

Now we rearrange the expression of (17) and obtain

$$\frac{f^{'}(z)}{f(z)}+\frac{\sum\limits_{k=1}^{n}f^{'}(z+c_k)\prod\limits_{\substack{j\neq k\\ j=1}}^{n}f(z+c_j)}{\prod\limits_{j=1}^{n}f(z+c_j)}=\frac{q^*(z)}{q(z)}-b\cdot\frac{q^*(z)}{q(z)}\cdot\frac{1}{f(z)\prod\limits_{j=1}^{n}f(z+c_j)}. \tag{20}$$

Since we suppose that $f(z)$ has infinitely many multiple zeros, there must exist a multiple zero $z_0$ such that $|z_0|$ is sufficiently large and $q(z_0) \neq 0, q^*(z_0) \neq 0$. From this, we obtain that the right side of (20) has a multiple pole $z_0$, and the left side of (20) has at most a simple pole $z_0$, a contradiction. Thus, $\Psi(z)$ assumes every value $b \in \mathbb{C}$ infinitely often.

## §4 Difference polynomials of entire functions with few zeros

When we check Example 2.3 again, we notice that $\lambda(f) = \sigma(f) = 1$. Does it imply that the zeros of $f(z)$ may play an important role? If we assume that $f(z)$ has few zeros, for example, $N\left(r, \frac{1}{f}\right) = O(r^{\rho-1+\varepsilon}) + S(r,f)$, whether $\Psi(z) = f(z)f(z+c_1)\cdots f(z+c_n)$ assumes every value $b \in \mathbb{C}$ infinitely often or not? We answer this question and obtain

***Theorem 4.1*** *Suppose that $f(z)$ is a transcendental entire function of finite order $\rho$, and $c_j \neq 0, j = 1, 2, \cdots, n$ are complex constants. If $N\left(r, \frac{1}{f}\right) = O(r^{\rho-1+\varepsilon}) + S(r,f)$ for all sufficiently small $\varepsilon > 0$, then*

$$\Psi(z) = f(z)f(z+c_1)\cdots f(z+c_n)$$

*assumes every non-zero value $b \in \mathbb{C}$ infinitely often.*

***Remark 4.1*** *In fact, Theorem 4.1 can be seen the corollary of Theorem 2.2. Here, we list it to show that the same problem can be proved in different ways. Thus, the expression of the theorems are somewhat different.*

Example 4.1 shows that the assumption $N\left(r, \frac{1}{f}\right) = O(r^{\rho-1+\varepsilon}) + S(r,f)$ in Theorem 4.1 can not be omitted.

***Example 4.1*** *Assume that $f(z) = e^z + 1$. Then $\lambda(f) = \sigma(f) = 1$. Thus $N\left(r, \frac{1}{f}\right) \neq O(r^{\rho-1+\varepsilon}) + S(r,f)$ and the value 1 is a Borel exceptional value of*

$$\Psi(z) = f(z)f(z+\pi i)f\left(z+\frac{\pi}{2}i\right)f\left(z+\frac{3\pi}{2}i\right) = 1 - e^{4z}.$$

Halburd and Korhonen proved difference counterparts, see [ [8], Theorem 3.1 and Theorem 3.2], for the well-known Clunie and Mohon'ko lemmas in Nevanlinna theory. Laine and Yang prove difference counterpart, Theorem 2.3 in [14] to the Yang-Ye theorem, see Theorem 1 in [20].

***Lemma 4.1*** [14] *Let $f(z)$ be a transcendental meromorphic solution of finite order $\rho$ of a difference equation of the form*

$$U(z,f)P(z,f) = Q(z,f),$$

*where $U(z,f), P(z,f)$ and $Q(z,f)$ are difference polynomials with the all coefficients $\alpha_\lambda(z)$ are small functions as understood in the usual Nevanlinna theory, i.e. $T(r,\alpha_\lambda) = O(r^{\rho-1+\varepsilon}) + S(r,f)$. The maximum total degree $\deg_f U(z,f) = n$ in $f(z)$ and its shifts, and $\deg_f Q(z,f) \leq n$. Moreover, we assume that $U(z,f)$ contains just one term of maximal total degree in $f(z)$ and its shifts. Then for each $\varepsilon > 0$ ,*

$$m(r, P(z,f)) = O(r^{\rho-1+\varepsilon}) + S(r,f),$$

*possibly outside of an exceptional set of finite logarithmic measure.*

We now proceed to prove Theorem 4.1.

***Proof of Theorem 4.1*** Suppose that the assertion does not hold. Then there exists a non-zero value $b \in \mathbb{C}$, such that

$$\Psi(z) - b = f(z)f(z+c_1)\cdots f(z+c_n) - b = q(z)e^{p(z)}, \tag{21}$$

where $q(z)$ is polynomial, and $p(z)$ is an entire function. It follows from Lemma 2.5 that $\sigma(e^{p(z)}) = \sigma(\Psi(z) - b) = \sigma(f)$.

Differentiating (21) and eliminating $e^{p(z)}$, we obtain

$$f^{'}(z)\prod_{j=1}^{n} f(z+c_j) + f(z)\left[\sum_{k=1}^{n} f^{'}(z+c_k)\prod_{\substack{j\neq k\\ j=1}}^{n} f(z+c_j)\right] = \frac{q^*(z)}{q(z)}\left[f(z)\prod_{j=1}^{n} f(z+c_j) - b\right], \tag{22}$$

where $q^*(z) = q^{'}(z) + q(z)p^{'}(z)$. Similar to the proof of Theorem 3.1, we have $q^*(z) \not\equiv 0$.

Now we rearrange the expression of (22) and obtain

$$f^2(z)P(z, f(z)) = -bq^*(z), \tag{23}$$

where

$$\begin{aligned} P(z, f(z)) = q(z)&\left[\frac{f^{'}(z)}{f(z)}\frac{f(z+c_1)}{f(z)}\prod_{j=2}^{n} f(z+c_j) + \sum_{k=1}^{n}\frac{f^{'}(z+c_k)}{f(z+c_k)}\cdot\frac{f(z+c_k)}{f(z)}\prod_{\substack{j\neq k\\ j=1}}^{n} f(z+c_j)\right] \\ &- q^*(z)\frac{f(z+c_1)}{f(z)}\prod_{j=2}^{n} f(z+c_j). \end{aligned}$$

Since $f(z)$ is entire function of finite order $\rho$ and $N\left(r, \frac{1}{f}\right) = O(r^{\rho-1+\varepsilon}) + S(r, f)$, we obtain that

$$\begin{aligned} &m\left(r, \frac{f^{'}(z)}{f(z)}\right) = S(r, f), \quad m\left(r, \frac{f^{'}(z+c_k)}{f(z+c_k)}\right) = S(r, f), \\ &m\left(r, \frac{f(z+c_k)}{f(z)}\right) = O(r^{\rho-1+\varepsilon}), (k = 1, 2, \cdots, n), \\ &N\left(r, \frac{f^{'}(z)}{f(z)}\right) = O(r^{\rho-1+\varepsilon}) + S(r, f), \\ &N\left(r, \frac{f^{'}(z+c_k)}{f(z+c_k)}\right) = O(r^{\rho-1+\varepsilon}) + S(r, f), \\ &N\left(r, \frac{f(z+c_k)}{f(z)}\right) = O(r^{\rho-1+\varepsilon}) + S(r, f), (k = 1, 2, \cdots, n), \end{aligned}$$

and so, we have

$$\begin{aligned} &T\left(r, \frac{f^{'}(z)}{f(z)}\right) = S(r, f), \quad T\left(r, \frac{f^{'}(z+c_k)}{f(z+c_k)}\right) = S(r, f), \\ &T\left(r, \frac{f(z+c_k)}{f(z)}\right) = O(r^{\rho-1+\varepsilon}) + S(r, f), (k = 1, 2, \cdots, n). \end{aligned} \tag{24}$$

It follows from (24) that we have the all coefficients $\alpha_\lambda$ of $P(z, f(z))$ are small functions in the usual sense of Nevanlinna theory, i.e. $T(r, \alpha_\lambda) = O(r^{\rho-1+\varepsilon}) + S(r, f)$. Thus, we obtain from Lemma 4.1 that

$$m(r, P(z, f)) = O(r^{\rho-1+\varepsilon}) + S(r, f), \tag{25}$$

and

$$m(r, fP(z,f)) = O(r^{\rho-1+\varepsilon}) + S(r,f). \tag{26}$$

Since $N\left(r, \frac{1}{f}\right) = O(r^{\rho-1+\varepsilon}) + S(r,f)$ and $aq^*(z)$ is an entire function by (21), we have

$$N(r, P(z,f)) = O(r^{\rho-1+\varepsilon}) + S(r,f). \tag{27}$$

It follows form (25)−(25) that we have

$$T(r, P(z,f)) = O(r^{\rho-1+\varepsilon}) + S(r,f), \tag{28}$$

and

$$T(r, fP(z,f)) = O(r^{\rho-1+\varepsilon}) + S(r,f). \tag{29}$$

Thus, (28) and (29) give

$$T(r,f) = O(r^{\rho-1+\varepsilon}) + S(r,f),$$

a contradiction.

## Declarations

**Conflict of interest** The authors declare no conflict of interest.

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