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Decay of correlations for weakly expanding dynamical systems with Dini potentials under optimal quasi-gap condition

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Abstract

We find an optimal quasi-gap condition for a weakly expanding dynamical system associated with Dini potential. Under this optimal quasi-gap condition, we prove the Ruelle operator theorem and further the decay of the correlations for any weakly expanding dynamical systems with Dini potentials.

Keywords: weakly expanding dynamical system, weakly contractive iterated function system, Dini potential, Ruelle operator theorem, decay of correlations

Mathematics Subject Classification numbers: primary 37C30, secondary 37D25, 37D35.

1. Introduction

An important problem in dynamical systems is to understand the mixing of a dynamical system on a probability space. One way to describe the mixing is to use the decay of correlations. Suppose X is a non-empty compact metric space. Let $\mathcal{C}(X)$ be the space of all continuous real functions on X with the maximal norm. Let $\mathcal{M}(X)$ be the dual space of $\mathcal{C}(X)$. By the Riesz representation theorem, $\mathcal{M}(X)$ is the space of all measures on X. Then we have

$$\langle \nu, \phi \rangle = \int_X \phi \, d\nu, \quad \phi \in \mathcal{C}(X), \ \nu \in \mathcal{M}(X). \tag{1.1}$$

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Suppose $f : X \to X$ is a continuous map. We use $f^n = \underbrace{f \circ \cdots \circ f}_{n}$ to denote the n-iteration of f for every positive integer n and $f^0 = Id$, the identity. Then we can consider the dynamical system $\{f^n\}_{n=0}^{\infty}$. We simply call f a dynamical system. Given a test function $\phi \in \mathcal{C}(X)$ and a probability measure $\nu \in \mathcal{M}(X)$, we have a sequence of random variables

$$\{X_n = \phi \circ f^n\}_{n=0}^{\infty}$$

on the probability space (X, ν). Correlations of f on (X, ν) are

$$C_\phi(n) = \langle \nu, (\phi \circ f^n) \cdot \phi \rangle - (\langle \nu, \phi \rangle)^2, \quad n = 0, 1, 2, \ldots.$$

They measure the independence of these random variables and the mixing of the dynamical system. The correlations can be studied by using the transfer operator $\mathcal{L}$ associated with f and a potential ψ (see [4]) defined as

$$\mathcal{L}\phi(x) = \sum_{y \in f^{-1}(x)} \psi(y)\phi(y), \quad \phi \in \mathcal{C}(X). \tag{1.2}$$

The dual operator $\mathcal{L}^* : \mathcal{M}(X) \to \mathcal{M}(X)$ is defined, by using (1.1), as

$$\langle \mathcal{L}^*\nu, \phi \rangle = \langle \nu, \mathcal{L}\phi \rangle, \quad \nu \in \mathcal{M}(X),\ \phi \in \mathcal{C}(X). \tag{1.3}$$

Suppose ν is an eigen-measure corresponding to an eigenvalue ϱ, that is, $\mathcal{L}^*\nu = \varrho\nu$. Then we have that

$$\langle \nu, (\phi \circ f^n) \cdot \phi \rangle = \langle \varrho^{-n}(\mathcal{L}^*)^n \nu, \phi \circ f^n \cdot \phi \rangle$$
$$= \langle \nu, \varrho^{-n}\mathcal{L}^n(\phi \circ f^n \cdot \phi) \rangle = \langle \nu, \phi \cdot \varrho^{-n}\mathcal{L}^n\phi \rangle = \langle \phi\nu, \varrho^{-n}\mathcal{L}^n\phi \rangle.$$

Thus the decay of correlations relates to the convergence speed of

$$\varrho^{-n}\mathcal{L}^n\phi \to \langle \nu, \phi \rangle h, \quad \text{as} \quad n \to \infty,$$

where $\mathcal{L}h = \varrho h$ and $\langle \phi, \nu \rangle = 1$. To study the convergence speed, we need first to make sure the existence of an eigen-measure corresponding to an eigenvalue. For the spectral radius as the maximal eigenvalue, the existence of the probability eigen-measure is called the Ruelle operator theorem (see e.g. [2, 12]).

It is known that for a uniformly expanding dynamical system f with an α-Hölder continuous potential ψ, the sequence $\{\varrho^{-n}\mathcal{L}^n\phi\}_{n=1}^{\infty}$ converges to a constant multiple of h at a geometric rate for every α-Hölder continuous function ϕ (see e.g. [2, 4]). This is because the operator $\mathcal{L}$ acting on the space of α-Hölder continuous functions has a spectral gap, that means that the essential spectral radius of $\mathcal{L}$ is strictly less than the spectral radius of $\mathcal{L}$.

There is no spectral gap for the operator $\mathcal{L}$ associated with a uniformly expanding dynamical system f with an only Dini continuous potential ψ (see [3, 11]). However, in [5–7], it is still possible to have a Ruelle operator theorem and the decay of correlation. There is no spectral gap too for the transfer operator associated with a weakly expanding dynamical system f with a Hölder continuous potential ψ (see [10, 18, 22] for other references). However, in [10], it is still possible to have a Ruelle operator theorem and the decay of correlation. We would like to note that in the above studies, no similar spectral gap consideration is involved. There are many interesting works recently to study the decay of correlation in various cases. We give a partial list of other papers in the literature [16, 17, 19, 21, 23, 24]. Therefore, the study of a Ruelle operator theorem and the decay of correlation becomes a difficult and important problem for a weakly expanding dynamical system with a Dini continuous potential, in particular, under the consideration of a similar idea of the spectral gap. We started to study in this direction following a similar idea of the spectral gap in [12, 13, 22].

In this paper, we investigate a best general formula which we call an optimal quasi-gap condition (2.8) or (2.9) for a weakly expanding dynamical system with a Dini potential so that the Ruelle operator theorem holds and the decay of correlations can be obtained even there is no spectral gap in this situation. This optimal quasi-gap condition is the one we have searched for in a long-time research project (see [12, 13, 22]). This paper eventually completes this long-time project. Moreover, this paper generalises a result obtained in [9] where the potential is required to be Hölder continuous for a weakly expanding dynamical system and results in [5–7] where the dynamical system is required to be uniformly expanding. We would like to point out that there is no any differentiability assumption on the dynamical systems in the paper, we remove the conformal assumption on a weakly expanding dynamical system with Dini potential and, thus, theorem 2.5 generalises the main result of paper [22] (see corollary 2.9).

The paper is organised as follows. In section 2, we state our optimal quasi-gap conditions (2.8) and (2.9) for a weakly expanding dynamical system with a Dini potential. We state our main results (theorems 2.5, 2.7, and corollary 2.11) in section 2 too. Theorem 2.5 shows that the Ruelle operator theorem holds under the optimal quasi-gap condition. Theorem 2.7 is about the convergence speed and corollary 2.11 is about the decay of correlations. We state corollary 2.9 to close a gap in [22]. In section 3, we prove theorem 2.5 and in section 4, we prove theorem 2.7. In section 5, we prove corollary 2.9. We prove corollary 2.11 in section 6. Our main results are very general for weakly expanding dynamical systems with Dini potentials. In section 7, we present two examples of weakly expanding dynamical system with Dini potential which satisfies the optimal quasi-gap condition. Furthermore, we discuss the weak Gibbs measure and the central limit theorem as two more applications of the decay of correlations in section 7.

2. Optimal quasi-gap condition and statements of main results

Suppose $\mathbb{R}^d$ is the d-dimensional Euclidean space with norm $|\cdot|$ and suppose $X \subseteq \mathbb{R}^d$ is a non-empty compact connected subset with $\overline{\mathring{X}} = X$, where $\mathring{X}$ is the interior set of X. Without loss of generality, assume that the diameter

$$|X| := \max_{x,y\in X}|x-y| = 1.$$

In the following we always assume map $f : X \to X$ satisfies the following Markov property: there exist finite number of non-empty compact connected sets $\{X_j\}_{j=1}^m$ satisfying that

(a) $\overline{\mathring{X}_i} = X_i$ for all $1 \leqslant i \leqslant m$;
(b) $X = \cup_{i=1}^m X_i$;
(c) $\mathring{X}_i \cap X_j = \emptyset$ for all $1 \leqslant i \neq j \leqslant m$;
(d) $f|_{\mathring{X}_i} : \mathring{X}_i \to f\left(\mathring{X}_i\right)$ is a homeomorphism with continuous extension to X_i; and
(e) $\overline{f\left(\mathring{X}_i\right)} = X$.

If further

$$\lambda(t) := \min_{1\leqslant i\leqslant m} \inf_{\substack{x,y\in\mathring{X}_i \\ |x-y|\geqslant t}} |f(x)-f(y)| > t, \quad \forall\ 0 < t \leqslant \min_{1\leqslant i\leqslant m}|X_i|. \tag{2.1}$$

The dynamical system (X, f) is called *weakly expanding*.

Remark 2.1. The reason that we use a compact set in $\mathbb{R}^d$ in this paper is because that in the proof of corollary 2.9 in section 5, we need to use the Fréchet derivative of a map. Other than that all other calculations and results will be held for a general compact metric space with a metric $d(\cdot, \cdot)$ and then, in many places, the notation $|\cdot|$ will be changed to $d(\cdot, \cdot)$.

Throughout the paper, we consider such a class of weakly expanding dynamical systems. At this chance, we would like to remark that there is not any differentiability assumption imposed on f; and the weakly expanding dynamical system under consideration is somehow similar to the *non-uniformly expanding dynamical system*. For this, the readers may refer to paper [1] for example.

Let $g_i : X \to X_i$ be the continuous extension of the inverse of the map $f|_{\mathring{X}_i} : \mathring{X}_i \to X$ for every $1 \leqslant i \leqslant m$. Then we can define an iteration function system

$$\mathcal{W} = \langle g_1, \ldots, g_m \rangle.$$

Therefore, the study of the iterated function system $\mathcal{W}$ and the study of the dynamical system (X, f) is the same. We say the iteration function system $\mathcal{W}$ is *weakly contractive* if

$$\theta(t) := \max_{1\leqslant i\leqslant m} \sup_{\substack{x,y\in X\\ |x-y|\leqslant t}} |g_i(x) - g_i(y)| < t, \quad \forall t > 0.$$

The reader can check that (X, f) is weakly expanding if and only if $\mathcal{W}$ is weakly contractive.

We say a dynamical system (X, f) is *uniformly expanding* if there exists $a > 1$ such that

$$\lambda(t) = \min_{1\leqslant i\leqslant m} \inf_{\substack{x,y\in \mathring{X}_i\\ |x-y|\geqslant t}} |f(x) - f(y)| > at, \quad \forall\ 0 < t \leqslant \min_{1\leqslant i\leqslant m} |X_i|. \tag{2.2}$$

From the definition (2.1) and (2.2), we see that the uniformly expanding implies the weakly expanding. Therefore, the study of this paper includes all uniformly expanding dynamical systems.

Let $X = [0, 1]$, $X_1 = [0, 1/2]$ and $X_2 = [1/2, 1]$. Let $0 < \alpha \leqslant 1$, and let

$$f(x) = \begin{cases} x + 2^\alpha x^{1+\alpha}, & \text{if } 0 \leqslant x \leqslant 1/2; \\ 2x - 1, & \text{if } 1/2 < x \leqslant 1. \end{cases}$$

Then (X, f) is a weakly expanding dynamical system. We would like to point out that, by modifying this example with a small perturbation, we can get a weakly expanding dynamical system on X which is not differentiable on $\mathring{X}_1 \bigcup \mathring{X}_2$.

We need symbolic codings. Let

$$\Sigma_n = \{I = i_0 i_1 \ldots i_{n-1} \mid i_k \in \{1, \ldots, m\}, k = 0, 1, \ldots, n-1\}$$

be the space of all n-strings of i's with $1 \leqslant i \leqslant m$. Let $\Sigma = \bigcup_{n=0}^\infty \Sigma_n$. For each $I = i_0 i_1 \ldots i_{n-1} \in \Sigma_n$, define

$$g_I = g_{i_0} \circ g_{i_1} \circ \cdots \circ g_{i_{n-1}}.$$

In the following we always let $X_I = g_I(X)$. And denote

$$\tau_n = \max_{I\in\Sigma_n} |X_I|. \tag{2.3}$$

For a weakly expanding dynamical system (X, f), we have $0 \leqslant \tau_{n+1} < \tau_n$. It follows that (see e.g. [8, theorem 3.2] or [21, proposition 2.2])

$$\lim_{n\to\infty} \tau_n = 0. \tag{2.4}$$

When (X, f) is a uniformly expanding dynamical system, τ_n tends to 0 at geometric rate. However, for a general weakly expanding dynamical system, we may not have any expression of τ_n. Luckily, for example 7.3 in section 7, one can check that $\tau_n = \frac{1}{1+n}$.

By considering the iteration function system $\mathcal{W}$ associated with a system of positive continuous functions $\mathcal{P} = \{p_i\}_{i=1}^n$, we can define a transfer operator

$$\mathcal{L}\phi(x) = \sum_{i=1}^{m} p_i(x)\phi(g_i(x)) : \mathcal{C}(X) \to \mathcal{C}(X). \tag{2.5}$$

In particular, for a single positive function ψ, if we take $p_i = \psi \circ g_i$, then the transfer operator defined in (2.5) is the one defined in (1.2). We call $\mathcal{P}$ or ψ a potential. Henceforth, when we talk about a transfer operator associated with a dynamical system and a potential, we mean the one defined as in (2.5). Let $\mathcal{L}^* : \mathcal{M}(X) \to \mathcal{M}(X)$ be the dual operator of $\mathcal{L}$ defined as in (1.3).

Given $I = i_0 i_1 \ldots i_{n-1} \in \Sigma_n$, define

$$p_I(x) = p_{i_0}(g_{i_1} \circ g_{i_2} \circ \cdots \circ g_{i_{n-1}}(x)) \ldots p_{i_{n-2}}(g_{i_{n-1}}(x)) p_{i_{n-1}}(x).$$

We use $\mathcal{P}^n$ to denote the system of positive functions $\{p_I\}_{I\in\Sigma_n}$. One can check that for any $n \in \mathbb{N}$,

$$\mathcal{L}^n\phi(x) = \sum_{I\in\Sigma_n} p_I(x)\phi(g_I(x)),$$

is the transfer operator associated with $(f^n, \mathcal{P}^n)$.

The *modulus of continuity* of a continuous function $\phi \in \mathcal{C}(X)$ is defined as

$$\omega_\phi(t) := \sup_{\substack{x,y\in X\\ |x-y|\leqslant t}} |\phi(x) - \phi(y)|.$$

For a potential $\mathcal{P}$, we define its modulus as

$$\omega_{\mathcal{P}}(t) := \max_{1\leqslant i\leqslant m} \omega_{\log\, p_i}(t).$$

Define

$$\widetilde{\omega}_{\mathcal{P}}(t) = \int_0^t \frac{\omega_{\mathcal{P}}(s)}{s}\, \mathrm{d}s.$$

We call $\mathcal{P}$ Dini if $\widetilde{\omega}_{\mathcal{P}}(a) < \infty$ for some $0 < a \leqslant 1$. For the notational simplicity, we use $\omega(t)$ and $\widetilde{\omega}(t)$ to denote $\omega_{\mathcal{P}}(t)$ and $\widetilde{\omega}_{\mathcal{P}}(t)$, respectively, when there is no confusion.

In the Dini case, $\widetilde{\omega}(t) \to 0$ as $t \to 0^+$. Thus it is also a modulus of continuity but may not be Dini again. The following lemma is useful for us.

Lemma 2.2. *Suppose $\omega(t)$ is a Dini modulus of continuity. Suppose $0 < \theta < 1$ is a real number. Then for any $0 < t \leqslant 1$,*

$$\sum_{n=1}^{\infty} \omega(\theta^n t) \leqslant \frac{1}{-\log\theta} \int_0^t \frac{\omega(s)}{s}\, \mathrm{d}s \leqslant \sum_{n=0}^{\infty} \omega(\theta^n t). \tag{2.6}$$

Proof. Consider

$$\int_0^t \frac{\omega(s)}{s}\,\mathrm{d}s = \sum_{n=0}^{\infty}\int_{\theta^{n+1}t}^{\theta^n t}\frac{\omega(s)}{s}\mathrm{d}s \leqslant \sum_{n=0}^{\infty}\omega(\theta^n t)\int_{\theta^{n+1}t}^{\theta^n t}\frac{1}{s}\mathrm{d}s$$
$$\leqslant (-\log\,\theta)\sum_{n=0}^{\infty}\omega(\theta^n t).$$

For the left-sided inequality, we have

$$\sum_{n=1}^{\infty}\omega(\theta^n t) \leqslant \int_0^{\infty}\omega(\theta^x t)\mathrm{d}x = -\frac{1}{\log\,\theta}\int_0^t\frac{\omega(s)}{s}\mathrm{d}s.$$

In the last equality, we use the change of coordinate $s = \theta^x t$. □

Throughout the paper, we always assume (X, f) is a weakly expanding dynamical system and $\mathcal{P}$ is a Dini potential. We call the triple $(X, f, \mathcal{P})$ a *weakly expansive Dini system*. Let $\mathcal{L}$ be the transfer operator associated with the system $(X, f, \mathcal{P})$ defined as in (2.5). By using Gelfand's formula, the spectral radius $\varrho = \varrho(\mathcal{L})$ of the operator $\mathcal{L}$ is

$$\varrho = \lim_{n\to\infty}\|\mathcal{L}^n\|^{\frac{1}{n}}.$$

Since $\mathcal{L}$ is a positive operator, we have

$$\varrho = \lim_{n\to\infty}\|\mathcal{L}^n 1\|^{\frac{1}{n}}.$$

This gives us a lower bound for ϱ

$$\varrho \geqslant \min_{x\in X}\sum_{i=1}^{m}p_i(x). \tag{2.7}$$

Since all p_i are positive functions on a compact metric space, we see that $\varrho > 0$.

Definition 2.3. We say a weakly expansive Dini system $(X, f, \mathcal{P})$ satisfies the optimal quasi-gap condition if

$$\sup_{x\in X}\sum_{i=1}^{m}p_i(x)\left(\sup_{\substack{y\in X\\ 0<|x-y|\leqslant b}}\frac{|g_i(y)-g_i(x)|}{|y-x|}\right) < \varrho \tag{2.8}$$

for some $b > 0$. More general, we say a weakly expansive Dini system $(X, f, \mathcal{P})$ satisfies the *optimal quasi-gap condition* if for some integer $q > 0$, $(X, f^q, \mathcal{P}^q)$ satisfies (2.8), that is,

$$\sup_{x\in X}\sum_{I\in\Sigma_q}p_I(x)\left(\sup_{\substack{y\in X\\ 0<|x-y|\leqslant b}}\frac{|g_I(y)-g_I(x)|}{|y-x|}\right) < \varrho^q \tag{2.9}$$

for some $b > 0$.

Remark 2.4. For a weakly expanding dynamical system f and the corresponding weakly contractive iteration function system

$$\mathcal{W} = \langle g_1, \ldots, g_m\rangle,$$

each g_i is a global Lipschitz function, that is, we have a smallest constant $0 < \mathrm{Lip}(g_i) \leqslant 1$ such that

$$|g_i(x) - g_i(y)| \leqslant \mathrm{Lip}(g_i)|x - y|, \quad \forall\, x, y \in X.$$

Given a potential $\mathcal{P} = \{p_1, \ldots, p_m\}$, one can consider a geometric condition

$$\sup_{x\in X} \sum_{i=1}^{m} p_i(x)\mathrm{Lip}(g_i) < \varrho \tag{2.10}$$

which implies our quasi-gap condition (2.8). However, this condition is not preserved by a simple change in the system $(g_1, \ldots, g_m; p_1, \ldots, p_m)$, in particular, from uniformly expanding dynamical systems into a weakly expanding dynamical system. For example, let $X = [0, 1]$, define $f : [0, 1] \to [0, 1]$ such that

$$g_1(x) = \frac{x}{1+x} : [0, 1] \to \left[0, \frac{1}{2}\right] \quad \text{and} \quad g_2(x) = \frac{1}{2-x} : [0, 1] \to \left[\frac{1}{2}, 1\right].$$

Construct a family $\{f_\epsilon\}_{0\leqslant\epsilon\leqslant\frac{1}{4}}$ of maps such that

$$g_{1,\epsilon}(x) = \frac{x}{1+x}\left(1 + \epsilon(x-1)\right) \quad \text{and} \quad g_{2,\epsilon}(x) = \frac{1}{2-x}\left(1 + \epsilon x(1-x)\right).$$

Then for every $0 \leqslant \epsilon \leqslant \frac{1}{4}$,

$$g_{1,\epsilon}(0) = 0, \quad g_{1,\epsilon}(1) = 1/2 \quad \text{and} \quad \mathrm{Lip}(g_{1,\epsilon}) = 1 - \epsilon;$$

and

$$g_{2,\epsilon}(0) = \frac{1}{2}, \quad g_{2,\epsilon}(1) = 1 \quad \text{and} \quad \mathrm{Lip}(g_{2,\epsilon}) = 1 - \epsilon.$$

Moreover, for $1 \leqslant j \leqslant 2$, $g_{j,0} = g_j$ and $g_{j,\epsilon} \to g_j$ as $\epsilon \to 0$. Thus we have a family of dynamical systems such that f_ϵ is a uniformly expanding dynamical system for every $0 < \epsilon \leqslant \frac{1}{4}$ and $f_0 = f$ is a weakly expanding dynamical system and $f_\epsilon \to f$ as $\epsilon \to 0$. Let

$$p_1(x) = p_2(x) = \frac{1}{2}, \quad x \in [0, 1].$$

Then $\mathcal{P} = \{p_1, p_2\}$ is a potential. Let $\mathcal{L}_\epsilon$ be the transfer operator associated with the system $(g_{1,\epsilon}, g_{2,\epsilon}; p_1, p_2)$. The spectral radius ϱ_ϵ of $\mathcal{L}_\epsilon$ is 1 for all $0 \leqslant \epsilon < \frac{1}{4}$. For any $\epsilon > 0$, we have the condition (2.10) but for $\epsilon = 0$, the condition (2.10) fails.

On the other hand, it is clearly that when $0 < \epsilon \leqslant \frac{1}{4}$, our quasi-gap condition (2.8) holds. When $\epsilon = 0$, take $b = 1/2$, we have that for any $x \in [0, 1]$,

$$\sup_{\substack{y\in[0,1]\\ 0<|x-y|\leqslant b}} \frac{|g_1(x) - g_1(y)|}{|x-y|} \leqslant \frac{1}{1+x} \quad \text{and} \quad \sup_{\substack{y\in[0,1]\\ 0<|x-y|\leqslant b}} \frac{|g_2(x) - g_2(y)|}{|x-y|} \leqslant \frac{1}{2-x}.$$

This implies that

$$\sup_{x\in[0,1]}\left(p_1(x)\sup_{\substack{y\in[0,1]\\0<|x-y|\leqslant b}}\frac{|g_1(x)-g_1(y)|}{|x-y|}+p_2(x)\sup_{\substack{y\in[0,1]\\0<|x-y|\leqslant b}}\frac{|g_2(x)-g_2(y)|}{|x-y|}\right)$$

$$\leqslant\frac{1}{2}\sup_{x\in[0,1]}\left(\frac{1}{1+x}+\frac{1}{2-x}\right)=\frac{3}{4}<1.$$

Thus our quasi-gap condition (2.8) also holds when $\epsilon=0$.

Under the optimal quasi-gap condition, we will prove the following results.

Theorem 2.5. *Suppose a weakly expansive Dini system $(X,f,\mathcal{P})$ satisfies the optimal quasi-gap condition (2.9). Then there exists a unique positive function $h\in\mathcal{C}(X)$ and a unique probability measure $\nu\in\mathcal{M}(X)$ such that*

$$\mathcal{L}h=\varrho h,\quad \mathcal{L}^*\nu=\varrho\nu,\quad \langle\nu,h\rangle=1.$$

And moreover, for any $\phi\in\mathcal{C}(X)$,

$$\lim_{n\to\infty}\|\varrho^{-n}\mathcal{L}^n\phi-\langle\nu,\phi\rangle h\|=0. \tag{2.11}$$

We say that the *Ruelle operator theorem for $\mathcal{L}$ holds* if the assertion of theorem 2.5 holds.

Lemma 2.6. *If the Ruelle operator theorem for $\mathcal{L}^q$ holds for some $q\geqslant 2$, then the Ruelle operator theorem for $\mathcal{L}$ holds.*

Proof. The spectral radius for $\mathcal{L}^q$ is ϱ^q. Since the Ruelle operator theorem for $\mathcal{L}^q$ holds, we have a unique positive function $h\in\mathcal{C}(X)$ and a unique probability measure $\nu\in\mathcal{M}(X)$ such that

$$\mathcal{L}^qh=\varrho^qh,\quad (\mathcal{L}^q)^*\nu=\varrho^q\nu,\quad \langle\nu,h\rangle=1.$$

And moreover, for any $\phi\in\mathcal{C}(X)$,

$$\lim_{k\to\infty}\|(\varrho^q)^{-k}(\mathcal{L}^q)^k\phi-\langle\nu,\phi\rangle h\|=0.$$

This implies that for $\phi=\mathcal{L}h$, we have that

$$\lim_{k\to\infty}\varrho^{-kq}\mathcal{L}^{kq}\left(\mathcal{L}h\right)=\langle\nu,\mathcal{L}h\rangle h=\lim_{k\to\infty}\mathcal{L}\left(\varrho^{-kq}\mathcal{L}^{kq}h\right)=\langle\nu,h\rangle\mathcal{L}h=\mathcal{L}h.$$

This implies that

$$\mathcal{L}h=\langle\nu,\mathcal{L}h\rangle h.$$

And then,

$$\mathcal{L}^qh=(\langle\nu,\mathcal{L}h\rangle)^qh=\varrho^qh.$$

This implies that

$$\mathcal{L}h=\varrho h.$$

Similarly, we have that $\mathcal{L}^*\nu=\varrho\nu$.

For the convergence, take any $\phi \in \mathcal{C}(X)$, we have that

$$\lim_{k\to\infty} \|\varrho^{-kq-j}\mathcal{L}^{kq+j}\phi - \varrho^{-j}\langle \nu, \mathcal{L}^j\phi\rangle h\| = 0 \quad \text{for any } 0 \leqslant j < q.$$

Note that

$$\langle \nu, \mathcal{L}^j\phi\rangle = \langle (\mathcal{L}^*)^j\nu, \phi\rangle = \varrho^j\langle \nu, \phi\rangle.$$

We get that

$$\lim_{k\to\infty} \|\varrho^{-k}\mathcal{L}^{k}\phi - \langle \nu, \phi\rangle h\| = 0.$$

Hence, the Ruelle operator theorem for $\mathcal{L}$ holds. □

Furthermore, we have the convergence speed of $\varrho^{-n}\mathcal{L}^n\phi$ to $\langle \nu, \phi\rangle h$.

Theorem 2.7. *Suppose a weakly expansive Dini system $(X, f, \mathcal{P})$ satisfies the optimal quasi-gap condition (2.8). Let $h \in \mathcal{C}(X)$ and $\nu \in \mathcal{M}(X)$ be from theorem 2.5. Then there exist constants $A > 0$, $0 < \gamma_0 < 1$, and $\ell_0 \in \mathbb{N}$ such that for any $\phi \in \mathcal{C}(X)$ and $n \geqslant k\ell$ with $\ell \geqslant \ell_0$,*

$$\|\varrho^{-n}\mathcal{L}^n\phi - \langle \nu, \phi\rangle h\| \leqslant A\left(\omega_\phi(\tau_\ell) + \|\phi\|\left(\gamma_0^k + \gamma_0^\ell + \widetilde{\omega}(\tau_\ell)\right)\right),$$

where τ_ℓ is the number defined in (2.3) and $\tau_\ell \to 0$ as $\ell \to \infty$.

Remark 2.8. If a weakly expansive Dini system $(X, f, \mathcal{P})$ satisfies the optimal quasi-gap condition (2.9), then we replace n by qn in theorem 2.7.

We take this opportunity to point out that there is a gap in paper [22] since the proof of [22, lemma 4.2] works only with the assumption of the conformal condition on f. theorem 2.5 successfully removes the assumption of the conformal assumption on f. Hence, the gap in [22] is closed by this paper as follows.

We say further f is piecewise C^1 if every $f|_{\mathring{X}_i}$ is C^1, $1 \leqslant i \leqslant m$. Then all g_i are continuously Fréchet differentiable on $\mathring{X}$. In this case, the optimal quasi-gap condition (2.9) is equivalent to

$$\sup_{x\in X} \sum_{I\in\Sigma_q} p_I(x) \cdot \|D_x g_I\| < \varrho^q, \tag{2.12}$$

where $D_x g_I$ means the derivative of g_I at x; $\|D_x g_I\|$ is the norm of $D_x g_I$ as an operator.

Corollary 2.9. *Suppose a weakly expansive Dini system $(X, f, \mathcal{P})$ satisfies the condition (2.12) and suppose f is C^1. Then there exists a unique positive function $h \in \mathcal{C}(X)$ and a unique probability measure $\nu \in \mathcal{M}(X)$ such that*

$$\mathcal{L}h = \varrho h, \quad \mathcal{L}^*\nu = \varrho\nu, \quad \langle \nu, h\rangle = 1.$$

Moreover, there exist constants $A > 0$, $0 < \gamma_0 < 1$, $\ell_0 \in \mathbb{N}$ such that for any $\phi \in \mathcal{C}(X)$ and $qn \geqslant k\ell$ with $\ell \geqslant \ell_0$,

$$\|\varrho^{-qn}\mathcal{L}^{qn}\phi - \langle \nu, \phi\rangle h\| \leqslant A\left(\omega_\phi(\tau_\ell) + \|\phi\|\left(\gamma_0^k + \gamma_0^\ell + \widetilde{\omega}(\tau_\ell)\right)\right),$$

where τ_ℓ is the number defined in (2.3) and $\tau_\ell \to 0$ as $\ell \to \infty$.

Using the lower bound of the spectral radius ϱ in (2.7), we can use the following calculable condition (2.13) to replace the optimal quasi-gap condition (2.8):

$$\sup_{x\in X}\sum_{i=1}^{m} p_i(x)\left(\sup_{\substack{y\in X\\ 0<|x-y|\leqslant b}} \frac{|g_i(y)-g_i(x)|}{|y-x|}\right) < \min_{x\in X}\sum_{i=1}^{m} p_i(x) \tag{2.13}$$

for some $b > 0$.

Corollary 2.10. *Suppose a weakly expansive Dini system $(X, f, \mathcal{P})$ satisfies the condition (2.13). Then there exists a unique positive function $h \in \mathcal{C}(X)$ and a unique probability measure $\nu \in \mathcal{M}(X)$ such that*

$$\mathcal{L}h = \varrho h, \quad \mathcal{L}^*\nu = \varrho\nu, \quad \langle \nu, h\rangle = 1.$$

Moreover, there exist constants $A > 0$, $0 < \gamma_0 < 1$, $\ell_0 \in \mathbb{N}$ such that for any $\phi \in \mathcal{C}(X)$ and $n \geqslant k\ell$ with $\ell \geqslant \ell_0$,

$$\|\varrho^{-n}\mathcal{L}^n\phi - \langle \nu, \phi\rangle h\| \leqslant A\left(\omega_\phi(\tau_\ell) + \|\phi\|\left(\gamma_0^k + \gamma_0^\ell + \widetilde{\omega}(\tau_\ell)\right)\right),$$

where τ_ℓ is the number defined in (2.3) and $\tau_\ell \to 0$ as $\ell \to \infty$.

An important consequence of theorem 2.7 is the decay of correlations.

Corollary 2.11. *Suppose a weakly expansive Dini system $(X, f, \mathcal{P})$ satisfies the optimal quasi-gap condition (2.8). Let $\nu \in \mathcal{M}(X)$ be the probability measure from theorem 2.5. Then there exist constants $A > 0$, $0 < \gamma_0 < 1$ and $\ell_0 \in \mathbb{N}$ such that for any $n \geqslant k\ell$ with $\ell \geqslant \ell_0$ and $\phi \in \mathcal{C}(X)$, we have*

$$|C_\phi(n)| \leqslant A\|\phi\|\left(\omega_\phi(\tau_\ell) + \|\phi\|\left(\gamma_0^k + \gamma_0^\ell + \widetilde{\omega}(\tau_\ell)\right)\right),$$

where τ_ℓ is the number defined in (2.3) and $\tau_\ell \to 0$ as $\ell \to \infty$.

3. Proof of theorem 2.5

We prove theorem 2.5 through several lemmas.

Lemma 3.1. *The followings hold*

(*a*) $\min_{x\in X}\varrho^{-n}\mathcal{L}^n 1(x) \leqslant 1 \leqslant \max_{x\in X}\varrho^{-n}\mathcal{L}^n 1(x)$ *for all $n > 0$;*

(*b*) *if there exist $\lambda > 0$ and $0 < h \in \mathcal{C}(X)$ such that $\mathcal{L}h = \lambda h$, then $\lambda = \varrho$ and there exist $A, B > 0$ such that*

$$A \leqslant \varrho^{-n}\mathcal{L}^n 1(x) \leqslant B \quad \textit{for all } \ n > 0;$$

(*c*) *if $h \gneqq 0$ is a ϱ-eigenfunction of $\mathcal{L}$, then $h > 0$;*

(*d*) $\dim\{h \in \mathcal{C}(X) \mid \mathcal{L}h = \varrho h,\ h \geqslant 0\} \leqslant 1$.

Proof. We prove the second inequality in (a) by contradiction. Suppose it is not true, then there exists an integer k such that $\|\mathcal{L}^k 1\| < \varrho^k$. From Gelfand's formula, $\varrho^k = (\varrho(\mathcal{L}))^k = \varrho(\mathcal{L}^k)$, we have that

$$\varrho = \left(\varrho(\mathcal{L}^k)\right)^{\frac{1}{k}} \leqslant \|\mathcal{L}^k\|^{\frac{1}{k}} = \|\mathcal{L}^k 1\|^{\frac{1}{k}} < \varrho,$$

which is a contradiction. The proof of the first inequality in (a) is similar.

To prove (b), let $a_1 = \min_{x\in X} h(x)$ and $a_2 = \max_{x\in X} h(x)$. Then

$$0 < \frac{a_1}{a_2} \leqslant \frac{h(x)}{a_2} = \frac{\lambda^{-n}}{a_2}\mathcal{L}^n h(x) \leqslant \lambda^{-n}\mathcal{L}^n 1(x) \leqslant \lambda^{-n}\|\mathcal{L}^n\|.$$

Similarly we can show that $\lambda^{-n}\|\mathcal{L}^n\| \leqslant a_2/a_1$. Hence

$$\varrho = \lim_{n\to\infty}\|\mathcal{L}^n\|^{\frac{1}{n}} = \lambda.$$

To prove (c), we first note that if $0 \not\equiv \phi \in \mathcal{C}(X)$ and $\phi \geqslant 0$, then for any $x \in X$, there is an integer $n > 0$ such that $\mathcal{L}^n\phi(x) > 0$. This is because that if we let $V = \{y \in X | \phi(y) > 0\}$. Then V is a non-empty open set. And there exists an $I_0 \in \Sigma$ such that $g_{I_0}(x) \in V$ for any $x \in X$. Then

$$\mathcal{L}^n\phi(x) = \sum_{|I|=|I_0|} p_I(x)\phi(g_I(x)) \geqslant p_{I_0}(x)\phi(g_{I_0}(x)) > 0.$$

From the above argument, we can get (c).

To prove (d), suppose $h_1, h_2 \in \mathcal{C}(X)$ are two strictly positive ϱ-eigenfunctions. Without loss of generality, assume $0 < h_1 \leqslant h_2$ and $h_1(x_0) = h_2(x_0)$ for some $x_0 \in X$. Consider $h = h_2 - h_1 \geqslant 0$. It is a ϱ-eigenfunction with $h(x_0) = 0$. From (c), we have that $h \equiv 0$, that is, $h_1 \equiv h_2$. □

The next result is interesting since for the transfer operator associated with a weakly expanding dynamical system and a Dini potential, we have, in general, the essential spectrum radius $\varrho_{\text{ess}} = \varrho$. We first give a basic criterion for the existence of the eigenfunction corresponding to the spectrum radius ϱ in this case.

From proposition 3.1 of [14], we have

Lemma 3.2. *Suppose that*

(*a*) *There exist* $A, B > 0$ *such that* $A \leqslant \varrho^{-n}\mathcal{L}^n 1(x) \leqslant B$ *for all* $x \in X$ *and* $n > 0$ *and*

(*b*) *For any* $\phi \in \mathcal{C}(X)$, $\{\varrho^{-n}\mathcal{L}^n\phi\}_{n=1}^{\infty}$ *is an equicontinuous sequence.*

Then the Ruelle operator theorem for $\mathcal{L}$ *holds.*

Lemma 3.3. *Let* $0 < \theta < 1$ *and* $a = \sum_{i=0}^{\infty}\omega(\theta^i)$. *For any* $n > 0$ *and any* $x, y \in X$, *if*

$$|g_{j_i}\circ\cdots\circ g_{j_{n-1}}(x) - g_{j_i}\circ\cdots\circ g_{j_{n-1}}(y)| \leqslant \theta^{n-1-i} \quad \forall\, 0 \leqslant i < n-1,$$

then

$$p_J(x) \leqslant e^a p_J(y),$$

where $J = j_0 j_1 \ldots j_{n-1} \in \Sigma_n$.

Proof. Note the definition of ω. The inequality follows from the estimate that

$$\begin{aligned}\left|\log\frac{p_J(x)}{p_J(y)}\right| &\leqslant \sum_{i=0}^{n-1}\left|\log\, p_{j_i}\left(g_{j_{i+1}}\circ\cdots\circ g_{j_{n-1}}(x)\right) - \log\, p_{j_i}\left(g_{j_{i+1}}\circ\cdots\circ g_{j_{n-1}}(y)\right)\right| \\ &\leqslant \sum_{i=0}^{n-1}\omega(\theta^{n-1-i}) \leqslant a.\end{aligned}$$

□

Lemma 3.4. *Suppose*

(*a*) $r := \min_{1\leqslant i\leqslant m}\sup_{x\in X}\sup_{\substack{y\in X\\ 0<|y-x|\leqslant b_0}} \frac{|g_i(x)-g_i(y)|}{|x-y|} < 1$ *for some* $b_0 > 0$ *and suppose*

(*b*) *There exist constants* $A, B > 0$ *such that*

$$A \leqslant \varrho^{-n}\mathcal{L}^n 1(x) \leqslant B \text{ for any } x \in X \text{ and } n > 0.$$

Then the Ruelle operator theorem for $\mathcal{L}$ *holds.*

To prove lemma 3.4, we need the following lemma that is modified from lemma 3.3 of [14]. Let $\mathcal{C}^+(X) := \{0 < \phi \in \mathcal{C}(X)\}$.

Lemma 3.5. *Suppose*

(*a*) $\sup_n \|\varrho^{-n}\mathcal{L}^n\| < \infty$ *and suppose*

(*b*) *There exists a* $b_0 > 0$ *and a dense subset* $\mathcal{D}$ *of* $\mathcal{C}^+(X)$ *such that for each* $\phi \in \mathcal{D}$, *there exists a continuous function* Φ (*depends on* ϕ) *defined on* $[0, 1]$ *with* $\Phi(0) = 0$ *such that*

$$0 < \mathcal{L}^n\phi(x) \leqslant \mathcal{L}^n\phi(y)e^{\Phi(|x-y|)} \quad \forall\ n \geqslant 0 \quad \textit{and} \quad x, y \in X \quad \textit{with } |x-y| \leqslant b_0.$$

Then for each $\phi \in \mathcal{C}(X)$, $\{\varrho^{-n}\mathcal{L}^n\phi\}_{n=1}^{\infty}$ *is a bounded equicontinuous sequence.*

Proof. Let $\phi \in \mathcal{D}$ and $\varphi \in C(X)$. For any $x, y \in X$ with $|x - y| \leqslant b_0$ and $n > 0$,

$$\begin{aligned}
&|\varrho^{-n}\mathcal{L}^n\varphi(x) - \varrho^{-n}\mathcal{L}^n\varphi(y)|\\
&\quad\leqslant \|\varrho^{-n}\mathcal{L}^n\phi\| \cdot \left|1 - \frac{\mathcal{L}^n\phi(y)}{\mathcal{L}^n\phi(x)}\right| + 2\|\varrho^{-n}\mathcal{L}^n\| \cdot \|\phi - \varphi\|\\
&\quad\leqslant B\left(\|\phi\|(e^{\Phi(|x-y|)} - 1) + 2\|\phi - \varphi\|\right).
\end{aligned}$$

where $B = \sup_n \|\varrho^{-n}\mathcal{L}^n\|$. By the assumptions on $\mathcal{D}$ and Φ, we can show that for each $\phi \in \mathcal{C}^+(X)$, $\{\varrho^{-n}\mathcal{L}^n\phi\}_{n=1}^{\infty}$ is a bounded equicontinuous subset of $\mathcal{C}(X)$.

For $\phi \in \mathcal{C}(X)$, we can choose $a > 0$ such that $\phi + a > 0$. Then

$$\{\varrho^{-n}\mathcal{L}^n(\phi + a)\}_{n=1}^{\infty}$$

and $\{\varrho^{-n}\mathcal{L}^n a\}_{n=1}^{\infty}$ are bounded equicontinuous subsets of $\mathcal{C}^+(X)$, hence $\{\varrho^{-n}\mathcal{L}^n\phi\}_{n=1}^{\infty}$ is also a bounded equicontinuous subset of $\mathcal{C}(X)$. □

Proof of lemma 3.4. Let

$$\mathcal{D} = \{\phi \in \mathcal{C}^+(X) : \ \phi(x) \leqslant \phi(y)e^{c|x-y|} \ \text{ for all } x, y \in X \text{ for some } \ c > 0\}.$$

Then $\mathcal{D}$ is dense in $\mathcal{C}^+(X)$.

For any $\phi \in \mathcal{D}$ there exist $c, c_1 > 0$ such that

$$\phi(x) \leqslant \phi(y)e^{c|x-y|}\ \forall x, y \in X \quad \text{and} \quad c_1^{-1} \leqslant \phi(x) \leqslant c_1.$$

This, together with assumption (b), implies that

$$Ac_1^{-1} \leqslant \varrho^{-n}\mathcal{L}^n\phi(x) \leqslant Bc_1.$$

Combining with the strictly positivity of p_i, it is direct to show that

$$0 < b := \inf_{n\geqslant 1} \min_{x,i} \frac{p_i(x)\mathcal{L}^{n-1}\phi(g_i(x))}{\mathcal{L}^n\phi(x)} < 1. \tag{3.1}$$

Since $\mathcal{P}$ is a Dini potential, we can choose $k \geqslant 1$ large enough such that $kb \geqslant 1$ and define

$$\Phi(t) = \frac{k+c}{1-r}\int_0^t \frac{\omega\left(\frac{x}{r}\right)}{x}\,\mathrm{d}x. \tag{3.2}$$

By a direct calculation, we have

$$ct \leqslant \Phi(t), \quad k\omega(t) + \Phi(rt) \leqslant \Phi(t), \tag{3.3}$$

and hence $\phi(x) \leqslant \phi(y)e^{\Phi(|x-y|)}$ for any $x, y \in X$ with $|x - y| \leqslant b_0$. We will prove that for any $n > 0$

$$\mathcal{L}^n\phi(x) \leqslant \mathcal{L}^n\phi(y)e^{\Phi(|x-y|)} \quad \forall x, y \in X \quad \text{with } |x-y| \leqslant b_0.$$

Let $\omega_{\log\phi}(t)$ be the modulus of continuity of $\log\phi$. Then for any $x, y \in X$ with $|x - y| \leqslant b_0$,

$$\begin{aligned}
\mathcal{L}\phi(y) &= \mathcal{L}\phi(x)\sum_{i=1}^m \frac{p_i(y)\phi(g_i(y))}{\mathcal{L}\phi(x)} \\
&\geqslant \mathcal{L}\phi(x)\sum_{i=1}^m \frac{p_i(x)\phi(g_i(x))}{\mathcal{L}\phi(x)} e^{-\omega(|x-y|)-\omega_{\log\phi}(|g_i(x)-g_i(y)|)} \\
&\geqslant \mathcal{L}\phi(x)e^{-\omega(t)-S} \qquad \text{(by the convexity of } e^x\text{)}
\end{aligned}$$

where $t = |x - y|$ and

$$S = \sum_{i=1}^m \frac{p_i(x)\phi(g_i(x))}{\mathcal{L}\phi(x)}\omega_{\log\phi}(|g_i(x) - g_i(y)|).$$

From (a) we can assume, without loss of generality, that

$$\sup_{x\in X} \sup_{\substack{y\in X \\ 0<|y-x|\leqslant b_0}} \frac{|g_1(x) - g_1(y)|}{t} \leqslant r,$$

then

$$\omega_{\log\phi}(|g_1(x) - g_1(y)|) \leqslant \Phi(rt),$$

and by the weak contractivity of g_i, $2 \leqslant i \leqslant m$, we have

$$\omega_{\log\phi}(|g_i(x) - g_i(y)|) \leqslant \Phi(t).$$

We continue the above estimate on S:

$$\begin{aligned} S &\leqslant \frac{p_1(x)\phi(g_1(x))}{\mathcal{L}\phi(x)}\left(\Phi(rt)-\Phi(t)\right) + \Phi(t) \\ &\leqslant -bk\cdot\omega(t)+\Phi(t) \qquad\qquad \text{(by (3.1), (3.3))} \\ &\leqslant -\omega(t)+\Phi(t). \end{aligned}$$

Hence, $\mathcal{L}\phi(x) \leqslant \mathcal{L}\phi(y)e^{\Phi(|x-y|)}$. Inductively we prove that for any $n \in \mathbb{N}$

$$\mathcal{L}^n\phi(x) \leqslant \mathcal{L}^n\phi(y)e^{\Phi(|x-y|)} \quad \forall x, y \in X \quad \text{with } |x-y| \leqslant b_0. \tag{3.4}$$

The result now follows from lemmas 3.5 and 3.2. □

We would like to point out that the condition (a) of lemma 3.4 is a generalisation of the condition (a) of theorem 4.2 in [14]. We modify it in the current form so that the system considered in this paper satisfies the condition (a) of lemma 3.4.

For any integer n, we let

$$D_n = \{(n_1, n_2, \ldots, n_k) : 0 < n_i < n_{i+1} \text{ and } n_k \leqslant n\} \bigcup \{(0)\}.$$

For any $J = j_0 j_1 \ldots j_{n-1} \in \Sigma_n$ and any $0 \leqslant k < l \leqslant n$, we define

$$J|_l^k = j_{n-l} j_{n-l+1} \ldots j_{n-k-1}.$$

We let $J|_l^k = \emptyset$ if $k = l$.

Let $b_0 > 0$ be fixed. For any multi-index $J \in \Sigma$ and $x \in X$, we let

$$\gamma_J(x) = \sup_{\substack{y\in X \\ 0<|y-x|\leqslant b_0}} \frac{|g_J(x)-g_J(y)|}{|x-y|}. \tag{3.5}$$

For convenience, we let $\gamma_J(x) = 1$ and $p_J(x) = 1$ if $|J| = 0$.

Lemma 3.6. *Let $\{D(k)\}_{k=1}^{\ell}$ be a partition of Σ_n, and let*

$$0 = n_0^{(k)} < n_1^{(k)} < \cdots < n_{t_k}^{(k)} = n \quad \forall 1 \leqslant k \leqslant \ell. \tag{3.6}$$

Then for any $x \in X$,

$$\sum_{k=1}^{\ell}\sum_{J\in D(k)} p_J(x)\cdot\prod_{t=1}^{t_k}\gamma_{J|_{n_t^{(k)}}^{n_{t-1}^{(k)}}}\left(g_{J|^0_{n_{t-1}^{(k)}}}(x)\right) \leqslant a^n,$$

provided that

$$\sup_{x\in X}\sum_{i=1}^{m} p_i(x)\cdot\gamma_i(x) \leqslant a. \tag{3.7}$$

Proof. Note the fact that for any multi-index $J = j_0 j_1 \ldots j_{n-1} \in \Sigma_n$, $n \in \mathbb{N}$, and $x \in X$, we have

$$\frac{|g_J(x)-g_J(y)|}{|x-y|} = \prod_{i=1}^{n}\frac{|g_{j_{n-i}}\left(g_{J|^0_{i-1}}(x)\right) - g_{j_{n-i}}\left(g_{J|^0_{i-1}}(y)\right)|}{|g_{J|^0_{i-1}}(x) - g_{J|^0_{i-1}}(y)|}, \quad \forall y \neq x,$$

and

$$p_J(x) = \prod_{i=1}^{n} p_{j_{n-i}}\left(g_{J|_{i-1}^0}(x)\right).$$

Hence, from the weakly contraction of the system $\mathcal{W}$, we conclude that

$$\gamma_J(x) \leqslant \prod_{i=1}^{n} \gamma_{j_{n-i}}\left(g_{J|_{i-1}^0}(x)\right). \tag{3.8}$$

And then this, combined with (3.7), implies that

$$\sum_{J\in\Sigma_n} p_J(x) \cdot \prod_{i=0}^{n-1} \gamma_{J|_{i+1}^i}\left(g_{J|_i^0}(x)\right) \leqslant a^n. \tag{3.9}$$

From the assumption (3.6), using the same argument as that in (3.8), we can deduce that for any $J \in \Sigma_n$,

$$\prod_{t=1}^{t_k} \gamma_{J|_{n_t^{(k)}}^{n_{t-1}^{(k)}}}\left(g_{J|_{n_{t-1}^{(k)}}^0}(x)\right) \leqslant \prod_{i=0}^{n-1} \gamma_{J|_{i+1}^i}\left(g_{J|_i^0}(x)\right). \tag{3.10}$$

Note that $\{D(k)\}_{k=1}^{\ell}$ is a partition of $\Sigma_n (= \{J : |J| = n\})$. It follows that

$$\begin{aligned}
&\sum_{k=1}^{\ell} \sum_{J\in D(k)} p_J(x) \cdot \prod_{t=1}^{t_k} \gamma_{J|_{n_t^{(k)}}^{n_{t-1}^{(k)}}}\left(g_{J|_{n_{t-1}^{(k)}}^0}(x)\right) \\
&\quad \leqslant \sum_{J\in\Sigma_n} p_J(x) \cdot \prod_{i=0}^{n-1} \gamma_{J|_{i+1}^i}\left(g_{J|_i^0}(x)\right) \qquad \text{(by (3.10))} \\
&\quad \leqslant a^n \qquad \text{(by (3.9))}.
\end{aligned}$$

Thus, the conclusion follows. □

As a consequence of lemma 3.4, we have

Lemma 3.7. *Suppose*

(a) *There exists q such that*

$$\sup_{x\in X} \sum_{J\in\Sigma_q} p_J(x) \cdot \gamma_J(x) < \varrho^q$$

and suppose

(b) *There exist constants $A, B > 0$ such that $A \leqslant \varrho^{-n}\mathcal{L}^n 1(x) \leqslant B$ for any $x \in X$ and $n > 0$.*

Then the Ruelle operator theorem for $\mathcal{L}$ holds.

Proof. By (a) there exists a $0 < \eta < 1$ such that

$$\sup_{x\in X} \sum_{J\in\Sigma_q} p_J(x) \cdot \gamma_J(x) \leqslant \eta\varrho^q$$

From this, we conclude, by applying lemma 3.6, that for any $x \in X$ and $n \in \mathbb{N}$,

$$\sum_{J\in\Sigma_{nq}} p_J(x) \cdot \prod_{t=1}^{n} \gamma_{J|_{tq}^{(t-1)q}} \left(g_{J|_{(t-1)q}^{0}}(x) \right) \leqslant \eta^n \varrho^{nq}.$$

By using the argument similar to (3.8), we can prove that for any muti-index $J \in \Sigma_{nq}$,

$$\gamma_J(x) \leqslant \prod_{t=1}^{n} \gamma_{J|_{tq}^{(t-1)q}} \left(g_{J|_{(t-1)q}^{0}}(x) \right).$$

It follows that

$$\sum_{J\in\Sigma_{nq}} p_J(x) \cdot \gamma_J(x) \leqslant \eta^n \varrho^{nq}. \tag{3.11}$$

We claim that

$$\sup_{x\in X} \inf_{n\in\mathbb{N}} \min_{J\in\Sigma_{nq}} \gamma_J(x) = 0.$$

Otherwise, we suppose that

$$\sup_{x\in X} \inf_{n\in\mathbb{N}} \min_{J\in\Sigma_{nq}} \gamma_J(x) > 0.$$

Then, there exists a $c_0 > 0$ and a $x_0 \in X$ such that

$$\inf_{n\in\mathbb{N}} \min_{J\in\Sigma_{nq}} \gamma_J(x_0) \geqslant c_0.$$

This, combined with (3.11) and (b), implies that for any $n \in \mathbb{N}$,

$$\begin{aligned}\eta^n &\geqslant \varrho^{-nq} \sum_{J\in\Sigma_{nq}} p_J(x_0) \cdot \gamma_J(x_0) \geqslant c_0 \cdot \varrho^{-nq} \sum_{J\in\Sigma_{nq}} p_J(x_0) \\ &= c_0 \cdot \varrho^{-nq} \mathcal{L}^{nq} 1(x_0) \geqslant c_0 A. \qquad \text{(by (b))}\end{aligned}$$

This contradicts to the choice of $0 < \eta < 1$. Then, the claim follows. And thus, there exists a $n_0 \in \mathbb{N}$ and a $J_0 \in \Sigma_{n_0 q}$ such that $\sup_{x\in X} \gamma_{J_0}(x) < 1$. From this, by applying lemma 3.4, we conclude that the Ruelle operator theorem for $\mathcal{L}^{n_0 q}$ holds. lemma 2.6 implies that the Ruelle operator theorem for $\mathcal{L}$ holds. $\square$

Now we can complete our proof of theorem 2.5.

Proof of theorem 2.5. The optimal quasi-gap condition (2.9) says that the condition (a) of lemma 3.7 is satisfied. Hence, to finish the proof, we need only to prove that condition (b) of lemma 3.7 is also satisfied, i.e., there exist $A, B > 0$ such that

$$A \leqslant \varrho^{-n} \sum_{J\in\Sigma_n} p_J(x) \leqslant B \quad \forall\ x \in X \text{ and } n \in \mathbb{N}. \tag{3.12}$$

From lemma 2.6, we know that if the Ruelle operator theorem for $\mathcal{L}^q$ holds for some $q \geqslant 2$, then the Ruelle operator theorem for $\mathcal{L}$ holds. Without loss of generality, we may assume that the optimal quasi-gap condition (2.8) is satisfied, i.e.,

$$\sup_{x\in X} \sum_{j=1}^{m} p_j(x) \cdot \gamma_j(x) < \varrho. \tag{3.13}$$

By (3.13), we can find $0 < \eta < 1$ such that

$$\sup_{x\in X}\sum_{j=1}^{m} p_j(x)\cdot\gamma_j(x) \leqslant \eta\varrho. \tag{3.14}$$

For any $x \in X$ and any $r > 0$, we let

$$B(x;r) = \{y \in X : |y - x| < r\}.$$

It is obvious that

$$X \subseteq \bigcup_{x\in X} B\left(x;\frac{b_0}{4}\right).$$

From the compactness of X, we conclude that there exists a finite subset $\{z_i\}_{i=1}^{\ell_0}$ of X such that

$$X \subseteq \bigcup_{i=1}^{\ell_0} B\left(z_i;\frac{b_0}{4}\right). \tag{3.15}$$

It follows that

$$\max_{1\leqslant i\leqslant \ell_0}\sup_{x\in B\left(z_i;\frac{b_0}{2}\right)}\sum_{j=1}^{m} p_j(x)\cdot\gamma_j(x) = \sup_{x\in X}\sum_{j=1}^{m} p_j(x)\cdot\gamma_j(x).$$

Hence it follows from (3.14) that

$$\max_{1\leqslant i\leqslant \ell_0}\sup_{x\in B\left(z_i;\frac{b_0}{2}\right)}\sum_{j=1}^{m} p_j(x)\cdot\gamma_j(x) < \eta\varrho. \tag{3.16}$$

For any fixed $x \in X$, we let $\gamma_J(x)$, $J \in \Sigma$, be defined as in (3.5). Choose θ such that $0 < \eta < \theta < 1$. Let $\delta_0 = \frac{\eta}{\theta}$. Then $0 < \delta_0 < 1$. And for any integer n and $J \in \Sigma_n$, since f is weakly expanding, we have the largest integer $n_1(x) \geqslant 0$ such that

$$\gamma_{J|^0_{n_1(x)}}(x) \geqslant \theta^{n_1(x)},$$

and let $n_2(x)(> n_1(x))$ be the largest integer such that

$$\gamma_{J|^{n_1(x)}_{n_2(x)}}\left(g_{J|^0_{n_1(x)}}(x)\right) \geqslant \theta^{n_2(x)-n_1(x)},$$

and so on. Then, we find a (finite) sequence $\{n_i(x)\}_{i=1}^{t_J}$ such that

$$\gamma_{J|^{n_i(x)}_{n_{i+1}(x)}}\left(g_{J|^0_{n_i(x)}}(x)\right) \geqslant \theta^{n_{i+1}(x)-n_i(x)} \quad \forall\ 1 \leqslant i \leqslant n_{t_J}(x) - 1,$$

and

$$\gamma_{J|^{n_{t_J}(x)}_{i}}\left(g_{J|^0_{n_{t_J}(x)}}(x)\right) < \theta^{i-n_{t_J}(x)} \quad \forall\ n_{t_J}(x) < i \leqslant n. \tag{3.17}$$

Define $\sigma_x : \Sigma_n \to D_n$ by

$$\sigma_x(J) = \left(n_1(x), n_2(x), \ldots, n_{t_J}(x)\right).$$

Then $\#\sigma_x(\Sigma_n) < \infty$. Denote $\sigma_x(\Sigma_n) = \{A_k(x)\}_{k=1}^{\ell(x)}$, where $A_k(x) \in D_n$. Let

$$D_x(k) = \{J : \sigma_x(J) = A_k(x)\}, \quad \forall 1 \leqslant k \leqslant \ell(x).$$

It is clear that

$$D_x(i) \bigcap D_x(j) = \emptyset, \quad \forall i \neq j.$$

Hence, $\{D_x(k)\}_{k=1}^{\ell(x)}$ is a partition of Σ_n. Let

$$\Omega_x(n,k) = \{J \in \Sigma_n : n_{t_J}(x) = k\}, \quad 1 \leqslant k \leqslant n,$$

$$\Omega_x(n,0) = \{J \in \Sigma_n : n_{t_J}(x) = 0\}.$$

Then

$$\Sigma_n = \bigcup_{k=0}^{n} \Omega_x(n,k).$$

We, sometimes, use A_k, ℓ and $\Omega(\cdot,\cdot)$ to denote $A_k(x)$, $\ell(x)$ and $\Omega_x(\cdot,\cdot)$, respectively, for the simplicity if there is no confusion causes. For any $1 \leqslant k \leqslant \ell$, let $A_k = (n_1^{(k)}, n_2^{(k)}, \ldots, n_{t_{k-1}}^{(k)})$. For convenience, we let $n_0^{(k)} = 0$ and let $n_{t_k}^{(k)} = n$. We conclude from lemma 3.6, by making use of (3.14), that

$$S_0 := \sum_{k=1}^{\ell} \sum_{J \in D(k)} p_J(x) \cdot \prod_{t=1}^{t_k} \gamma_{J|_{n_t^{(k)}}^{n_{t-1}^{(k)}}} \left(g_{J|^0_{n_{t-1}^{(k)}}}(x) \right) \leqslant (\eta\varrho)^n. \tag{3.18}$$

Without loss of generality, we assume that $\Omega(n,n) = \{D(k)\}_{k=1}^{\ell_1}$, where $\ell_1 \leqslant \ell = \ell(x)$. And we let

$$S_1 := \sum_{k=1}^{\ell_1} \sum_{J \in D(k)} p_J(x) \cdot \prod_{t=1}^{t_k} \gamma_{J|_{n_t^{(k)}}^{n_{t-1}^{(k)}}} \left(g_{J|^0_{n_{t-1}^{(k)}}}(x) \right).$$

For any $1 \leqslant k \leqslant \ell_1$ and any $J \in D(k)$, we have $n_{t_{k-1}}^{(k)} = n_{t_J} = n$, and this implies that

$$\prod_{t=1}^{t_k} \gamma_{J|_{n_t^{(k)}}^{n_{t-1}^{(k)}}} \left(g_{J|^0_{n_{t-1}^{(k)}}}(x) \right) \geqslant \prod_{t=1}^{t_k} \theta^{n_t^{(k)} - n_{t-1}^{(k)}} = \theta^n.$$

From this, we conclude that

$$S_1 \geqslant \sum_{k=1}^{\ell_1} \sum_{J \in D(k)} p_J(x) \cdot \theta^n = \sum_{J \in \Omega(n,n)} p_J(x) \cdot \theta^n.$$

This, combined with (3.18), implies that

$$\sum_{J \in \Omega_x(n,n)} p_J(x) \cdot \theta^n \leqslant S_1 \leqslant S_0 \leqslant (\eta\varrho)^n.$$

Thus, it follows that

$$\varrho^{-n} \sum_{J \in \Omega_x(n,n)} p_J(x) \leqslant \left(\frac{\eta}{\theta}\right)^n = \delta_0^n. \tag{3.19}$$

Remember that $\omega(t)$ is the modulus of continuity for $\mathcal{P}$ and

$$a := \sum_{k=0}^{\infty} \omega(\theta^k) < \infty.$$

For any $n \in \mathbb{N}$, let

$$\begin{cases} \Xi(n,k) & = \{J \in \Sigma_n : \max\limits_{1 \leqslant i \leqslant \ell_0} n_{t_J}(z_i) = k\}, \quad 1 \leqslant k \leqslant n, \\ \Xi(n,0) & = \{J \in \Sigma_n : \max\limits_{1 \leqslant i \leqslant \ell_0} n_{t_J}(z_i) = 0\}. \end{cases} \tag{3.20}$$

Then

$$\Sigma_n = \bigcup_{k=0}^{n} \Xi(n,k).$$

Remember that

$$\tau_n = \max_{J \in \Sigma_n} \sup_{x,y \in X} |g_J(x) - g_J(y)| \to 0, \quad \text{as } n \to \infty.$$

For any $n > 0$, we can make use of lemma 3.1(a) to find $x_n \in X$ such that

$$\varrho^{-n} \sum_{J \in \Sigma_n} p_J(x_n) \leqslant 1. \tag{3.21}$$

And from (3.15), it is easy to confirm that for any $J \in \Xi(n,n)$ and $x \in X$

$$p_J(x) \leqslant e^{\sum_{q=0}^{n-1} \omega(\tau_q)} \cdot p_J(z_i), \quad \forall 1 \leqslant i \leqslant \ell_0.$$

From this, together with (3.19), we conclude that for any $x \in X$,

$$\begin{aligned} \varrho^{-n} \sum_{J \in \Xi(n,n)} p_J(x) &\leqslant \varrho^{-n} e^{\sum_{q=0}^{n-1} \omega(\tau_q)} \max_{1 \leqslant i \leqslant \ell_0} \sum_{J \in \Omega_{z_i}(n,n)} p_J(z_i) \\ &\leqslant e^{\sum_{q=0}^{n-1} \omega(\tau_q)} \cdot \delta_0^n. \end{aligned} \tag{3.22}$$

For any $J = j_0 j_1 \dots j_{n-1} \in \Xi(n,k)$, we have $J|_k^0 \in \Omega_{z_i}(k,k)$ for some $1 \leqslant i \leqslant \ell_0$. For any $x, y \in X$, it follows from (3.15) that there exist $\{i_j\}_{j=1}^{\ell} \subseteq \{1, \dots, \ell_0\}$, which depends on $g_{J|_k^0}(x)$ and y, such that

$$\begin{cases} (a) & 1 \leqslant \ell \leqslant \ell_0; \\ (b) & B\left(z_{i_j}; \dfrac{b_0}{2}\right) \bigcap B\left(z_{i_{j+1}}; \dfrac{b_0}{2}\right) \neq \emptyset \quad \text{for all } 1 \leqslant j \leqslant \ell - 1; \\ (c) & g_{J|_k^0}(x) \in B\left(z_{i_1}; \dfrac{b_0}{2}\right) \text{ and } y \in B\left(z_{i_\ell}; \dfrac{b_0}{2}\right). \end{cases}$$

Taking $x_j \in B(z_{i_j}; \frac{b_0}{2}) \bigcap B(z_{i_{j+1}}; \frac{b_0}{2})$, $1 \leqslant j \leqslant \ell - 1$. And let $x_0 = g_{J|_k^0}(x)$, $y = x_\ell$. Then we get a chain from x_0 to y:

$$g_{J|_k^0}(x) = x_0 \to z_{i_1} \to x_1 \to z_{i_2} \to \cdots \to x_{\ell-1} \to z_{i_\ell} \to x_\ell = y.$$

It is obvious that

$$x_{j-1}, x_j \in B\left(z_{i_j}; \frac{b_0}{2}\right) \qquad \text{for any } 1 \leqslant j \leqslant \ell.$$

Note that the definition of $\Xi(n,k)$. From this, together with lemma 3.3, we can deduce, by using (3.17) repeatedly 2ℓ times, that

$$p_{J|_n^k}(x_0) \leqslant e^a p_{J|_n^k}(z_{i_1}) \leqslant e^{2a} p_{J|_n^k}(x_1) \leqslant \cdots \leqslant e^{2\ell a} p_{J|_n^k}(x_\ell),$$

i.e., $p_{J|_n^k}\left(g_{J|_k^0}(x)\right) \leqslant e^{2\ell a} p_{J|_n^k}(y)$. (We use $|X| = 1$ here.) This implies that

$$p_{J|_n^k}\left(g_{J|_k^0}(x)\right) \leqslant e^{2\ell_0 a} p_{J|_n^k}(y) \quad \forall\ x, y \in X. \tag{3.23}$$

Hence

$$p_J(x) = p_{J|_n^k}\left(g_{J|_k^0}(x)\right) \cdot p_{J|_k^0}(x) \leqslant e^{2\ell_0 a} p_{J|_n^k}(x_{n-k}) \cdot p_{J|_k^0}(x). \tag{3.24}$$

It follows that

$$\begin{aligned} \varrho^{-n} \sum_{J \in \Xi(n,k)} p_J(x) &\leqslant \varrho^{-n} e^{2\ell_0 a} \sum_{J \in \Xi(n,k)} p_{J|_n^k}(x_{n-k}) p_{J|_k^0}(x) \qquad \text{(by (3.24))} \\ &\leqslant e^{2\ell_0 a} \left(\varrho^{-n+k} \sum_{J' \in \Sigma_{n-k}} p_{J'}(x_{n-k})\right) \left(\varrho^{-k} \sum_{J'' \in \Xi(k,k)} p_{J''}(x)\right) \\ &\leqslant e^{2\ell_0 a} \delta_0^k e^{\sum_{q=0}^{k-1} \omega(\tau_q)}. \qquad \text{(by (3.21) and (3.22))} \end{aligned} \tag{3.25}$$

This further implies that

$$\begin{aligned} \varrho^{-n} \sum_{J \in \Sigma_n} p_J(x) &= \varrho^{-n} \sum_{k=0}^{n} \sum_{J \in \Xi(n,k)} p_J(x) \\ &\leqslant e^{2\ell_0 a} \sum_{k=0}^{n} \prod_{q=0}^{k-1} \left(\delta_0 \cdot e^{\omega(\tau_q)}\right). \end{aligned} \tag{3.26}$$

From the fact that

$$\lim_{k\to\infty} \frac{\prod_{q=0}^{k} \left(\delta_0 \cdot e^{\omega(\tau_q)}\right)}{\prod_{q=0}^{k-1} \left(\delta_0 \cdot e^{\omega(\tau_q)}\right)} = \lim_{k\to\infty} \left(\delta_0 \cdot e^{\omega(\tau_k)}\right) = \delta_0 < 1,$$

we conclude that the last term of (3.26) is bounded by

$$B := e^{2\ell_0 a} \sum_{k=0}^{\infty} \prod_{q=0}^{k-1} \left(\delta_0 \cdot e^{\omega(\tau_q)}\right) < \infty.$$

This concludes our upper bound estimation for (3.12).

For the lower bound estimation for (3.12), we note that lemma 3.1(a) and (3.26) implies that for any $n > 0$, there exists $y_n \in$ such that

$$1 \leqslant C_n := \varrho^{-n} \sum_{J\in\Sigma_n} p_J(y_n) \leqslant B.$$

For any fixed $x \in X$, we let

$$\omega_J = \sum_{i=0}^{n-1} \omega(|g_{J|_i^0}(x) - g_{J|_i^0}(y_n)|).$$

Then, we have

$$p_J(y_n) \leqslant p_J(x) e^{\omega_J}.$$

By making use of (3.23), we get that

$$\omega_J \leqslant 2\ell_0 a + \sum_{i=0}^{k-1} \omega(\tau_i) \leqslant 2\ell_0 a + k\omega(1) \quad \text{for all } J \in \Xi(n,k). \tag{3.27}$$

And then, we have

$$\begin{aligned}
\varrho^{-n} \sum_{J\in\Sigma_n} p_J(y_n)\omega_J &= \varrho^{-n} \sum_{k=0}^{n} \sum_{J\in\Xi(n,k)} p_J(y_n)\omega_J \\
&\leqslant \varrho^{-n} \sum_{k=0}^{n} (2\ell_0 a + k\omega(1)) \sum_{J\in\Xi(n,k)} p_J(y_n) \\
&\leqslant \sum_{k=0}^{n} (2\ell_0 a + k\omega(1)) \cdot \left(\delta_0^k \cdot e^{2\ell_0 a + \sum_{q=0}^{k-1} \omega(\tau_q)}\right) \quad \text{(by (3.25))} \\
&\leqslant B_1,
\end{aligned}$$

where $B_1 := e^{2\ell_0 a} \sum_{k=0}^{\infty} \delta_0^k (2\ell_0 a + k\omega(1)) \cdot e^{\sum_{q=0}^{k-1} \omega(\tau_q)} < \infty$. By the convexity of function e^x, we have

$$\begin{aligned}
\varrho^{-n} \sum_{J\in\Sigma_n} p_J(x) &\geqslant \varrho^{-n} \sum_{J\in\Sigma_n} p_J(y_n) e^{-\omega_J} \geqslant \frac{\varrho^{-n}}{C_n} \sum_{J\in\Sigma_n} p_J(y_n) e^{-\omega_J} \\
&\geqslant e^{-\frac{1}{C_n} \varrho^{-n} \sum_{J\in\Sigma_n} p_J(y_n)\omega_J} \geqslant A := e^{-B_1}.
\end{aligned}$$

This concludes our lower bound estimation for (3.12).

Now lemma 3.7 completes the proof of theorem 2.5. □

4. Proof of theorem 2.7

From theorem 2.5, we can normalise the operator $\mathcal{L}$ as follows. Suppose ϱ and h and ν are eigenvalue and the unique eigenfunction and the unique eigen-measure for $\mathcal{L}$ and $\mathcal{L}^*$ from theorem 2.5, that is,

$$\mathcal{L}h = \varrho h, \quad \mathcal{L}^*\nu = \varrho\nu \quad \text{and} \quad \langle \nu, h\rangle = 1.$$

Let $\mu = h\nu$, and let

$$\widetilde{p}_i(x) = \frac{h(g_i(x))}{\varrho h(x)} p_i(x). \tag{4.1}$$

Then we have a normalised transfer operator $\widetilde{\mathcal{L}}$ defined as

$$\widetilde{\mathcal{L}}\phi(x) = \sum_{i=1}^{m} \widetilde{p}_i(x)\phi(g_i(x)). \tag{4.2}$$

The normalisation means that

$$\widetilde{\mathcal{L}}1 = \sum_{i=1}^{m} \widetilde{p}_i = 1 \quad \text{and} \quad \widetilde{\mathcal{L}}^*\mu = \mu.$$

Moreover, we have that for any $\phi \in \mathcal{C}(X)$ and $n \geqslant 1$

$$\widetilde{\mathcal{L}}^n\phi = \varrho^{-n}(1/h)\mathcal{L}^n(\phi h) = \varrho^{-n}\mathcal{L}^n\left(\frac{\phi h}{h \circ f^n}\right). \tag{4.3}$$

Furthermore, we define

$$\mathcal{P}_n\phi = (\widetilde{\mathcal{L}}^n\phi) \circ f^n, \quad \forall \phi \in \mathcal{C}(X), \ \forall\, n \geqslant 1.$$

And let

$$\mathrm{Im}\,\mathcal{P}_n = \mathcal{P}_n\left(\mathcal{C}(X)\right).$$

The sequence of operators $\mathcal{CMP} = \{\mathcal{P}_n\}_{n=1}^{\infty}$ is a compatible chain of Markovian projections, that is, it satisfies that

(a) $\mathcal{P}_n\mathcal{P}_m = \mathcal{P}_m\mathcal{P}_n = \mathcal{P}_n$ for all $m \geqslant n \geqslant 1$ and
(b) For any $\phi \in C(X)$ and $\chi \in \Gamma_n = \mathrm{Im}\,\mathcal{P}_n$,

$$\mathcal{P}_n(\chi\phi) = \chi\mathcal{P}_n\phi.$$

Then μ is a g-measure for this chain of Markovian projections, that is, $\mathcal{P}_n^*\mu = \mu$ for all $n \geqslant 1$. We would like to note that μ is the unique solution for the eigenvalue problem $\mathcal{P}_n^*\mu = \mu$ for all $n \geqslant 1$. Then convergence (2.11) is equivalent to the convergence condition:

$$\mathcal{P}_n\phi(x) \to \langle \mu, \phi\rangle, \quad n \to \infty, \ \forall\, \phi \in \mathcal{C}(X), \ \forall\, x \in X,$$

due to a standard result in probability theory. It is worth to note that the convergence implies that the eigenvalue problem $\mathcal{P}_n^*\mu = \mu$ should have the unique solution. Actually the convergence problem and the eigenvalue problem are equivalent. Note that

$$\|\mathcal{P}_n\phi\| = \|\widetilde{\mathcal{L}}^n\phi\| = \varrho^{-n}\|\mathcal{L}^n\left(\frac{\phi h}{h \circ f^n}\right)\|, \quad \forall \phi \in \mathcal{C}(X), \ \ \forall n \geqslant 1. \tag{4.4}$$

So if we can get a formula for the convergence speed for $\mathcal{P}_n$, then we can have a convergence speed for $\mathcal{L}^n$. We divide the proof into several steps stated as lemmas and propositions.

For every $n > 0$, consider the σ-algebra $\mathcal{B}_n$ generated by $\{X_I | I \in \Sigma_n\}$. Recall that $X_I (I \in \Sigma)$ is defined in section 2 just before (2.3). Let $E_n(\cdot) = E(\cdot|\mathcal{B}_n)$, $n > 0$, be the conditional expectation in the probability space $(X,\ \mu)$. And let, in particular, $\mathbb{E}(\cdot) = E(\cdot|X)$. For every $\phi \in \mathcal{C}(X)$, we have that

$$E_n(\phi)(x) = \frac{\int_{X_I} \phi \, \mathrm{d}\mu}{\nu\,(X_I)}, \quad \forall x \in X_I,\ \forall I \in \Sigma_n.$$

Then

$$\widetilde{\mathcal{L}}^n E_n(\phi)(x) = \sum_{I \in \Sigma_n} \widetilde{p}_I(x) \frac{\int_{X_I} \phi \, \mathrm{d}\mu}{\nu(X_I)}, \quad \forall\ x \in X. \tag{4.5}$$

$$\mathcal{P}_n E_n(\phi)(x) = \sum_{I \in \Sigma_n} \widetilde{p}_I(f^n(x)) \frac{\int_{X_I} \phi \, \mathrm{d}\mu}{\nu(X_I)}, \quad \forall\ x \in X. \tag{4.6}$$

Recall τ_ℓ $(\ell \in \mathbb{N})$ defined in (2.3) and assume $\tau_\ell < b$ where b is in definition 2.3 for the quasi-gap condition. For any $\ell, n \in \mathbb{N}$, denote

$$S_n^\ell = \max_{|x-y| \leqslant \tau_\ell} \sum_{I \in \Sigma_n} |\widetilde{p}_I(x) - \widetilde{p}_I(y)|\,.$$

Lemma 4.1. *For any $n = k\ell$ $(k, \ell \in \mathbb{N})$ and any $\phi \in \mathcal{C}(X)$, we have*

$$\|\mathcal{P}_n \phi\| \leqslant \omega_\phi(\tau_\ell) + \sum_{j=2}^{k-1} S^\ell_{(j-1)\ell} \cdot \|\prod_{i=1}^{j-2} \mathcal{P}_{i\ell} E_{i\ell}(\phi)\| + \|\prod_{i=1}^{k} \mathcal{P}_{i\ell} E_{i\ell}(\phi)\|.$$

Proof. Since $\mathcal{P}_n 1 = 1$, we have that

$$\|\mathcal{P}_n \phi\| \leqslant \|\phi\|.$$

Let

$$\mathrm{var}_n(\phi) = \max_{I \in \Sigma_n} \max_{x,y \in X_I} |\phi(x) - \phi(y)|$$

be the variation of ϕ on the σ-algebra $\mathcal{B}_n$. Then we have

$$\|\mathcal{P}_n(\mathbb{E} - E_k)(\phi)\| \leqslant \|(\mathbb{E} - E_k)(\phi)\| \leqslant \mathrm{var}_k(\phi).$$

By (4.6) and the $\mathcal{B}_n$-measurability of $E_n(\phi)$, we have

$$\mathrm{var}_{n+\ell}\,(\mathcal{P}_n E_n(\phi)) \leqslant \mathrm{var}_\ell\,(\mathcal{L}^n E_n(\phi)) \leqslant S_n^\ell \cdot \|\phi\|.$$

Then for any $j \geqslant 2$

$$\|\mathcal{P}_n(\mathbb{E} - E_{j\ell}) \left(\prod_{i=1}^{j-1} \mathcal{P}_{i\ell} E_{i\ell} \right) \phi\| \leqslant S^\ell_{(j-1)\ell} \| \left(\prod_{i=1}^{j-2} \mathcal{P}_{i\ell} E_{i\ell} \right) (\phi)\|. \tag{4.7}$$

Since $n = k\ell$, we get that

$$\mathcal{P}_n = \mathcal{P}_n\left((\mathbb{E}-E_\ell) + \sum_{j=2}^{k-1}(\mathbb{E}-E_{j\ell})\prod_{i=1}^{j-1}\mathcal{P}_{i\ell}E_{i\ell} + \prod_{i=1}^{k}\mathcal{P}_{i\ell}E_{i\ell}\right).$$

Note that

$$\|\mathcal{P}_n(\mathbb{E}-E_\ell)(\phi)\| \leqslant \mathrm{var}_\ell(\phi) \leqslant \omega_\phi(\tau_\ell).$$

From this, together with (4.7), we get the conclusion of the lemma. □

From this lemma, we see that the proof of theorem 2.7 depends on estimations of $\|\mathcal{P}_nE_n(\phi)\|$ and S^n_{jn}. We will give these estimations in two key propositions (proposition 4.4 and proposition 4.5).

Lemma 4.2. *Let $C>0$ be a constant such that for $I\in\Sigma_n$*

$$\widetilde{p}_I(x) \leqslant C\widetilde{p}_I(y), \quad \forall x,y\in X.$$

Then

$$C^{-1} \leqslant \frac{\widetilde{p}_I(x)}{\mu(X_I)} \leqslant C, \quad \forall x\in X.$$

Proof. Since for any $x\in\mathring{X}$,

$$\widetilde{\mathcal{L}}^n 1_{X_I}(x) = \sum_{J\in\Sigma_n}\widetilde{p}_J(x)\cdot 1_{X_I}(g_J(x))) = \widetilde{p}_I(x)$$

and since $\mathcal{L}^*\mu=\mu$ and $<\mu,1>=1$, we have that

$$C^{-1}\widetilde{p}_I(x) \leqslant \mu(X_I) = \langle\mu,\widetilde{\mathcal{L}}^n 1_{X_I}\rangle = \langle\mu,\widetilde{p}_I(\cdot)\rangle \leqslant C\widetilde{p}_I(x).$$

This gives the lemma. □

Lemma 4.3. *Suppose $a\geqslant 1$ is a constant. Let $c_I(x)$ be a function for any $I\in\Sigma_n$. Suppose*

$$a^{-1} \leqslant c_I(x) \leqslant a, \quad \forall x\in X, \quad \forall I\in\Sigma_n.$$

Then, for any $\phi(x)\in\mathcal{C}(X)$ with $\int_X\phi\mathrm{d}\mu=0$,

$$|\sum_{I\in\Sigma_n}c_I(x)\int_{X_I}\phi\,\mathrm{d}\mu| \leqslant (1-a^{-2})\|\phi\|\sum_{I\in\Sigma_n}c_I(x)\mu(X_I).$$

Before proving this lemma, we would like to point out that the factor $(1-a^{-2})$ in the last inequality will play an important role in the later argument.

Proof. Let

$$\Sigma_n^+ = \left\{I\in\Sigma_n\ \middle|\ \int_{X_I}\phi\,\mathrm{d}\mu>0\right\} \quad\text{and}\quad \Sigma_n^- = \Sigma_n\backslash\Sigma_n^+.$$

Since

$$\sum_{I\in\Sigma_n}\int_{X_I}\phi\,\mathrm{d}\mu = \int_X\phi\,\mathrm{d}\mu = 0,$$

we have that

$$A = \sum_{I\in\Sigma_n^+} \int_{X_I} \phi \, \mathrm{d}\mu = -\sum_{I\in\Sigma_n^-} \int_{X_I} \phi \, \mathrm{d}\mu.$$

Without lost of generality, we assume $A = 1$. Set

$$b_1 = \sum_{I\in\Sigma_n^+} c_I(x) \int_{X_I} \phi \, \mathrm{d}\mu, \qquad b_2 = -\sum_{I\in\Sigma_n^-} c_I(x) \int_{X_I} \phi \, \mathrm{d}\mu.$$

We have that

$$a^{-1} \leqslant b_i \leqslant a, \qquad i = 1, 2.$$

So one can show that

$$\frac{|b_1 - b_2|}{b_1 + b_2} \leqslant \frac{a - a^{-1}}{a + a^{-1}}.$$

This implies that

$$\left| \sum_{I\in\Sigma_n} c_I(x) \int_{X_I} \phi \, \mathrm{d}\mu \right| = |b_1 - b_2| \leqslant \frac{a - a^{-1}}{a + a^{-1}} (b_1 + b_2)$$

$$\leqslant \frac{a - a^{-1}}{a + a^{-1}} \sum_{I\in\Sigma_n} c_I(x) \left| \int_{X_I} \phi \, \mathrm{d}\mu \right| \leqslant (1 - a^{-2}) \|\phi\| \sum_{I\in\Sigma_n} c_I(x)\mu(X_I).$$

We prove the lemma. □

Now we prove the first key proposition.

Proposition 4.4. *There exists $0 < \gamma < 1$ such that for any $\phi \in \mathcal{C}(X)$ with $\int_X \phi \mathrm{d}\mu = 0$, we have*

$$\|\mathcal{P}_n E_n(\phi)\| \leqslant \gamma \|\phi\|, \qquad \forall \, n \geqslant 1.$$

Proof. From (4.1), we can deduce that there exist two constants $c_1, c_2 > 0$ such that

$$c_1 \varrho^{-n} p_I(x) \leqslant \widetilde{p}_I(x) \leqslant c_2 \varrho^{-n} p_I(x), \qquad \forall I \in \Sigma_n, \qquad \forall n \geqslant 1 \ \forall x \in X. \tag{4.8}$$

We know from theorem 2.5 and lemma 3.1(b) that there exists a constant $B > 0$ such that

$$\varrho^{-n} \sum_{I\in\Sigma_n} p_I(x) \leqslant B \qquad \text{for all } x \in X \text{ and } n > 0.$$

By condition (2.8), we can find a $0 < \eta < 1$ such that

$$\sup_{x\in X} \sum_{i=1}^{m} p_i(x) \cdot \left(\sup_{\substack{y\in X \\ |y-x|\leqslant b}} \frac{|g_i(y) - g_i(x)|}{|y - x|} \right) \leqslant \eta \varrho.$$

Then we know from (3.11) that

$$\sum_{I\in\Sigma_n} p_I(x) \cdot \left(\sup_{\substack{y\in X \\ |y-x|\leqslant b}} \frac{|g_I(y) - g_I(x)|}{|y - x|} \right) \leqslant (\eta\varrho)^n, \qquad \forall \, x \in X, \ \forall n \geqslant 1.$$

Let $0 < \delta_0 = \frac{\eta}{\theta} < 1$ be as in the proof of theorem 2.5. From (4.8) and (3.25), we get a constant $c_3 > 0$ such that

$$\sum_{I\in\Xi(n,k)} \widetilde{p}_I(x) \leqslant c_3\delta_0^k e^{\sum_{q=0}^{k-1}\omega(\tau_q)} \quad \text{for all } x \in X.$$

Recall $\Xi(n, j)$ defined in (3.20) in the proof of theorem 2.5. Let

$$\mathcal{A}(n,k) = \bigcup_{j=k}^{n} \Xi(n, j) \quad \text{and} \quad \mathcal{B}(n,k) = \bigcup_{j=0}^{k-1} \Xi(n, j).$$

Then

$$\Sigma_n = \mathcal{A}(n,k) \bigcup \mathcal{B}(n,k) = \bigcup_{k=0}^{n} \Xi(n,k).$$

Hence

$$\begin{aligned}\sum_{I\in\mathcal{A}(n,k)} \widetilde{p}_I(x) &= \sum_{j=k}^{n} \sum_{I\in\Xi(n,j)} \widetilde{p}_I(x) \leqslant \sum_{j=k}^{n} c_3\delta_0^j e^{\sum_{q=0}^{j-1}\omega(\tau_q)} \\ &\leqslant c_3 \sum_{j=k}^{\infty} \prod_{q=0}^{j-1} \left(\delta_0 e^{\omega(\tau_q)}\right) := d_k.\end{aligned}$$

Note that

$$\lim_{j\to\infty} \frac{\prod_{q=0}^{j} \left(\delta_0 e^{\omega(\tau_q)}\right)}{\prod_{q=0}^{j-1} \left(\delta_0 e^{\omega(\tau_q)}\right)} = \delta_0 < 1.$$

It follows that there exist $c > 0$ and $\delta_0 < \delta < 1$ such that $d_k \leqslant c\delta^k$ for all $k \in \mathbb{N}$. This implies that for any $x \in X$

$$\sum_{I\in\mathcal{A}(n,k)} \widetilde{p}_I(x) \leqslant d_k \leqslant c\delta^k \quad \text{for all } k, n \in \mathbb{N} \text{ with } k \leqslant n. \tag{4.9}$$

For any $I \in \Xi(n,k)$, we conclude from (3.23) that for any $x, y \in X$

$$|\log\, p_I(x) - \log\, p_I(y)| \leqslant 2\ell_0 a + \sum_{j=0}^{k-1} \omega(\tau_j). \tag{4.10}$$

For any $I \in \mathcal{B}(n,k)$, there exists some $\ell < k$ such that $I \in \Xi(n,\ell)$. Then from (4.10), we get that

$$|\log\, p_I(x) - \log\, p_I(y)| \leqslant 2\ell_0 a + \sum_{j=0}^{\ell-1} \omega(\tau_j) \leqslant 2\ell_0 a + \sum_{j=0}^{k-1} \omega(\tau_j).$$

Let $\omega_{\log h}(\cdot)$ be the modulus of continuity of $\log h$. Denote

$$a_k = \exp\left(2\ell_0 a + 2\omega_{\log\, h}(1) + \sum_{j=0}^{k-1} \omega(\tau_j)\right).$$

From this, together with (4.1), we conclude that for any $I \in \mathcal{B}(n,k)$,

$$\widetilde{p}_I(x) \leqslant a_k \cdot \widetilde{p}_I(y), \quad \forall x, y \in X. \tag{4.11}$$

(We use $|X| = 1$ here.)

From (2.4), we know that $\lim_{n\to\infty}\tau_n = 0$. Since ω is the modulus of continuity, we have that $\lim_{n\to\infty}\omega(\tau_n) = 0$. This, combined with the fact that $0 < \delta < 1$, implies that there exists integer $k_0 > 0$ such that $a_{k_0}^{-2} - 3c\delta^{k_0} > 0$. Fixing such a k_0, and let

$$\gamma_1 = 1 - a_{k_0}^{-2} + 3c\delta^{k_0}.$$

Then $0 < \gamma_1 < 1$.

Claim. For any $\phi \in \mathcal{C}(X)$ with $\int_X \phi \mathrm{d}\mu = 0$,

$$\|\widetilde{\mathcal{L}}^n E_n(\phi)\| \leqslant \gamma_1 \|\phi\| \quad \forall n > k_0. \tag{4.12}$$

Proof of claim. Indeed for any $n > k_0$, we let

$$A_n = \sum_{I\in\mathcal{A}(n,k_0)} \int_{X_I} \phi\, \mathrm{d}\mu,$$

and let

$$B_n(x) = \sum_{I\in\mathcal{B}(n,k_0)} \frac{\widetilde{p}_I(x)}{\mu(X_I)} \int_{X_I} \phi\, \mathrm{d}\mu.$$

Note that

$$\mu(X_I) = \int_{X_I} \widetilde{p}_I(x) \mathrm{d}\mu.$$

This, together with (4.9), implies that

$$|A_n| \leqslant \sum_{I\in\mathcal{A}(n,k_0)} \mu(X_I)\|\phi\| \leqslant c\delta^{k_0}\|\phi\|.$$

It follows, again from (4.9), that

$$|\widetilde{\mathcal{L}}E_n(\phi)(x) - B_n(x)| = \left|\sum_{I\in\mathcal{A}(n,k_0)} \frac{\widetilde{p}_I(x)}{\mu(X_I)} \int_{X_I} \phi\, \mathrm{d}\mu\right|$$
$$\leqslant \sum_{I\in\mathcal{A}(n,k_0)} \widetilde{p}_I(x)\|\phi\| \leqslant c\delta^{k_0}\|\phi\|.$$

Hence, we conclude that

$$\left|\widetilde{\mathcal{L}}E_n(\phi)(x) - A_n - B_n(x)\right| \leqslant |A_n| + |\widetilde{\mathcal{L}}E_n(\phi)(x) - B_n(x)| \leqslant 2c\delta^{k_0}\|\phi\|.$$

Define

$$c_I(x) = \begin{cases} 1, & \text{if } \ I \in \mathcal{A}(n, k_0); \\ \dfrac{\widetilde{p}_{I(x)}}{\mu(X_I)}, & \text{if } \ I \in \mathcal{B}(n, k_0). \end{cases}$$

From (4.11) and lemma 4.2, we deduce that

$$a_{k_0}^{-1} \leqslant \frac{\widetilde{p}_I(x)}{\mu(X_I)} \leqslant a_{k_0} \quad \forall I \in \mathcal{B}(n, k_0).$$

This implies that for any $n > k_0$,

$$a_{k_0}^{-1} \leqslant c_I(x) \leqslant a_{k_0} \quad \forall I \in \Sigma_n, \ \forall x \in X.$$

From lemmas 4.3 and 4.2, we have that

$$\begin{aligned} |A_n + B_n(x)| &= \left| \sum_{I \in \Sigma_n} c_I(x) \int_{X_I} \phi \, \mathrm{d}\mu \right| \\ &\leqslant (1 - a_{k_0}^{-2}) \|\phi\| \left(\sum_{I \in \mathcal{A}(n,k_0)} \mu(X_I) + \sum_{I \in \mathcal{B}(n,k_0)} \widetilde{p}_I(x) \right) \\ &\leqslant (1 - a_{k_0}^{-2})(1 + c\delta^{k_0}) \|\phi\| \leqslant (1 - a_{k_0}^{-2} + c\delta^{k_0}) \|\phi\|. \end{aligned}$$

Thus for any $n > k_0$

$$\begin{aligned} \left| \widetilde{\mathcal{L}} E_n(\phi)(x) \right| &\leqslant \left| \widetilde{\mathcal{L}} E_n(\phi)(x) - A_n - B_n(x) \right| + |A_n + B_n(x)| \\ &\leqslant 2c\delta^{k_0} \|\phi\| + (1 - a_{k_0}^{-2} + c\delta^{k_0}) \|\phi\| = \gamma_1 \|\phi\|. \end{aligned}$$

The claim is proved. □

Therefore, from (4.4), we have that for any $\phi \in C(X)$ with $\int_X \phi \mathrm{d}\mu = 0$,

$$\|\mathcal{P}_n E_n(\phi)\| \leqslant \gamma_1 \|\phi\| \quad \forall n > k_0.$$

For any $n \leqslant k_0$, let

$$b = \exp\left(2\omega_{\log\, h}(1) + \sum_{j=0}^{k_0} \omega(\tau_j) \right),$$

and let $\gamma_2 = 1 - b^{-2}$. Then $0 < \gamma_2 < 1$. For $n \leqslant k_0$, we have

$$|\log\, p_I(x) - \log\, p_I(y)| \leqslant \sum_{j=0}^{k_0} \omega(\tau_j), \quad \forall I \in \Sigma_n, \ \forall x, y \in X.$$

This, combined with (4.8) and lemma 4.2, implies that

$$b^{-1} \leqslant \frac{\widetilde{p}_I(x)}{\mu(X_I)} \leqslant b, \quad \forall I \in \Sigma_n \quad \text{and} \quad x \in X.$$

Then again from (4.5) and lemma 4.3, we conclude that

$$|\widetilde{\mathcal{L}}E_n(\phi)(x)| \leqslant \gamma_2\|\phi\|, \quad \text{for any } n \leqslant k_0.$$

Let $\gamma = \max\{\gamma_1, \gamma_2\}$. Then $0 < \gamma < 1$. This, combined with (4.12), implies that

$$\|\widetilde{\mathcal{L}}E_n(\phi)\| \leqslant \gamma\|\phi\|, \quad \forall n \geqslant 1.$$

From (4.4), we finally have that

$$\|\mathcal{P}_n E_n(\phi)\| \leqslant \gamma\|\phi\|, \quad \forall n \geqslant 1.$$

We proved the proposition. □

Now we prove our second key proposition.

Proposition 4.5. *There exist $C > 0$, $0 < \varepsilon < 1$ and $\ell_0 \in \mathbb{N}$ such that for any $\ell \geqslant \ell_0$*

$$S^{\ell}_{j\ell} \leqslant C\left(\varepsilon^{j\ell} + \widetilde{\omega}(\tau_\ell)\right), \quad \forall j \in \mathbb{N}.$$

Proof. We use the same notation as those in the proof of proposition 4.4.

For any $I \in \Sigma_n$ and $x, y \in X$, let

$$\alpha_I(x,y) = 2\omega_{\log\, h}(|x-y|) + \sum_{k=0}^{n} \omega\left(|g_{I|^0_k}(x) - g_{I|^0_k}(y)|\right).$$

From (4.1), we conclude that

$$\widetilde{p}_I(x) \leqslant \exp(\alpha_I(x,y)) \cdot \widetilde{p}_I(y). \tag{4.13}$$

Let $\Phi(\cdot)$ be defined as in (3.2). Then we have $\Phi(t) \approx \widetilde{\omega}(t)$ as $t \to 0^+$. From lemma 3.4 (in particular, (3.4)) and theorem 2.5, we conclude that there exists $a > 0$ such that

(a) $\frac{\varrho^{-n}\mathcal{L}^n 1(x)}{\varrho^{-n}\mathcal{L}^n 1(y)} \leqslant e^{a\cdot\widetilde{\omega}(|x-y|)} \quad \forall x, y \in X$ with $|x-y| \leqslant b_0$;
(b) $\lim_{n\to\infty} \max_{x\in X} |\varrho^{-n}\mathcal{L}^n 1(x) - h(x)| = 0$.

From this, we conclude that

$$\sup_{t>0}\left(\frac{\omega_{\log\, h}(t)}{\widetilde{\omega}(t)}\right) < \infty. \tag{4.14}$$

Then from (2.6) and (4.14), we deduce that there exists $c_1 > 0$ such that for any $I \in \Xi(n,k)$,

$$\alpha_I(x,y) \leqslant (n-k)\omega(|x-y|) + c_1\widetilde{\omega}(|x-y|). \tag{4.15}$$

Remember that $0 < \delta_0 = \eta/\theta < \delta < 1$. Let $t = |x-y|$. Then, by (4.9), (4.13) and (4.15), we have

$$\sum_{I\in\Xi(n,k)} |\widetilde{p}_I(x) - \widetilde{p}_I(y)| \leqslant \left(\max_{I\in\Xi(n,k)} \exp(\alpha_I(x,y)) - 1\right) \cdot \sum_{I\in\Xi(n,k)} \widetilde{p}_I(x)$$
$$\leqslant \exp(c_1\widetilde{\omega}(t))(\delta\, \exp(\omega(t)))^{n-k} - \delta^{n-k}.$$

Take $t_0 > 0$ such that $\delta\exp(\omega(t_0)) < 1$. Then for any $0 \leqslant t = |x-y| \leqslant t_0$,

$$\sum_{I\in\Sigma_n} |\widetilde{p}_I(x) - \widetilde{p}_I(y)| = \sum_{k=0}^{n} \sum_{I\in\Xi(n,k)} |\widetilde{p}_I(x) - \widetilde{p}_I(y)| \leqslant S + \frac{\delta^n}{1-\delta},$$

where

$$S := \frac{\exp(c_1\widetilde{\omega}(t))}{1-\delta\ \exp(\omega(t))} - \frac{1}{1-\delta}.$$

From (2.6) and the fact that $\lim_{t\to 0^+}\widetilde{\omega}(t) = 0$, we deduce that there exists $c_2 > 0$ such that

$$\exp(c_1\widetilde{\omega}(t)) \leqslant 1 + c_2\widetilde{\omega}(t), \quad \forall\ t \leqslant t_0.$$

We continue the above estimate on S:

$$S \leqslant \frac{1+c_2\widetilde{\omega}(t)}{1-\delta\ \exp(\omega(t))} - \frac{1}{1-\delta} \leqslant c_3\widetilde{\omega}(t) \quad \text{for some } c_3 > 0.$$

Take $\varepsilon := \max\{\theta, \delta\} < 1$. Then there exists $C > 0$ such that for $|x-y| \leqslant t_0$

$$\sum_{I\in\Sigma_n} |\widetilde{p}_I(x) - \widetilde{p}_I(y)| \leqslant C\left(\varepsilon^n + \widetilde{\omega}(|x-y|)\right) \quad \forall n \in \mathbb{N}.$$

Since our system $\mathcal{W}$ is weakly contractive, there exists integer $\ell_0 > 0$ such that $\tau_\ell \leqslant \tau_{\ell_0} \leqslant t_0$ for any $\ell \geqslant \ell_0$. Hence, in particular,

$$S^\ell_{j\ell} \leqslant C(\varepsilon^{j\ell} + \widetilde{\omega}(\tau_\ell)), \quad \forall j \in \mathbb{N}.$$

We completed the proof. □

Now we are ready to complete the proof of theorem 2.7.

Proof of theorem 2.7. Let $0 < \gamma < 1$ be given in proposition 4.4. Let $C > 0$, $0 < \varepsilon < 1$ and $\ell_0 > 0$ be given by proposition 4.5. Then $0 < \gamma_0 := \max\{\varepsilon, \gamma\} < 1$. Hence for any $j \geqslant 1$ and $\ell \geqslant \ell_0$

$$S^\ell_{j\ell} \leqslant C\left(\gamma_0^{j\ell} + \widetilde{\omega}(\tau_\ell)\right) \leqslant C\left(\gamma_0^\ell + \widetilde{\omega}(\tau_\ell)\right).$$

For any $\phi \in \mathcal{C}(X)$ with $\int \phi \mathrm{d}\mu = 0$. By using

$$\|\mathcal{P}_n E_n(\phi)\| \leqslant \gamma_0 \|\phi\|$$

repeatedly, we prove that

$$\|\left(\prod_{i=1}^{k'} \mathcal{P}_{i\ell}E_{i\ell}\right)(\phi)\| \leqslant \gamma_0^{k'}\|\phi\|, \quad \forall\ 1 \leqslant k' \leqslant k.$$

Observe that $\sum_{j=0}^\infty \gamma_0^j < \infty$. This, together with Lemma 4.1, implies that

$$\begin{aligned}\|\mathcal{P}_{k\ell}(\phi)\| &\leqslant \omega_\phi(\tau_\ell) + \sum_{j=2}^{k-1} S^\ell_{(j-1)\ell}\cdot\gamma_0^{j-2}\|\phi\| + \gamma_0^k\|\phi\| \\ &\leqslant \omega_\phi(\tau_\ell) + A\|\phi\|\left(\gamma_0^\ell + \gamma_0^k + \widetilde{\omega}(\tau_\ell)\right) \quad \text{for some } A > 0.\end{aligned}$$

Consequently for any $n \geqslant k\ell$ with $\ell \geqslant \ell_0$

$$\|\widetilde{\mathcal{L}}^n\phi\| \leqslant \|\widetilde{\mathcal{L}}^{k\ell}\phi\| = \|\mathcal{P}_{k\ell}(\phi)\| \leqslant \omega_\phi(\tau_\ell) + A\|\phi\| \left(\gamma_0^k + \gamma_0^\ell + \widetilde{\omega}(\tau_\ell)\right).$$

For any $\phi \in \mathcal{C}(X)$, let $\phi_0 = \phi - \langle \nu, \phi \rangle h$. Then we have that

$$\langle \mu, \phi_0/h \rangle = 0.$$

Note that

$$\varrho^{-n}\mathcal{L}^n\phi - \langle \nu, \phi \rangle h = \varrho^{-n}\mathcal{L}^n\phi_0 = h\widetilde{\mathcal{L}}^n(\phi_0/h).$$

It follows that

$$\|\varrho^{-n}\mathcal{L}^n\phi - \langle \nu, \phi \rangle h\| \leqslant \|h\| \left(\omega_{\phi_0/h}(\tau_\ell) + A\|\phi_0/h\| \left(\gamma_0^k + \gamma_0^\ell + \widetilde{\omega}(\tau_\ell)\right).$$

By (4.14), there exists $C > 0$ such that

$$\omega_{\phi_0/h}(\tau_\ell) = \omega_{\phi/h}(\tau_\ell) \leqslant C \left(\omega_\phi(\tau_\ell) + \|\phi\|\widetilde{\omega}(\tau_\ell)\right).$$

Note that $\|\phi_0/h\| \leqslant \left(1 + (h_{\min})^{-1}\right) \|\phi\|$. Therefore, we get a positive constant, which we still denote as A, such that

$$\|\varrho^{-n}\mathcal{L}^n\phi - \langle \nu, \phi \rangle h\| \leqslant A \left(\omega_\phi(\tau_\ell) + \|\phi\|(\gamma_0^k + \gamma_0^\ell + \widetilde{\omega}(\tau_\ell))\right).$$

This completes the proof of theorem 2.7. □

5. The proof of corollary 2.9

From (2.12), it follows that there exists $0 < \eta_1 < 1$ such that

$$\sup_{x\in X} \sum_{I\in\Sigma_q} p_I(x) \cdot \|D_x g_I\| < \eta_1 \cdot \varrho^q. \tag{5.1}$$

Without loss of generality, we assume that X is a compact convex set and all g_i's are Fréchet differentiable on $\mathring{X}$.

For any multi-index I and $y, z \in X$ with $y \neq z$, there exists $0 \leqslant \theta \leqslant 1$ such that

$$|g_I(y) - g_I(z)| \leqslant |g_I'(y + \theta(z - y))(y - z)|.$$

From this, it follows that

$$\frac{|g_I(y) - g_I(z)|}{|y - z|} \leqslant \frac{|\left(D_{y+\theta(z-y)}g_I\right)(y - z)|}{|y - z|} \leqslant \|D_{y+\theta(z-y)}g_I\|.$$

From this, together with the continuity of $\|D_x g_I\|$'s and (5.1), we deduce that for any $\eta_1 < \eta_2 < 1$ and for any $x \in X$ there exists $r_x > 0$ such that

$$\sum_{I\in\Sigma_q} p_I(x) \cdot \left(\sup_{\substack{y\neq z \\ y,z\in B(x;r_x)}} \frac{|g_I(y) - g_I(z)|}{|y - z|} \right) < \eta_2 \cdot \varrho^q. \tag{5.2}$$

Note that $X \subset \bigcup_{x\in X} B(x; r_x)$ and X is compact. By Lebesgue's number lemma, there exists $b_0 > 0$ (the Lebesgue number) such that for any $y, z \in X$, if $|y - z| \leqslant b_0$, then $y, z \in B(x, r_x)$ for some $x \in X$. From this, together with (5.2), we deduce that

$$\sup_{x\in X} \sum_{I\in\Sigma_q} p_I(x) \cdot \left(\sup_{\substack{y\in X \\ 0<|x-y|\leqslant b_0}} \frac{|g_I(y) - g_I(x)|}{|y - x|} \right) < \varrho^q.$$

This is condition (2.9). By applying theorems 2.5 and 2.7, we finish the proof of the corollary.

6. The proof of corollary 2.11

For any $\phi \in \mathcal{C}(X)$,

$$\mathcal{L}^n(\phi \circ f^n \cdot \phi) = \phi \mathcal{L}^n \phi.$$

Then

$$\langle \nu, (\phi \circ f^n)\phi \rangle = \langle \varrho^{-n} (\mathcal{L}^*)^n \nu,\ \phi \circ f^n \cdot \phi \rangle = \langle \nu, \varrho^{-n} \mathcal{L}^n (\phi \circ f^n \cdot \phi) \rangle$$

$$= \langle \nu, \phi \varrho^{-n} \mathcal{L}^n \phi \rangle = \langle \nu, \phi h \widetilde{\mathcal{L}}^n (\phi/h) \rangle = \langle \mu, \phi \widetilde{\mathcal{L}}^n (\phi/h) \rangle.$$

Let $\phi_0 = \phi/h - \langle \nu, \phi \rangle$. Then $<\mu, \phi_0 >= 0$. Now theorem 2.7 says that there exist constants $A > 0$, $0 < \gamma_0 < 1$ and $\ell_0 \in \mathbb{N}$ such that

$$C_\phi(n) = |\langle \mu, \phi \widetilde{\mathcal{L}}^n(\phi_0) \rangle| \leqslant \|\phi\| \cdot \|\widetilde{\mathcal{L}}^n(\phi_0)\|$$
$$\leqslant \|\phi\| \left(\omega_{\phi_0}(\tau_\ell) + A\|\phi_0\| \left(\gamma_0^k + \gamma_0^\ell + \widetilde{\omega}(\tau_\ell) \right) \right).$$

By the same way to the estimates of modulus of continuity as we did in the proof of theorem 2.7, we get for a different positive constant, which we still denote as A, such that

$$C_\phi(n) \leqslant A\|\phi\| \left(\omega_\phi(\tau_\ell) + \|\phi\| \left(\gamma_0^k + \gamma_0^\ell + \widetilde{\omega}(\tau_\ell) \right) \right).$$

This completes the proof.

7. Examples and applications

Our main results, theorems 2.5, 2.7, and corollary 2.11 are useful for much more general weakly expanding Dini systems. Here we give two examples. For this purpose, we need the following result.

Proposition 7.1. *Let (X,f) be a weakly expanding dynamical system with $f|_{\mathring{X}_j}$ being expanding for all $2 \leqslant j \leqslant m$. Let $\mathcal{P}$ be a Dini potential. If*

$$\max_{x\in X} p_1(x) < \min_{x\in X} \left(\sum_{j=1}^{m} p_j(x) \right),$$

then there exists an integer $q > 0$ such that

$$\max_{x\in X} \sum_{I\in\Sigma_q} p_I(x) \cdot \left(\sup_{\substack{y,z\in X \\ y\neq z}} \frac{|g_I(y) - g_I(z)|}{|y - z|} \right) < \varrho^q. \tag{7.1}$$

And hence, the weakly expansive Dini system $(X, f, \mathcal{P})$ *satisfies the optimal quasi-gap condition* (2.9).

Proof. It is easy to confirm that for any $I = i_0 i_1 \dots i_{n-1} \in \Sigma_n$

$$\sup_{\substack{y,z\in X\\ y\neq z}} \frac{|g_I(y) - g_I(z)|}{|y - z|} \leqslant \prod_{k=0}^{n-1} \sup_{\substack{y,z\in X\\ y\neq z}} \frac{|g_{i_k}(y) - g_{i_k}(z)|}{|y - z|}.$$

From this, together with assumptions on the system $(X, f, \mathcal{P})$, we conclude, by applying [14, theorem 4.7], that (7.1) is satisfied. Note that

$$\sup_{\substack{y\in X\\ 0<|x-y|\leqslant b}} \frac{|g_I(y) - g_I(x)|}{|y - x|} \leqslant \sup_{\substack{y,z\in X\\ y\neq z}} \frac{|g_I(y) - g_I(z)|}{|y - z|}.$$

From this, together with (7.1), we conclude that (2.9) is satisfied. □

Example 7.2. Let $X = [0, 1]$, $X_1 = [0, 1/2]$ and $X_2 = [1/2, 1]$. Let $0 < \alpha \leqslant 1$, and let

$$f(x) = \begin{cases} x + x^{1+\alpha} + o(x^{1+\alpha}), & \text{if } x \in X_1 \text{ near } 0; \\ 2x - 1, & \text{if } x \in X_2. \end{cases}$$

be a weakly expanding dynamical system with $\overline{f(X_i)} = X$ for $i = 1, 2$. Let $\{p_i\}_{i=1}^2$ be positive Dini potentials satisfying the following condition:

$$1 = p_1(0) = \max_{x\in X} p_1(x) < p_1(x) + p_2(x). \tag{7.2}$$

Then the weakly expanding dynamical system (X, f) associated with the Dini potential $\mathcal{P} = \{p_1, p_2\}$ satisfies the optimal quasi-gap condition (2.9).

Proof. Since $f_1 = f|_{\mathring{X}_1} : \mathring{X}_1 \to f_1\left(\mathring{X}_1\right)$ and $f_2 = f|_{\mathring{X}_2} : \mathring{X}_2 \to f_2\left(\mathring{X}_2\right)$ are two diffeomorphisms, we can consider the continuous extensions of their inverses:

$$f_1^{-1} : f_1\left(\mathring{X}_1\right) \to X_1 \quad \text{and} \quad f_2^{-1} : f_2\left(\mathring{X}_2\right) \to X_2.$$

And denoted by g_1, g_2, respectively. Then $\mathcal{W} =< g_1, g_2 >$ is a weakly contractive iteration function system. The potential $\mathcal{P}$ is a Dini potential. From this, together with (7.2), it follows that

$$p_1(0) = 1 = \max_{x\in X} p_1(x) < \min_{x\in X} \left(p_1(x) + p_2(x)\right).$$

By applying proposition 7.1, we conclude that the weakly expansive Dini system $(X, f, \mathcal{P})$ satisfies the optimal quasi-gap condition (2.9). □

We would like to remark that the system (X, f) in example 7.2 is not assumed to be smooth. Now we present an example which does not satisfy the condition of proposition 7.1.

Example 7.3. Let $X = [0, 1]$ and $X_1 = [0, 1/2]$ and $X_2 = [1/2, 1]$. Consider

$$f(x) = \begin{cases} \dfrac{x}{1-x}, & \text{if } 0 \leqslant x \leqslant 1/2; \\ 2x - 1, & \text{if } 1/2 < x \leqslant 1. \end{cases}$$

It is a weakly expanding dynamical system. For $1/2 < a < 3/4$, let

$$p_1(x) = \begin{cases} -\log\left(a + (-\log x + \sqrt{8})^{-2}\right), & 0 < x \leqslant 1; \\ -\log a, & x = 0 \end{cases}$$

and

$$p_2(x) = \begin{cases} -\log\left(1 - a - (-\log x + \sqrt{8})^{-2}\right), & 0 < x \leqslant 1; \\ -\log(1-a), & x = 0 \end{cases}$$

be two positive functions in $\mathcal{C}(X)$. Then $\mathcal{P} = \{p_1, p_2\}$ is a Dini potential with the modulus $\widetilde{\omega}(t) = O\left(\frac{-1}{\log t}\right)$ as $t \to 0^+$. Then the weakly expansive Dini system $(X, f, \mathcal{P})$ satisfies the optimal quasi-gap condition (2.8).

Proof. Since $f_1 = f|_{\mathring{X}_1} : \mathring{X}_1 \to f_1\left(\mathring{X}_1\right)$ and $f_2 = f|_{\mathring{X}_2} : \mathring{X}_2 \to f_2\left(\mathring{X}_2\right)$ are two diffeomorphisms, we can consider the continuous extensions of their inverses:

$$f_1^{-1} : f_1\left(\mathring{X}_1\right) \to X_1 \quad \text{and} \quad f_2^{-1} : f_2\left(\mathring{X}_2\right) \to X_2.$$

And denoted by g_1, g_2, respectively. Then $\mathcal{W} =< g_1, g_2 >$ is a weakly contractive iteration function system with the derivatives

$$g_1'(x) = \frac{1}{(1+x)^2} \quad \text{and} \quad g_2'(x) = \frac{1}{2}.$$

Let ϱ be the spectral radius of the transfer operator $\mathcal{L}$ in (2.5). We have that

$$\begin{aligned} &\min_{x\in X}\{p_1(x) + p_2(x)\} \\ &= \min_{x\in X}\left\{-\log\left(a + (-\log x + \sqrt{8})^{-2}\right) - \log\left(1 - a - (-\log x + \sqrt{8})^{-2}\right)\right\} \\ &= -\log a(1-a) \leqslant \varrho. \end{aligned}$$

Now

$$\max_{x\in X}\{p_1(x)|g_1'(x)| + p_2(x)|g_2'(x)|\}$$

$$= \max_{x\in X}\left\{-\frac{1}{(1+x)^2}\log\left(a + (-\log x + \sqrt{8})^{-2}\right) - \frac{1}{2}\log\left(1 - a - (-\log x + \sqrt{8})^{-2}\right)\right\}$$

$$\leqslant \max_{x\in X}\left\{-\frac{1}{(1+x)^2}\log\left(a + (-\log x + \sqrt{8})^{-2}\right)\right\}$$

$$+\max_{x\in X}\left\{-\frac{1}{2}\log\left(1-a-\left(-\log\ x+\sqrt{8}\right)^{-2}\right)\right\}$$

$$=-\log\ a-\frac{1}{2}\log\left(1-a-\frac{1}{8}\right)<-\log\ a(1-a)\leqslant\varrho.$$

Since

$$g_1'(x)=\lim_{y\to x}\frac{g_1(y)-g_1(x)}{y-x}\quad\text{and}\quad g_2'(x)=\lim_{y\to x}\frac{g_2(y)-g_2(x)}{y-x}$$

are continuous functions and since X is a compact metric space, we have a real number $b>0$ such that

$$\sup_{x\in X}\sum_{i=1}^{2}p_i(x)\left(\sup_{\substack{y\in X\\ 0<|x-y|\leqslant b}}\frac{|g_i(y)-g_i(x)|}{|y-x|}\right)<\varrho$$

This is the optimal quasi-gap condition (2.8) for system $(X,f,\mathcal{P})$. □

A probability measure μ on X is said to be a *weak Gibbs measure* for $(X,f,\mathcal{P})$ if there exists a sub-exponential sequence of real numbers $K_n>1$ (i.e. $\lim_{n\to\infty}(1/n)\log K_n=0$) such that for any $n>0$ and $I\in\Sigma_n$,

$$\frac{1}{K_n}\leqslant\frac{\mu(X_I)}{\exp(\log\ p_I(x)-n\ \log\ \varrho)}\leqslant K_n\quad\text{for all } x\in X.\tag{7.3}$$

Corollary 7.4. *Suppose a weakly expansive Dini system $(X,f,\mathcal{P})$ satisfies the optimal quasi-gap condition (2.8). Let $0<h\in\mathcal{C}(X)$, $\nu\in\mathcal{M}(X)$ and $\varrho>0$ be from theorem 2.5, that is,*

$$\mathcal{L}h=\varrho h,\quad \mathcal{L}^*\nu=\varrho\nu,\quad\textit{and}\quad\langle\nu,h\rangle=1.$$

Let $\mu=h\nu$. Then we have that

(a) μ is a weak Gibbs measure;
(b) μ is the unique equilibrium state;
(c) μ is mixing.

Proof.

(a) Define

$$\xi_n=\sum_{k=0}^{n-1}\omega(\tau_k),\quad\text{where } \tau_0=1.$$

Since $\lim_{k\to\infty}\omega(\tau_k)=0$, we have that

$$\lim_{n\to\infty}\frac{\xi_n}{n}=0.$$

For any $I=i_0i_1\ldots i_{n-1}\in\Sigma_n$, we have that for any $x,y\in X$,

$$
\begin{aligned}
|\log\ p_I(x) - \log\ p_I(y)| &\leqslant \sum_{k=1}^{n} |\log\ p_{i_{n-k}}(g_{I|_{k-1}^0}(x)) - \log\ p_{i_{n-k}}(g_{I|_{k-1}^0}(y))| \\
&\leqslant \xi_n.
\end{aligned}
$$

Let 1_{X_I} be the characteristic function on X_I. Then we have that

$$\mathcal{L}^n 1_{X_I}(x) = p_I(x), \quad \forall\, x \in \mathring{X}.$$

This implies that

$$\nu(X_I) = \langle \varrho^{-n}\mathcal{L}^{*n}\nu, 1_{X_I}\rangle = \langle \nu, \varrho^{-n}\mathcal{L}^n 1_{X_I}\rangle = \langle \nu, \varrho^{-n} p_I\rangle.$$

So we have that for any $n > 0$ and $I \in \Sigma_n$,

$$\exp(-\xi_n) \leqslant \frac{\nu(X_I)}{\exp(\log\ p_I(x) - n\ \log\ \varrho)} \leqslant \exp(\xi_n) \quad \text{for all } x \in X.$$

Let $\mu = h\nu$. From this, together with $0 < h \in \mathcal{C}(X)$ and $<\nu, h>= 1$, we can get (7.3). Thus μ is a weak Gibbs measure for the weakly expanding dynamical system f with the Dini potential $\mathcal{P}$.

(b) From (4.3) and (2.11), we can deduce, similarly to [20, Theorem 2.1], that the system has a unique equilibrium state $\mu = h\nu$.

The proof of (c) is standard and can be modified from [4] on the Hölder continuous system. We omit it. □

Another important statistical property for a dynamical system f on a probability space $(X,\ \mu)$ is so called the central limit theorem. This property for a weakly expanding dynamical system now follows theorem 2.7.

Corollary 7.5. *Suppose a weakly expansive Dini system $(X, f, \mathcal{P})$ satisfies the optimal quasi-gap condition (2.8). Let $0 < h \in \mathcal{C}(X)$, $\nu \in \mathcal{M}(X)$ and $\varrho > 0$ be from theorem 2.5, that is,*

$$\mathcal{L}h = \varrho h, \quad \mathcal{L}^*\nu = \varrho\nu, \quad \textit{and} \quad \langle \nu, h\rangle = 1.$$

Let $\mu = h\nu$. Suppose we have two strictly increasing sequences of integers $l_n, k_n \to \infty$ such that $n \geqslant k_n l_n$ and

$$\sum_{n=1}^{\infty} \omega(\tau_{l_n}) < \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} \widetilde{\omega}(\tau_{l_n}) < \infty.$$

Then we have the central limit theorem for f on $(X,\ \mu)$, that is,

$$\lim_{n\to\infty} \mu\left\{x : \sum_{j=0}^{n-1} \phi\circ f^j - n\int \phi \mathrm{d}\mu \leqslant t\sqrt{n}\right\} = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{t} \exp\left(-\frac{x^2}{2\sigma^2}\right)\mathrm{d}x,$$

where $\sigma^2 = -\mathbb{E}\phi^2 + 2\sum_{j=0}^{\infty}\mathbb{E}(\phi\cdot\phi\circ f^j)$ and $\mathbb{E}\phi = \langle \mu, \phi\rangle$.

Proof. Without loss of generality, assume $\int \phi \mathrm{d}\mu = 0$. Let $\mathcal{B}$ be the Borel σ-field. For $n \geqslant 1$, let $\mathcal{B}_n = f^{-n}\mathcal{B}$. Define $V\phi = \phi\circ f$ for $\phi \in L^2(\mu)$. Let V^* be the adjoint operator of $V : L^2(\mu) \to$

$L^2(\mu)$. By theorem 1.1 of [15], it suffices to show the convergences of the following two series

$$\sum_{n=0}^{\infty} |\mathbb{E}(\phi V^n \phi)| < \infty, \qquad \sum_{n=0}^{\infty} \mathbb{E}|V^{*n}\phi| < \infty. \tag{7.4}$$

Recall that $\tilde{\mathcal{L}}$ denotes the normalised transfer operator defined in (4.2). Then we have that $\tilde{\mathcal{L}}^*\mu = \mu$. Furthermore,

$$\mathbb{E}(\phi V^n \phi) = \langle \mu, \phi \cdot V^n \phi \rangle = \langle \tilde{\mathcal{L}}^{*n}\mu, \phi \cdot V^n \phi \rangle = \langle \mu, \tilde{\mathcal{L}}^n(\phi \cdot V^n \phi) \rangle = \langle \mu, \phi \tilde{\mathcal{L}}^n \phi \rangle.$$

So

$$|\mathbb{E}(\phi V^n \phi)| \leqslant \|\phi\| \cdot \|\tilde{\mathcal{L}}^n \phi\|.$$

Since $V^*\phi = \tilde{\mathcal{L}}\phi$, we have also

$$\mathbb{E}|V^{*n}\phi| \leqslant \|\tilde{\mathcal{L}}^n \phi\|.$$

From theorem 2.7, we know that

$$\|\tilde{\mathcal{L}}^n \phi\| \leqslant A\|\phi\| \left(\omega_\phi(\tau_{\ell_n}) + \|\phi\| \left(\gamma_0^{k_n} + \gamma_0^{\ell_n} + \widetilde{\omega}(\tau_{\ell_n})\right)\right).$$

Since $\sum_{n=1}^{\infty}\gamma_0^{k_n} < \infty$ and $\sum_{n=1}^{\infty}\gamma_0^{l_n} < \infty$, (7.4) depends on the finiteness of both of $\sum_{n=1}^{\infty}\omega_\phi(\tau_{l_n})$ and $\sum_{n=1}^{\infty}\widetilde{\omega}(\tau_{l_n})$. But that is our assumption. We proved the theorem. □

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