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A comparison of topological entropies for nonautonomous dynamical systems ☆

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ABSTRACT

We study the distance entropy, Bowen topological entropy, and the classical topological entropy of nonautonomous dynamical systems. We show, in particular, that the distance entropy, Bowen entropy, Pesin entropy and the classical entropy are equivalent when the system is weakly mixing. Furthermore, we investigate the relationship between distance entropy and Hausdorff dimension on subsets from several aspects in detail.

1. Introduction

Let X be a non-empty topological space, and f_i, $1 \leq i < \infty$, be continuous selfmaps of X. And we call $(X; \{f_i\}_{i=1}^{\infty})$ a nonautonomous dynamical system [9]. We are interested in the sequence of maps $f_{1,\infty} = \{f_i\}_{i=1}^{\infty}$ and its iteration $f_i^j := f_{i+j-1} \circ f_{i+j-2} \circ \cdots \circ f_i$. From a logical point of view, these systems naturally carry lots of similar objects as the autonomous systems (see e.g. [16]). For example, the Borel measures [1,6], topological entropies [8,17], topological pressures [7,11], etc.

It is well known that the topological entropy is one of most important topological invariants in dynamical systems. Bowen [2] defined the topological entropy on noncompact sets in a way resembling Hausdorff dimension. Pesin [14,15] developed C-structure, which was called Caratheodory-Pesin (C-P) structure later, to study topological entropy as well as the lower and upper capacity entropies. In a similar way to Bowen entropy, Dai and Jiang [3] proposed the distance entropy for dynamical systems. For nonautonomous systems, Li proposed the notions of Bowen entropy and Pesin topological entropy, which are similar to those used in classical dynamical systems [10]. He also briefly discussed the relationships among Bowen entropy, Pesin entropy and the classical entropy. Biś [1] constructed a C-P structure for nonautonomous systems

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and introduced the lower and upper capacity entropies; moreover, Biś introduced the entropy-like invariant for nonautonomous systems, which was distance entropy in actual. Following Feng and Huang's work [5] on the variational principle for topological entropies on subsets, Xu and Zhou [17] considered the measure-theoretical lower entropy for nonautonomous differential systems. Ju and Yang [8] discussed the properties of Pesin topological entropy of nonautonomous dynamical systems.

The main purpose of the paper is to explore the relationships of these various topological entropies of nonautonomous systems on subsets. To the best of our knowledge, there are a lot of works focusing on the entropies on compact space. Kolyada and Snoha [9] discussed many fundamental properties of the classical topological entropy for nonautonomous systems, for example, the power rule, topological equiconjugacy, entropy of uniformly convergent, etc. And the topological entropy on subsets of a compact space was given in their paper as well. Bowen's pioneering work demonstrated that the entropy on noncompact sets can be studied in a way similar to the Hausdorff dimension [2]. The distance entropy is similar to the Bowen entropy in many aspects, but has certain benefits over the Bowen entropy when describing the complexity of the dynamical behaviors [3].

Similar to the Hausdorff dimension, the distance entropy of the nonautonomous system $(X; f_{1,\infty})$ on subset Y was denoted by $ent_{H,d}(f_{1,\infty}, Y)$ temporarily. The distance entropy of nonautonomous system, in general, depends on the choice of metric d of the space. However, we have the following Proposition 1.1, which comes from Theorem 3.4 and its corollary.

Proposition 1.1. *Let d and d' be two uniformly equivalent metrics on X, and let $f_{1,\infty}$ be a sequence of continuous selfmaps on X. Then for any $Y \subseteq X$,*

$$ent_{H,d}(f_{1,\infty}, Y) = ent_{H,d'}(f_{1,\infty}, Y);$$

in particular, $ent_{H,d}(f_{1,\infty}) = ent_{H,d'}(f_{1,\infty})$.

The power rule of the entropy performs an important role in dynamical systems, which reveals the relationship between entropy $h(T)$ and entropy $h(T^m)$ for any $m \in \mathbb{N}$. The power rule of distance entropy and Bowen entropy is discussed in Theorem 3.7.

We suspect that, at least for compact phase space, the distance entropy, Bowen entropy and Pesin entropy are equivalent based on Li, Ju and Yang's findings [8,10]. This guess is proved to be correct (see Corollary 3.12), nevertheless we find out that the phase space can be moderately reduced to separable in some cases. That is also why the distance entropy is defined on separable metric spaces. A full discussion involving comparison of these entropies can be seen in Theorem 3.9 and Theorem 3.13.

Theorem 1.2. *Let X be a separable metric space, and let $f_{1,\infty}$ be a sequence of continuous selfmaps of X. Then the following statements hold:*

(1) *If X satisfies the Lebesgue finite covering property, then*

$$Bh(f_{1,\infty}, Y) \leq ent_H(f_{1,\infty}, Y), \ \forall Y \subseteq X;$$

(2) *if X satisfies the totally bounded property, then*

$$Bh(f_{1,\infty}, Y) \geq ent_H(f_{1,\infty}, Y), \quad \forall Y \subseteq X;$$

(3) *if X is compact, then*

$$Bh(f_{1,\infty}, Y) = ent_H(f_{1,\infty}, Y), \quad \forall Y \subseteq X.$$

Theorem 1.3. *Let (X, d) be a compact metric space, and let $f_{1,\infty}$ be a sequence of continuous selfmaps of X. Then for any compact subset $K \subseteq X$,*

$$ent_H(f_{1,\infty}, K) \leq h(f_{1,\infty}, K).$$

To obtain the equality, we consider the weakly mixing systems, then

Theorem 1.4. *Let (X, d) be a compact metric space, and let $(X; f_{1,\infty})$ be weakly mixing. Then*

$$ent_H(f_{1,\infty}) = Bh(f_{1,\infty}, X) = h(f_{1,\infty}, X).$$

When considering the Hausdorff dimension of subsets, Bowen's technique has certain drawbacks, and it also raises additional challenges, such as determining an appropriate Lebesgue number for a finite open cover of the phase space. However, based on the distance entropy, there is a natural Lebesgue number [3]. As applications, we consider the relationship between distance entropy and Hausdorff dimension on subsets from two perspectives, the expanding and Lipschitz systems. Lately, Xu and Zhou [17] considered the measure-theoretical lower entropy of the nonautonomous differential dynamical systems. We further discuss the lower bound of Hausdorff dimension of invariant measures.

The paper is organized as follows. In section 2, we give some terminologies of various types of topological entropies of nonautonomous dynamical systems. In section 3, the properties of these topological entropies are discussed. And the comparison of these entropies is also explored. As applications, we investigate the relationship between distance entropy and Hausdorff dimension on subsets in the last section.

2. Preliminaries

Throughout the paper we always let X be a non-empty topological space. And sometimes we assume X to be metric space if it is necessary. Let $f_i : X \to X$, $1 \leq i < \infty$. And denoted by $f_{1,\infty} = \{f_i\}_{i=1}^{\infty}$. Then we call $(X; f_{1,\infty})$ a *nonautonomous dynamical system* [9]. For $i, j \in \mathbb{N}$, let $f_i^0 = id_X$ be the identity map of X, $f_i^j := f_{i+j-1} \circ \cdots \circ f_i$ and $f_i^{-j} := (f_i^j)^{-1} = f_i^{-1} \circ \cdots \circ f_{i+j-1}^{-1}$. And we denote the sequence of maps $\{f_{ik+1}^k\}_{i=0}^{\infty}$ and $\{f_i^{-1}\}_{i=1}^{\infty}$ by $f_{1,\infty}^k$ and $f_{1,\infty}^{-1}$, respectively. For any $x \in X$, the *trajectory* and the *orbit* of x with starting map f_1 are the sequence $(f_1^n(x))_{n=0}^{\infty}$ and the set $\{f_1^n(x)\}_{n=0}^{\infty}$, respectively [9]. Here we would like to remark that, in general, we do not have

$$f_i^j = (f_i)^j \quad \text{and} \quad (f_i^j)^{-1} = (f_i^{-1})^j \quad \text{for all } i, j \in \mathbb{N}.$$

Kolyada and Snoha considered the topological entropy defined for nonautonomous dynamical systems [9]. And later we call it the *classical topological entropy*. In particular, if (X, d) is a compact metric space, for each $n \geq 1$ and $\varepsilon > 0$, we let

$$d_n(x, y) = \max_{0 \leq j \leq n-1} d(f_1^j(x), f_1^j(y)) \quad \forall x, y \in X.$$

Define

$$B_n(x, \varepsilon) = \{y \in X : d_n(x, y) < \varepsilon\}.$$

Let K be a non-empty subset of X, and let $r_n(f_{1,\infty}, K, \varepsilon)$ be the smallest number of $B_n(x, \varepsilon)$-balls with $x \in K$ needed to cover K. We set

$$r(f_{1,\infty}, K, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log r_n(f_{1,\infty}, K, \varepsilon).$$

Then the *classical topological entropy* of the sequence of maps $f_{1,\infty}$ on K is given by

$$h(f_{1,\infty}, K) = \lim_{\varepsilon \to 0} r(f_{1,\infty}, K, \varepsilon).$$

In the following we will present other three concepts.

2.1. Hausdorff dimension

In this subsection, we assume that X is a non-empty compact metric space. For any subset $K(\subseteq X)$, the diameter of K is given by

$$|K| = diam(A) = \sup\{d(x,y) : x, y \in K\}.$$

And we call a countable collection of subsets $\{U_i\}_{i=1}^{\infty}$ is a δ-cover of Y, if $0 < |U_i| \leq \delta$ and $Y \subset \bigcup_{i=1}^{\infty} U_i$.
For any $\delta > 0$, we define

$$\mathcal{H}_\delta^s(Y) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } Y\right\}. \tag{2.1}$$

The value $\mathcal{H}_\delta^s(Y)$ increases or at least does not decrease as $\delta \to 0$. We let

$$\mathcal{H}^s(Y) = \lim_{\delta \to 0} \mathcal{H}_\delta^s(Y).$$

Further, there exists a critical value such that $\mathcal{H}^s(Y)$ jumps from infinity to zero, and we let

$$\dim_{\mathcal{H}}(Y) = \inf\{s \geq 0 : \mathcal{H}^s(Y) = 0\} = \sup\{s : \mathcal{H}^s(Y) = +\infty\}.$$

This critical value $\dim_{\mathcal{H}}(Y)$ is called the *Hausdorff dimension* of Y [4].

2.2. Distance entropy

In this subsection, we assume that (X, d) is a non-empty separable metric space. The distance entropy for nonautonomous system was motivated by Dai and Jiang [3]. Biś [1] defined this entropy-like invariant for nonautonomous systems.

Let $f_{1,\infty}$ be a sequence of continuous selfmaps of X. For any $\varepsilon > 0$ and any $K \subseteq X$, let $n(f_{1,\infty}, K, \varepsilon)$ be the largest nonnegative integer such that

$$|f_1^j(K)| < \varepsilon \quad \forall j \in [0, n(f_{1,\infty}, K, \varepsilon)). \tag{2.2}$$

For the convenience, we let $n(f_{1,\infty}, K, \varepsilon) = 0$ if $|K| \geq \varepsilon$; and

$$n(f_{1,\infty}, K, \varepsilon) = +\infty \ \text{ if } \ |f_1^j(K)| < \varepsilon \ \forall j \in \mathbb{N}.$$

The number $n(f_{1,\infty}, K, \varepsilon)$ is called the *step length* of K with respect to $f_{1,\infty}$.

For any $\mathcal{K} = \{K_i\}_{i=1}^{\infty} \subseteq X$ and $s \geq 0$, we denote

$$diam_\varepsilon(f_{1,\infty}, K_i) = \exp(-n(f_{1,\infty}, K_i, \varepsilon)),$$

and

$$D_\varepsilon(f_{1,\infty},\mathcal{K},s)=\sum_{i=1}^{\infty}(diam_\varepsilon(f_{1,\infty},K_i))^s.$$

Now for any given $\varepsilon>0$ and $s\geq 0$, we define a measure for any subset $Y\subseteq X$ by

$$M_\varepsilon^s(f_{1,\infty},Y)=\inf\Big\{D_\varepsilon(f_{1,\infty},\mathcal{K},s):Y\subseteq\bigcup_{i=1}^{\infty}K_i,n(f_{1,\infty},K_i,\varepsilon)>-\log\varepsilon\Big\}.$$

Let

$$M^s(f_{1,\infty},Y):=\lim_{\varepsilon\to 0}M_\varepsilon^s(f_{1,\infty},Y).$$

Similar to the Hausdorff dimension, there is a critical value such that $M^s(f_{1,\infty},Y)$ jumps from $+\infty$ to 0. And we let

$$ent_H(f_{1,\infty},Y):=\inf\{s:M^s(f_{1,\infty},Y)=0\}=\sup\{s:M^s(f_{1,\infty},Y)=\infty\}.$$

We call $ent_H(f_{1,\infty},Y)$ the *topological distance entropy* (or *distance entropy*) of $f_{1,\infty}$ (respect to d) on the set Y. Especially, we denote $ent_H(f_{1,\infty},X)$ by $ent_H(f_{1,\infty})$ for short. In case to emphasize the metric d, we also denote the distance entropy by $ent_{H,d}(f_{1,\infty})$.

2.3. Bowen topological entropy

In this subsection, we assume that X is a non-empty topological space. Li [10] defined the Bowen topological entropy for nonautonomous systems.

Let $\mathcal{U}$ be a finite open cover of X. We write $K\prec\mathcal{U}$ if K is contained in some elements of $\mathcal{U}$. And we denote similarly that

$$\{K_i\}_{i\in I}\prec\mathcal{U}\ \text{ if } K_i\prec\mathcal{U} \text{ for each } i\in I.$$

Let

$$\ell(f_{1,\infty},\mathcal{U},K)=\max\{n\in\mathbb{N}:f_1^jK\prec\mathcal{U}\text{ for } 0\leq j\leq n-1\}.$$

For the convenience, we let $\ell(f_{1,\infty},\mathcal{U},K)=0$ if $K\not\prec\mathcal{U}$, and let

$$\ell(f_{1,\infty},\mathcal{U},K)=+\infty\ \text{ if } f_1^jK\prec\mathcal{U}\text{ for all } j.$$

Define

$$B(f_{1,\infty},\mathcal{U},K)=\exp(-\ell(f_{1,\infty},\mathcal{U},K)),$$

and for any $s\in\mathbb{R}$ and $\mathcal{K}=\{K_i\}_{i=1}^{\infty}$, we define

$$B(f_{1,\infty},\mathcal{U},\mathcal{K},s)=\sum_{i=1}^{\infty}(B(f_{1,\infty},\mathcal{U},K_i))^s.$$

For any given subset $Y\subseteq X$, we define a measure by

$$\mathcal{M}^s(f_{1,\infty},Y,\mathcal{U})=\lim_{\varepsilon\to 0}\inf\Big\{B(f_{1,\infty},\mathcal{U},\mathcal{K},s):Y\subset\bigcup_{i=1}^{\infty}K_i,B(f_{1,\infty},\mathcal{U},K_i)<\varepsilon,K_i\in\mathcal{K}\Big\}.$$

Similarly, there is a critical value such that $\mathcal{M}^s(f_{1,\infty}, Y, \mathcal{U})$ jumps from $+\infty$ to 0, and we define it as

$$Bh(f_{1,\infty}, Y, \mathcal{U}) = \inf\{s : \mathcal{M}^s(f_{1,\infty}, Y, \mathcal{U}) = 0\}.$$

The *Bowen topological entropy of $f_{1,\infty}$ restricted on Y* is given by

$$Bh(f_{1,\infty}, Y) = \sup_{\mathcal{U}}\{Bh(f_{1,\infty}, Y, \mathcal{U})\},$$

where $\mathcal{U}$ ranges over all finite open covers of X. We denote $Bh(f_{1,\infty}, X)$ by $Bh(f_{1,\infty})$ for short [10]. The Bowen topological entropy is called sometimes the *Bowen's dimension entropy* [3].

Remark 2.1. We would like to remark that, under the assumption that X is separable metric space, one of the differences between the Bowen topological entropy and distance entropy is that Bowen uses all finite open covers $\mathcal{U}$ of X, and the distance entropy covers are only by open ε-balls (see [3] and [12]).

3. The properties of various topological entropies

The property of the classical topological entropy has been discussed in Kolyada [9], we focus mainly on distance entropy and Bowen entropy in this section. We show that the distance entropy and Bowen entropy have some properties similar to the Hausdorff dimension.

Proposition 3.1. *Let (X, d) be a non-empty separable metric space, and let $f_{1,\infty}$ be a sequence of continuous selfmaps on X. Let $\mathcal{U}$ be a finite open cover of X. Then the following properties hold.*

(1) *For any $\varepsilon > 0$, $s > 0$,*

$$M^s_\varepsilon(f_{1,\infty}, \emptyset) = 0, \quad \mathcal{M}^s(f_{1,\infty}, \emptyset, \mathcal{U}) = 0;$$

(2) *Monotonicity: For any $E \subseteq F \subseteq X$, we have*

$$ent_H(f_{1,\infty}, E) \le ent_H(f_{1,\infty}, F), \quad Bh(f_{1,\infty}, E) \le Bh(f_{1,\infty}, F);$$

(3) *Countable subadditivity: Let $Z := \bigcup_{i=1}^\infty Z_i \subseteq X$. For any $\varepsilon > 0$ and $s > 0$, we have*

$$M^s_\varepsilon(f_{1,\infty}, Z) \le \sum_{i=1}^\infty M^s_\varepsilon(f_{1,\infty}, Z_i),$$

and

$$\mathcal{M}^s(f_{1,\infty}, Z, \mathcal{U}) \le \sum_{i=1}^\infty \mathcal{M}^s(f_{1,\infty}, Z_i, \mathcal{U});$$

(4) *For any given $\varepsilon > 0$ and $s > 0$, $M^s_\varepsilon(Z) := M^s_\varepsilon(f_{1,\infty}, Z)$ and $\mathcal{M}^s_{\mathcal{U}}(Z) := \mathcal{M}^s(f_{1,\infty}, Z, \mathcal{U})$ are outer measures on the space of all subsets of X;*

(5) *Countable stability: For any $Z = \bigcup_{i=1}^\infty Z_i \subseteq X$,*

$$ent_H(f_{1,\infty}, \cup_{i=1}^\infty Z_i) = \sup_{i\ge1} ent_H(f_{1,\infty}, Z_i);$$

$$Bh(f_{1,\infty}, Z) = \sup_{i\ge1} Bh(f_{1,\infty}, Z_i).$$

Proof. (1) is obvious. For (2) and (3), the monotonicity and countable subadditivity follow directly from the definitions.

(4) We need only to verify the monotonicity. Indeed, let $\mathcal{K} = \{K_i\}_{i=1}^{\infty}$. If $F \subseteq \cup_{i=1}^{\infty} K_i$ and $E \subseteq F \subseteq X$, then by definitions, it is easy to see that for any given $\varepsilon > 0, s > 0$ and $\mathcal{U}$, we have

$$M_\varepsilon^s(f_{1,\infty}, E) \le M_\varepsilon^s(f_{1,\infty}, F) \;\text{ and }\; \mathcal{M}^s(f_{1,\infty}, E, \mathcal{U}) \le \mathcal{M}^s(f_{1,\infty}, F, \mathcal{U}).$$

From this, together with (1) and (3), we conclude that both of $M_\varepsilon^s(Z) := M_\varepsilon^s(f_{1,\infty}, Z)$ and $\mathcal{M}_{\mathcal{U}}^s(Z) := \mathcal{M}^s(f_{1,\infty}, Z, \mathcal{U})$ are outer measures on the space of all subsets of X.

(5) By the monotonicity, we have

$$ent_H(f_{1,\infty}, \cup_{i=1}^{\infty} Z_i) \ge ent_H(f_{1,\infty}, Z_i), \quad \forall i \ge 1.$$

If $\sup_{i\ge1} ent_H(f_{1,\infty}, Z_i) = +\infty$, there is nothing to prove.
If $s := \sup_{i\ge1} ent_H(f_{1,\infty}, Z_i) < \infty$, then for any $\varepsilon > 0$,

$$ent_H(f_{1,\infty}, Z_i) < s + \varepsilon \text{ for all } i \ge 1.$$

From the definition of $ent_H(f_{1,\infty}, Z_i)$, it follows that

$$M^{s+\varepsilon}(f_{1,\infty}, Z_i) = 0.$$

By the countable subadditivity of the outer measure, we have

$$M_\varepsilon^{s+\varepsilon}(f_{1,\infty}, \cup_{i=1}^{\infty} Z_i) \le \sum_{i=1}^{\infty} M_\varepsilon^{s+\varepsilon}(f_{1,\infty}, Z_i) = 0.$$

This implies that

$$ent_H(f_{1,\infty}, \cup_{i=1}^{\infty} Z_i) \le s + \varepsilon.$$

From the arbitrariness of ε, it follows that

$$ent_H(f_{1,\infty}, \cup_{i=1}^{\infty} Z_i) \le \sup_{i\ge1} ent_H(f_{1,\infty}, Z_i).$$

Then

$$ent_H(f_{1,\infty}, \cup_{i=1}^{\infty} Z_i) = \sup_{i\ge1} ent_H(f_{1,\infty}, Z_i).$$

The second one can be proved similarly, and we omit it. □

Corollary 3.2. *Let (X, d) be a metric space and let $f_{1,\infty}$ be a sequence of uniformly continuous on X. Let K and $K_i, i \ge 1$, be compact subsets of X. If $K \subseteq \bigcup_{i=1}^{m} K_i$, then*

$$ent_H(f_{1,\infty}, K) \le \max_{1\le i\le m} ent_H(f_{1,\infty}, K_i), \quad Bh(f_{1,\infty}, K) \le \max_{1\le i\le m} Bh(f_{1,\infty}, K_i).$$

Definition 3.3. [8,9] Let $(X; f_{1,\infty})$ and $(Y; g_{1,\infty})$ be two nonautonomous dynamical systems. Denote by $\pi_{1,\infty} = \{\pi_i\}_{i=1}^{\infty}$ sequence of equicontinuous surjective maps from X to Y. If $\pi_{i+1} \circ f_i = g_i \circ \pi_i$ for all $i \ge 1$, we say that $\pi_{1,\infty}$ is a topological equisemiconjugacy between $f_{1,\infty}$ and $g_{1,\infty}$, and the dynamical system

$(X; f_{1,\infty})$ is topologically equisemiconjugate with $(Y; g_{1,\infty})$. Furthermore, if $\pi_{1,\infty}$ is an equicontinuous sequence of homeomorphisms such that the sequence $\pi_{1,\infty}^{-1} = \{\pi_i^{-1}\}_{i=1}^{\infty}$ of inverse homeomorphisms is also equicontinuous, we say that $\pi_{1,\infty}$ is a topological equiconjugacy between $f_{1,\infty}$ and $g_{1,\infty}$, and the dynamical system $(X; f_{1,\infty})$ is topologically equiconjugate with $(Y; g_{1,\infty})$.

Theorem 3.4. *Let (X, d) and (Y, ϱ) be metric spaces satisfying the second axiom of countability, $f_{1,\infty}$ be a sequence of continuous selfmaps of X, and $g_{1,\infty}$ be a sequence of continuous selfmaps of Y. If $(X; f_{1,\infty})$ is equisemiconjugate with $(Y; g_{1,\infty})$, then*

$$ent_H(f_{1,\infty}, Z) \geq ent_H(g_{1,\infty}, \pi_1(Z)).$$

Further if $(X; f_{1,\infty})$ is topological equiconjugate with $(Y; g_{1,\infty})$, then

$$ent_H(f_{1,\infty}, Z) = ent_H(g_{1,\infty}, \pi_1(Z)).$$

Proof. Since $\pi_{1,\infty}$ is a topological equisemiconjugacy between $f_{1,\infty}$ and $g_{1,\infty}$, we have

$$\pi_{i+1} \circ f_i = g_i \circ \pi_i \quad \text{for all} \quad i \geq 1,$$

i.e.,

$$\begin{array}{ccccc} X & \xrightarrow{f_1} & X & \xrightarrow{f_2} & X \\ \pi_1 \downarrow & & \downarrow \pi_2 & & \downarrow \pi_3 \; \cdots \\ Y & \xrightarrow{g_1} & Y & \xrightarrow{g_2} & Y \end{array}$$

We can check that

$$\pi_{j+1} \circ f_1^j = g_1^j \circ \pi_1 \quad \forall j \geq 1.$$

Note that $\pi_{1,\infty}$ is also equicontinuous, then for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $\delta < \varepsilon$ and

$$\rho(\pi_i(x_1), \pi_i(x_2)) < \varepsilon \quad \text{whenever } d(x_1, x_2) < \delta \text{ for any } x_1, x_2 \in X.$$

For any $Z \subset X$ and any countable cover $\mathcal{K} = \{K_i\}_{i\geq 1}$ of Z, note that $n(f_{1,\infty}, K_i, \delta)$ is the largest integer such that

$$\sup_{x_1, x_2 \in X} d(f_1^j(x_1), f_1^j(x_2)) < \delta \text{ for all } 0 \leq j < n(f_{1,\infty}, K_i, \delta) \text{ and } x_1, x_2 \in K_i.$$

This implies that for $0 \leq j < n(f_{1,\infty}, K_i, \delta)$ and $x_1, x_2 \in K_i$,

$$|g_1^j(\pi_1(K_i))| = |\pi_{j+1}(f_1^j(K_i))| = \sup_{x_1, x_2 \in K_i} \rho(\pi_{j+1}(f_1^j(x_1)), \pi_{j+1}(f_1^j(x_2))) < \varepsilon.$$

From this, we deduce

$$n(f_{1,\infty}, K_i, \delta) \leq n(g_{1,\infty}, \pi_1(K_i), \varepsilon).$$

It follows that for any $s > 0$,

$$D_\delta(f_{1,\infty}, \mathcal{K}, s) \geq D_\varepsilon(g_{1,\infty}, \pi_1(\mathcal{K}), s)$$

and

$$M_\delta^s(f_{1,\infty}, Z) \geq M_\varepsilon^s(g_{1,\infty}, \pi_1(Z)).$$

Hence

$$ent_H(f_{1,\infty}, Z) \geq ent_H(g_{1,\infty}, \pi_1(Z)).$$

If $\pi_{1,\infty}$ is further topological equiconjugate, then $f_i \circ \pi_i^{-1} = \pi_{i+1}^{-1} \circ g_i$ holds for all $i \geq 1$, hence

$$ent_H(g_{1,\infty}, \pi_1(Z)) \geq ent_H(f_{1,\infty}, \pi_1^{-1}(\pi_1(Z))) = ent_H(f_{1,\infty}, Z).$$

Therefore we proved that $ent_H(f_{1,\infty}, Z) = ent_H(g_{1,\infty}, \pi_1(Z))$. □

We consider a special case of topological equiconjugate, if there is a homeomorphism $\pi : X \to Y$ such that $\pi \circ f_i = g_i \circ \pi$ for all $i \geq 1$.

$$\begin{array}{ccccc} X & \xrightarrow{f_1} & X & \xrightarrow{f_2} & X \\ \pi\downarrow & & \downarrow\pi & & \downarrow\pi \quad \cdots \\ Y & \xrightarrow{g_1} & Y & \xrightarrow{g_2} & Y \end{array}$$

Then from Theorem 3.4, we get the following Corollary 3.5.

Corollary 3.5. *Let (X, d) be metric space satisfying the second axiom of countability, $f_{1,\infty}$ be a sequence of continuous selfmaps of X, and $g_{1,\infty}$ be a sequence of continuous selfmaps of Y. If the homeomorphism $\pi : X \to Y$ satisfies $\pi \circ f_i = g_i \circ \pi$ for all $i \geq 1$, then*

$$ent_H(f_{1,\infty}, Z) = ent_H(g_{1,\infty}, \pi(Z)).$$

It's well known that the Hausdorff dimension relies on the metric of the space. The distance entropy also depends strictly on the choice of the metric d of space X. We say two metrics d and d' on X are *uniformly equivalent* if

$$Id : (X, d) \to (X, d') \text{ and } Id : (X, d') \to (X, d)$$

are both uniformly continuous [16]. Then from Theorem 3.4, we get the following Corollary 3.6.

Corollary 3.6. *If d and d' are uniformly equivalent on X, and $f_{1,\infty}$ is a sequence of continuous selfmaps on X, then for any $Y \subseteq X$*

$$ent_{H,d}(f_{1,\infty}, Y) = ent_{H,d'}(f_{1,\infty}, Y);$$

in particular, $ent_{H,d}(f_{1,\infty}) = ent_{H,d'}(f_{1,\infty})$.

It's reasonable to neglect the metric d of $ent_H(f_{1,\infty})$, as long as we take the homeomorphism metric of d.

One of our research objects is the power rule, which can be used to study the relationship between the entropy or pressure of T^m and m times the entropy or pressure of T in dynamical systems. Kolyada and Snoha showed that the power rule equality holds for the classical topological entropy of nonautonomous systems when the sequence $f_{1,\infty}$ is periodic or equicontinuous [9]. The subsequently theorem is a generalization of power rule for distance entropy and Bowen entropy.

Theorem 3.7. *Let X be a separable metric space, and let $f_{1,\infty}$ be a sequence of continuous selfmaps of X. Then for any $m \in \mathbb{N}^+$ and any $Y \subseteq X$,*

(1) $ent_H(f_{1,\infty}^m, Y) \leq m \cdot ent_H(f_{1,\infty}, Y)$;
(2) $ent_H(f_{1,\infty}^m, Y) = m \cdot ent_H(f_{1,\infty}, Y)$ *if $f_{1,\infty}$ is equicontinuous.*

Proof. For the first inequality, for any $K \subseteq X$ and any $j, m \geq 1$, notice that $f_{1,\infty}^m = \{f_{im+1}^m\}_{i\geq 0}$ and

$$f_1^{mj}(K) = f_{(j-1)m+1}^m \circ \cdots \circ f_{m+1}^m \circ f_1^m(K).$$

From (2.2), we get that for any $\varepsilon > 0$,

$$n(f_{1,\infty}, K, \varepsilon) \leq m \cdot n(f_{1,\infty}^m, K, \varepsilon). \tag{3.1}$$

It is sufficient to prove the assertion under the assumption that $n(f_{1,\infty}, K, \varepsilon) > m$. As, otherwise, this inequality is apparent. The inequality follows immediately if we split $n(f_{1,\infty}, K, \varepsilon)$ into several groups with each group has m members and count them one by one.

From (3.1), it follows that for any $Y \subseteq X$, we have

$$M_\varepsilon^{ms}(f_{1,\infty}^m, Y) \leq M_\varepsilon^s(f_{1,\infty}, Y) \text{ and } M^{ms}(f_{1,\infty}^m, Y) \leq M^s(f_{1,\infty}, Y).$$

Then for any $s > ent_H(f_{1,\infty}, Y)$, we have

$$M^{ms}(f_{1,\infty}^m, Y) \leq M^s(f_{1,\infty}, Y) = 0.$$

This implies that $s \geq ent_H(f_{1,\infty}^m, Y)/m$. It follows that

$$ent_H(f_{1,\infty}^m, Y) \leq m \cdot ent_H(f_{1,\infty}, Y).$$

For the second equation, if $f_{1,\infty}$ is equicontinuous, then for every $\varepsilon > 0$, we take

$$\delta(\varepsilon) = \varepsilon + \sup_{i\geq 1} \max_{j=1,\cdots,m-1} \sup_{x,y\in X} \{d(f_i^j(x), f_i^j(y)) : d(x,y) < \varepsilon\},$$

then $\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0$ and

$$d(f_i^j(x), f_i^j(y)) \leq \delta(\varepsilon) \text{ whenever } i \geq 1 \text{ and } j = 1, \cdots, m-1 \text{ if } d(x,y) < \varepsilon.$$

Now we let $\mathcal{K} = \{K_i\}_{i=1}^\infty$ be a countable cover of Y, it follows that

$$m \cdot n(f_{1,\infty}^m, K_i, \varepsilon) \leq n(f_{1,\infty}, K_i, \delta(\varepsilon)) \text{ for } i = 1, 2, \cdots.$$

This means that for any $s > 0$ and $Y \subset X$, we have

$$M_\varepsilon^{sm}(f_{1,\infty}^m, Y) \geq M_{\delta(\varepsilon)}^s(f_{1,\infty}, Y) \text{ and } M^{sm}(f_{1,\infty}^m, Y) \geq M^s(f_{1,\infty}, Y).$$

Similarly, this implies that $ent_H(f_{1,\infty}^m, Y) \geq m \cdot ent_H(f_{1,\infty}, Y)$ and then the assertion follows. □

Corollary 3.8. *Let X be a topological space, and let $f_{1,\infty}$ be a sequence of continuous selfmaps of X. Then for any $m \in \mathbb{N}$ and any $Y \subseteq X$,*

$$Bh(f_{1,\infty}^m, Y) \le m \cdot Bh(f_{1,\infty}, Y).$$

In particular, if $f_{1,\infty}$ is periodic with period $m \in \mathbb{N}$, then

$$Bh(f_{1,\infty}^m, Y) = m \cdot Bh(f_{1,\infty}, Y) \quad \text{for any } Y \subseteq X.$$

Proof. The first part resembles the first inequality of Theorem 3.7, the key step is to figure out the relation between $\ell(f_{1,\infty}, \mathcal{U}, K)$ and $\ell(f_{1,\infty}^m, \mathcal{U}, K)$. Let $\mathcal{U}$ be a finite open cover of X. For any $K \subseteq X$, we denote by $l = \ell(f_{1,\infty}, \mathcal{U}, K)$ for simplicity, and split the set $\{f_1^j(K) \prec \mathcal{U}\}_{j=0}^{l-1}$ into several groups with each group has m members, we can check that

$$\ell(f_{1,\infty}, \mathcal{U}, K) \le m \cdot \ell(f_{1,\infty}^m, \mathcal{U}, K).$$

Then the rest proof is similar.

For the second part, if $f_{1,\infty}$ is periodic with period $m \in \mathbb{N}$, we would like to refer to Theorem 3.4 of [8] and Theorem 3.1 of [10] (see Lemma 3.11) for an equivalent discussion of Pesin topological entropy. □

Now we focus on the relationship between the distance entropy and the Bowen topological entropy. A metric space X is said to satisfy the *Lebesgue covering property* provided that for any (finite) open cover $\mathcal{U}$ of X there is a Lebesgue number δ such that each subset of X of diameter less than or equal to δ lies in some member of $\mathcal{U}$. A metric space is called *totally bounded* (or *precompact*) iff for any $\varepsilon > 0$ there is a finite cover which consists of Borel sets of diameter less than or equal to ε [3].

Theorem 3.9. *Let X be a separable metric space, and let $f_{1,\infty}$ be a sequence of continuous selfmaps of X. Then the following statements hold:*

(1) *If X satisfies the Lebesgue finite covering property, then*

$$Bh(f_{1,\infty}, Y) \le ent_H(f_{1,\infty}, Y) \quad \text{for all } Y \subseteq X;$$

(2) *if X satisfies the totally bounded property, then*

$$Bh(f_{1,\infty}, Y) \ge ent_H(f_{1,\infty}, Y) \quad \text{for all } Y \subseteq X;$$

(3) *if X is compact, then*

$$Bh(f_{1,\infty}, Y) = ent_H(f_{1,\infty}, Y) \quad \text{for all } Y \subseteq X.$$

Proof. (1) If X satisfies the Lebesgue finite covering property, for any finite open cover $\mathcal{U}$ of X, let $\delta > 0$ be a Lebesgue number of $\mathcal{U}$. By the definition of $ent_H(f_{1,\infty}, Y)$, for any $\varepsilon \le \delta$ and any cover $\mathcal{K} = \{K_i\}_{i=1}^{\infty}$ of Y with $e^{-n(f_{1,\infty}, K_i, \varepsilon)} < \varepsilon$, we have $|f_1^j(K_i)| < \varepsilon \le \delta$ and hence $f_1^j(K_i) \prec \mathcal{K}$. Therefore, we obtain that

$$n(f_{1,\infty}, K_i, \varepsilon) \le \ell(f_{1,\infty}, \mathcal{K}, K_i) \quad \text{for all } i \ge 1.$$

Hence for any $s \ge 0$,

$$M_\varepsilon^s(f_{1,\infty}, \mathcal{K}) = \sum_{i=1}^{\infty} e^{-s \cdot n(f_{1,\infty}, K_i, \varepsilon)} \ge \sum_{i=1}^{\infty} e^{-s \cdot \ell(f_{1,\infty}, \mathcal{K}, K_i)} = B(f_{1,\infty}, \mathcal{U}, \mathcal{K}, s).$$

This implies that

$$M_\varepsilon^s(f_{1,\infty}, Y) \geq \inf\Big\{B(f_{1,\infty}, \mathcal{U}, \mathcal{K}, s) : Y \subseteq \bigcup_{i=1}^{\infty} K_i, e^{-\ell(f_{1,\infty}, \mathcal{U}, K_i)} < \varepsilon\Big\}.$$

By letting $\varepsilon \to 0$, we get that

$$\mathcal{M}^s(f_{1,\infty}, Y, \mathcal{U}) \leq M^s(f_{1,\infty}, Y).$$

This apparently implies that

$$Bh(f_{1,\infty}, Y, \mathcal{U}) \leq ent_H(f_{1,\infty}, Y).$$

From this, together with the arbitrariness of the cover $\mathcal{U}$, we conclude that

$$Bh(f_{1,\infty}, Y) \leq ent_H(f_{1,\infty}, Y).$$

(2) We assume $Bh(f_{1,\infty}, Y) < \infty$ for convenience. For any $\varepsilon > 0$, since X satisfies the totally bounded property, we can take a finite open cover $\mathcal{U}$ of X consisting of r open balls $B(x_1, \varepsilon), \cdots, B(x_r, \varepsilon)$. For any $Z \subseteq X$, if $f_1^j(Z) \prec \mathcal{U}$ for $0 \leq j \leq \ell(f_{1,\infty}, \mathcal{U}, Z)$, then $f_1^j(Z)$ must be contained in some elements of $\mathcal{U}$, hence

$$|f_1^j(Z)| < 2\varepsilon \text{ for } 0 \leq j \leq \ell(f_{1,\infty}, \mathcal{U}, Z).$$

This implies that

$$\ell(f_{1,\infty}, \mathcal{U}, Z) \leq n(f_{1,\infty}, Z, 2\varepsilon).$$

Now for any $s > Bh(f_{1,\infty}, Y)$, by the definition of Bowen's topological entropy, we have

$$s > Bh(f_{1,\infty}, Y, \mathcal{U}) \text{ and } \mathcal{M}^s(f_{1,\infty}, Y, \mathcal{U}) = 0.$$

For any $\varepsilon' < \varepsilon$, if $\mathcal{Z} = \{Z_i\}_{i=1}^{\infty}$ covers Y and $B(f_{1,\infty}, \mathcal{U}, Z_i) < \varepsilon'$, then

$$n(f_{1,\infty}, Z, 2\varepsilon) \geq \ell(f_{1,\infty}, \mathcal{U}, Z) > -\log \varepsilon' > -\log \varepsilon,$$

and for any $s' > 0$,

$$\begin{aligned} B(f_{1,\infty}, \mathcal{U}, \mathcal{Z}, s') &= \sum_{i=1}^{\infty} (B(f_{1,\infty}, \mathcal{U}, Z_i))^{s'} \\ &= \sum_{i=1}^{\infty} e^{-s' \cdot \ell(f_{1,\infty}, \mathcal{U}, Z_i)} \\ &\geq \sum_{i=1}^{\infty} e^{-s' \cdot n(f_{1,\infty}, Z_i, 2\varepsilon)} \\ &= D_{2\varepsilon}(f_{1,\infty}, \mathcal{Z}, s'). \end{aligned}$$

This implies that

$$\inf\{B(f_{1,\infty}, \mathcal{U}, \mathcal{Z}, s) : B(f_{1,\infty}, \mathcal{U}, Z_i) < \varepsilon'\} \geq M_{2\varepsilon}^s(f_{1,\infty}, Y).$$

Letting $\varepsilon' \to 0$, we get that the left part is exactly $\mathcal{M}^s(f_{1,\infty}, Y, \mathcal{U})$. It follows that $0 \geq M_{2\varepsilon}^s(f_{1,\infty}, Y) \geq 0$. Hence

$$\lim_{\varepsilon\to 0} M^s_{2\varepsilon}(f_{1,\infty}, Y) = 0.$$

This implies that $s \geq ent_H(f_{1,\infty}, Y)$ and therefore we get

$$Bh(f_{1,\infty}, Y) \geq ent_H(f_{1,\infty}, Y).$$

(3) If X is compact, it naturally satisfies both the Lebesgue finite covering property and totally bounded property. This result follows from (1) and (2). $\square$

Theorem 3.10. *Let (X, d) be a compact metric space, and let $f_{1,\infty}$ be a sequence of continuous selfmaps of X. Then for any compact subset $K \subseteq X$,*

$$ent_H(f_{1,\infty}, K) \leq h(f_{1,\infty}, K).$$

Proof. Let $\mathcal{K}^\varepsilon_n = \{K_{n,i}\}_{i\geq 1}$ be a cover with $r_n(f_{1,\infty}, K, \varepsilon)$ members. Notice that $B_n(x, \varepsilon) = \{y \in X : d(f^j_1(x), f^j_1(y)) < \varepsilon, 0 \leq j \leq n-1\}$ and

$$D_{2\varepsilon}(f_{1,\infty}, K, s) = \sum_{i\geq 1} e^{-s\cdot n(f_{1,\infty}, K_{n,i}, 2\varepsilon)},$$

where $n(f_{1,\infty}, K_{n,i}, 2\varepsilon)$ is the largest number such that

$$diam(f^j_1(K_{n,i})) < 2\varepsilon \text{ for } 0 \leq j < n(f_{1,\infty}, K_{n,i}, 2\varepsilon).$$

Now for $x \in K$ and $y, z \in K_{n,i}$, then by the triangle inequality,

$$d(f^j_1(y), f^j_1(z)) < d(f^j_1(x), f^j_1(y)) + d(f^j_1(x), f^j_1(z)),$$

then we have $n(f_{1,\infty}, K_{n,i}, 2\varepsilon) \geq n$. It follows that

$$D_{2\varepsilon}(f_{1,\infty}, K, s) \leq r_n(f_{1,\infty}, K, \varepsilon) \cdot e^{-sn}.$$

Notice again that

$$r_n(f_{1,\infty}, K, \varepsilon) = \left[r_n(f_{1,\infty}, K, \varepsilon)^{\frac{1}{n}}\right]^n = \left[e^{\frac{1}{n}\log r_n(f_{1,\infty}, K, \varepsilon)}\right]^n.$$

Then

$$\begin{aligned} M^s_{2\varepsilon}(f_{1,\infty}, K) &\leq D_{2\varepsilon}(f_{1,\infty}, K, s) \leq \left[e^{-s+\frac{1}{n}\log r_n(f_{1,\infty}, K, \varepsilon)}\right]^n \\ &\leq \limsup_{n\to\infty} \left[e^{-s+\frac{1}{n}\log r_n(f_{1,\infty}, K, \varepsilon)}\right]^n. \end{aligned}$$

For $s > r(f_{1,\infty}, K, \varepsilon)$, we have $M^s_{2\varepsilon}(f_{1,\infty}, K) = 0$. Hence for $s > \lim_{\varepsilon\to 0} r(f_{1,\infty}, K, \varepsilon) = h(f_{1,\infty}, K)$, we have

$$M^s(f_{1,\infty}, K) = \lim_{\varepsilon\to 0} M^s_{2\varepsilon}(f_{1,\infty}, K) = 0.$$

This proves that $ent_H(f_{1,\infty}, K) \leq h(f_{1,\infty}, K)$. $\square$

Lately, Li [10], Ju and Yang [8] discussed the Pesin topological entropy on subsets for nonautonomous systems, we denote the Pesin entropy on subset Y of compact space X by $Ph(f_{1,\infty}, Y)$ here.

Lemma 3.11. *[10] Let $(X; f_{1,\infty})$ be a nonautonomous dynamical system on a compact metric space (X, d), then*

$$Bh(f_{1,\infty}, Y) = Ph(f_{1,\infty}, Y) \text{ for any } Y \subseteq X.$$

From this lemma, together with Theorem 3.9, we conclude that the distance entropy, Bowen entropy and Pesin entropy are actually equivalent in compact metric space.

Proposition 3.12. *Let (X, d) be a compact metric space, and let $f_{1,\infty}$ be a sequence of continuous selfmaps of X. Then*

$$ent_H(f_{1,\infty}) = Bh(f_{1,\infty}, X) = Ph(f_{1,\infty}, X) \leq h(f_{1,\infty}, X).$$

We would like to point out that, under the assumption that X is a compact space, Li [10] proposed a condition such that

$$ent_H(f_{1,\infty}) = Bh(f_{1,\infty}, X) = Ph(f_{1,\infty}, X) = h(f_{1,\infty}, X).$$

However, this condition is difficult to verify.

Let $f_i : X \to X$, $i \geq 1$, be continuous. Similar to [13], we call the nonautonomous dynamical system $(X; f_{1,\infty})$ to be *weakly mixing*, if for any nonempty open sets U and V and for any $N > 0$, there exists $k > N$ such that $f_1^k(U) \bigcap V \neq \emptyset$.

We recall that the classical topological entropy can also be defined by open covers [9]. Let $\mathcal{U}$ be a finite open cover of X. Note that $f_i : X \to X$ is continuous for each $i \geq 1$. Then for any $n \geq 1$ and for any $0 \leq j \leq n-1$, $f_1^{-j}(\mathcal{U})$ is also an open cover of X. Denote by

$$\mathcal{U}_1^n := \bigvee_{j=0}^{n-1} f_1^{-j}(\mathcal{U}) = \Big\{ \bigcap_{j=0}^{n-1} f_1^{-j}(A_j) : A_j \in \mathcal{U} \Big\},$$

and $\mathcal{N}(\mathcal{U})$ be the minimal possible cardinality of a subcover chosen from $\mathcal{U}$.

The *classical topological entropy of $f_{1,\infty}$ on the cover $\mathcal{U}$* is given by

$$h(f_{1,\infty}, \mathcal{U}) = \limsup_{n\to\infty} \frac{1}{n} \log \mathcal{N}(\mathcal{U}_1^n),$$

and the *classical topological entropy of $f_{1,\infty}$* is defined by

$$h(f_{1,\infty}, X) = \sup\{h(f_{1,\infty}, \mathcal{U}) : \mathcal{U} \text{ is an open cover of } X\}.$$

Theorem 3.13. *Let (X, d) be a compact metric space, and let $(X; f_{1,\infty})$ be weakly mixing. Then*

$$ent_H(f_{1,\infty}) = Bh(f_{1,\infty}, X) = Ph(f_{1,\infty}, X) = h(f_{1,\infty}, X).$$

Proof. To prove this theorem, by noting that Theorem 3.10 and Proposition 3.12, we need only to prove that

$$Bh(f_{1,\infty}, X) \geq h(f_{1,\infty}, X).$$

It is sufficient to that $h(f_{1,\infty}, X) \leq s$ for any $s > Bh(f_{1,\infty}, X)$. Indeed, let s be a number with $s > Bh(f_{1,\infty}, X)$. We know from paper [9] that

$$h(f_{1,\infty}, X) = \sup\{h(f_{1,\infty}, \mathcal{U}) : \mathcal{U} \text{ is finite open cover of } X\}.$$

Hence, we need only to prove that for any finite open cover $\mathcal{U}$ of X

$$h(f_{1,\infty}, \mathcal{U}) \leq s.$$

Now let $\mathcal{U}$ be an finite open cover of X. Note that for any $E \subseteq X$

$$f_1^{-n}(E) \prec X \quad \text{for all } n \in \mathbb{N}.$$

From this, it follows that

$$\ell(f_{1,\infty}, \mathcal{U}, E) = +\infty \quad \text{if } X \in \mathcal{U}.$$

This implies that for any $s > 0$

$$B(f_{1,\infty}, \mathcal{U}, \mathcal{K}, s) = 0.$$

Hence, without loss of generality, we assume that

$$(X \backslash V)^\circ \neq \emptyset \ \text{ for any } V \in \mathcal{U}.$$

For any $s > Bh(f_{1,\infty}, X, \mathcal{U})$, by the definition of $Bh(f_{1,\infty}, X, \mathcal{U})$, there exists a countable covering $\mathcal{K} = \{K_i\}_{i=1}^\infty$ of X so that

$$B(f_{1,\infty}, \mathcal{U}, \mathcal{K}, s) = \sum_{i=1}^\infty e^{-s \cdot \ell(f_{1,\infty}, \mathcal{U}, K_i)} \leq \frac{1}{4}.$$

Define

$$\triangle_1 = \{i : K_i \in \mathcal{K} \text{ and } \ell(f_{1,\infty}, \mathcal{U}, K_i) < \infty\},$$
$$\triangle_2 = \{i : K_i \in \mathcal{K} \text{ and } \ell(f_{1,\infty}, \mathcal{U}, K_i) = +\infty\}.$$

Then $\mathcal{K} = \{K_i : i \in \triangle_1 \bigcup \triangle_2\}$.

For any $i \in \triangle_1$, we may choose an open set E_i such that

$$K_i \subset E_i \quad \text{and} \quad \ell(f_{1,\infty}, \mathcal{U}, E_i) = \ell(f_{1,\infty}, \mathcal{U}, K_i).$$

And let

$$\hat{\mathcal{K}}_1 = \{E_i : i \in \triangle_1\}.$$

For any $i \in \triangle_2$, we have $\ell(f_{1,\infty}, \mathcal{U}, K_i) = +\infty$. This means that there exist open sets $U_{i_j} \in \mathcal{U}$, $j \in \mathbb{N}$, such that $f_1^j(K_i) \subseteq U_{i_j}$ for all $j \in \mathbb{N}$. In this case, we let

$$q_i = (i+2)\big([\frac{\log 2}{s}] + 1\big),$$

and define

$$E_i = \bigcap_{j=0}^{q_i} f_1^{-j}(U_{i_j}).$$

It is obvious that E_i is an open set and $K_i \subseteq \bigcap_{j=0}^{\infty} f_1^{-j}(U_{i_j}) \subseteq E_i$. Moreover, it follows that for any $0 \le \ell \le q_i$,

$$f_1^{\ell}(E_i) = f_1^{\ell}(\bigcap_{j=0}^{q_i} f_1^{-j}(U_{i_j})) \subseteq f_1^{\ell}(f_1^{-\ell}(U_{i_\ell})) = U_{i_\ell} \in \mathcal{U}.$$

From this, it follows that

$$\ell(f_{1,\infty}, \mathcal{U}, E_i) > q_i.$$

For any $V \in \mathcal{U}$, from the weakly mixingness of the nonautonomous dynamical system $(X; f_{1,\infty})$, we conclude that there exists a $k_V > q_i$ such that

$$f_1^{k_V}(E_i) \bigcap (X \backslash V) \neq \emptyset.$$

Let $k_i = \max\{k_V : V \in \mathcal{U}\}$. Note that $\#(\mathcal{U}) < \infty$. It follows that $k_i < \infty$. Hence

$$q_i \le \ell(f_{1,\infty}, \mathcal{U}, E_i) \le k_i < \infty.$$

Let

$$\hat{\mathcal{K}}_2 = \{E_i : i \in \triangle_2\}.$$

Define $\hat{\mathcal{K}} = \hat{\mathcal{K}}_1 \bigcup \hat{\mathcal{K}}_2$. We may assume, without loss of generality, that

$$\hat{\mathcal{K}} \prec \mathcal{U}.$$

Then $\hat{\mathcal{K}}$ is an open cover of X. And furthermore, it follows that

$$\begin{aligned} B(f_{1,\infty}, \mathcal{U}, \hat{\mathcal{K}}, s) &= B(f_{1,\infty}, \mathcal{U}, \hat{\mathcal{K}}_1, s) + B(f_{1,\infty}, \mathcal{U}, \hat{\mathcal{K}}_2, s) \\ &\le \frac{1}{4} + \sum_{E_i \in \hat{\mathcal{K}}_2} e^{-s \cdot \ell(f_{1,\infty}, \mathcal{U}, E_i)} \\ &\le \frac{1}{4} + \sum_{E_i \in \hat{\mathcal{K}}_2} e^{-s q_i} \\ &= \frac{1}{4} + \sum_{E_i \in \hat{\mathcal{K}}_2} e^{-s(i+2)([\frac{\log 2}{s}]+1)} \\ &\le \frac{1}{4} + \sum_{i=1}^{\infty} e^{-(i+2)\log 2} \\ &\le \frac{1}{4} + \frac{1}{2} < 1. \end{aligned} \tag{3.2}$$

As X is compact, the open cover $\hat{\mathcal{K}}$ contains a finite subcover

$$\mathcal{D} = \{D_1, D_2, \cdots, D_m\}.$$

Then, we have $\mathcal{D} \prec \mathcal{U}$. Let

$$M = \max_{1 \le i \le m} \ell(f_{1,\infty}, D_i, \mathcal{U}).$$

Then $M < \infty$.

For any $(j_1, \cdots, j_r) \in \{1, \cdots, m\}^r$, $r \in \mathbb{N}$, we let

$$\ell(f_{1,\infty}, \mathcal{U}, (D_{j_1}, \cdots, D_{j_s})) = \sum_{r=1}^{s} \ell(f_{1,\infty}, \mathcal{U}, D_{j_r}).$$

Define

$$\mathcal{C}(D_{j_1}, \cdots, D_{j_s}) = \big\{x \in X : f_1^{t_r+k}(x) \in D_{j_r} \text{ for all } 0 \le k < \ell(f_{1,\infty}, \mathcal{U}, D_{j_r}) \text{ and } 1 \le r \le s\big\},$$

where $t_r = \ell(f_{1,\infty}, \mathcal{U}, D_{j_1}) + \cdots + \ell(f_{1,\infty}, \mathcal{U}, D_{j_{r-1}})$ and $t_1 = 0$. From the continuity of f_n's, it follows that $\mathcal{C}(D_{j_1}, \cdots, D_{j_s})$ is open for any $(j_1, \cdots, j_s) \in \{1, \cdots, m\}^s$.

From (3.2), we conclude that

$$\Theta := \sum_{s=1}^{\infty} \sum_{j_1, \cdots, j_s} e^{-s \cdot \ell(f_{1,\infty}, \mathcal{U}, (D_{j_1}, \cdots, D_{j_s}))} = \sum_{k=1}^{\infty} B(f_{1,\infty}, \mathcal{U}, \hat{\mathcal{K}}, s)^k < \infty. \tag{3.3}$$

Note that $\mathcal{D} \prec \mathcal{U}$, and for any $1 \le j \le m$ there exists a $B_{i_j} \in \mathcal{U}$ such that

$$f_1^k(D_j) \subseteq B_{i_j} \quad \text{for all } 0 \le k < \ell(f_{1,\infty}, \mathcal{U}, D_j).$$

From this, we deduce that for any $n \le \ell(f_{1,\infty}, \mathcal{U}, (D_{j_1}, \cdots, D_{j_s}))$,

$$\begin{aligned} \mathcal{C}(D_{j_1}, \cdots, D_{j_s}) &= \bigcap_{r=0}^{s} \Big(\bigcap_{k=0}^{\ell(f_{1,\infty}, \mathcal{U}, D_{j_r})-1} f_1^{-(t_r+k)}(D_{j_r}) \Big) \\ &\subseteq \bigcap_{r=0}^{s} \Big(\bigcap_{k=0}^{\ell(f_{1,\infty}, \mathcal{U}, D_{j_r})-1} f_1^{-(t_r+k)}(B_{i_{j_r}}) \Big) \\ &\subseteq \bigcap_{i=0}^{n-1} f_1^{-i}(B_i) \quad \text{ for some } B_i \in \mathcal{U}. \end{aligned}$$

This implies that

$$\mathcal{C}(D_{j_1}, \cdots, D_{j_s}) \prec \bigvee_{i=0}^{n-1} f_1^{-i}(\mathcal{U}).$$

Remember that

$$M = \max_{1 \le i \le m} \ell(f_{1,\infty}, D_i, \mathcal{U}) < \infty.$$

From this, together with $X \subseteq \bigcup_{D \in \mathcal{D}}$ and $X \subseteq \bigcup_{B \in \mathcal{U}}$, we can deduce that

$$\nabla_n := \{\mathcal{C}(D_{j_1}, \cdots, D_{j_s}) : \ell(f_{1,\infty}, \mathcal{U}, (D_{j_1}, \cdots, D_{j_s})) \in [n, n+M), s \ge 1\}$$

forms a cover of X, and further

$$\nabla_n \prec \bigvee_{i=0}^{n-1} f_1^{-i}(\mathcal{U}).$$

From this, we conclude that

$$\mathcal{N}\Big(\bigvee_{j=0}^{n-1} f_1^{-j}(\mathcal{U})\Big) \leq \#\big(\nabla_n\big). \tag{3.4}$$

Let

$$A_n = \sum_{\mathcal{C}(D_{j_1},\cdots,D_{j_s})\in\nabla_n} e^{-s\cdot\ell(f_{1,\infty},\mathcal{U},(D_{j_1},\cdots,D_{j_s}))}.$$

Then

$$\#\big(\nabla_n\big)\cdot \min_{\mathcal{C}(D_{j_1},\cdots,D_{j_s})\in\nabla_n} e^{-s\cdot\ell(f_{1,\infty},\mathcal{U},(D_{j_1},\cdots,D_{j_s}))} \leq A_n. \tag{3.5}$$

Note that

$$e^{-s\cdot\ell(f_{1,\infty},\mathcal{U},(D_{j_1},\cdots,D_{j_s}))} \geq 0.$$

From this, together with (3.3), we deduce that

$$\Theta = \sum_{n=1}^{\infty} A_n < \infty.$$

This implies that there exists a $a_0 > 0$ such that

$$A_n \leq a_0 \quad \text{for all } n \in \mathbb{N}. \tag{3.6}$$

For any $(D_{j_1},\cdots,D_{j_s})$ with $\mathcal{C}(D_{j_1},\cdots,D_{j_s}) \in \nabla_n$, we have

$$n \geq \ell(f_{1,\infty},\mathcal{U},(D_{j_1},\cdots,D_{j_s})) - M.$$

It follows that

$$e^{-sn} \leq e^{sM-s\cdot\ell(f_{1,\infty},\mathcal{U},(D_{j_1},\cdots,D_{j_s}))}.$$

Hence

$$e^{-sn} \leq \min_{\mathcal{C}(D_{j_1},\cdots,D_{j_s})\in\nabla_n} e^{sM-s\cdot\ell(f_{1,\infty},\mathcal{U},(D_{j_1},\cdots,D_{j_s}))}.$$

From this, together with (3.4), (3.5) and (3.6), it follows that

$$\begin{aligned}\mathcal{N}\Big(\bigvee_{j=0}^{n-1} f_1^{-j}(\mathcal{U})\Big)\cdot e^{-sn} &\leq \#\big(\nabla_n\big)\cdot \min_{\mathcal{C}(D_{j_1},\cdots,D_{j_s})\in\nabla_n} e^{sM-s\cdot\ell(f_{1,\infty},\mathcal{U},(D_{j_1},\cdots,D_{j_s}))}\\ &\leq e^{Ms}A_n \leq a_0e^{Ms}.\end{aligned}$$

From this, we deduce that

$$h(f_{1,\infty}, \mathcal{U}) = \limsup_{n\to\infty} \frac{1}{n} \log \mathcal{N}\Big(\bigvee_{j=0}^{n-1} f_1^{-j}(\mathcal{U})\Big) \leq s.$$

From the above argument, we get the assertion. □

4. Applications

The next theorem shows that the Hausdorff dimension of any set Y of compact space X is governed by the distance entropy.

Theorem 4.1. *Let (X, d) be a compact metric space, and let $f_i : X \to X$ be expanding with skewness $\lambda > 1$ for any $i \geq 1$; namely*

$$d(f_i(x), f_i(y)) \geq \lambda d(x, y) \quad \textit{for any } x, y \in X \quad \textit{and } i \geq 1.$$

Then for any $Y \subseteq X$,

$$\dim_{\mathcal{H}}(Y) \log \lambda \leq ent_H(f_{1,\infty}, Y).$$

Proof. The proof is modified from [3]. Let Y be given. If $ent_H(f_{1,\infty}, Y) < \infty$, by the definition of distance entropy, we take some s such that $s \log \lambda > ent_H(f_{1,\infty}, Y)$, then $B(f_{1,\infty}, \mathcal{Z}, s \log \lambda, \varepsilon) < \epsilon$ for some arbitrary ε and ϵ, we have to show $\mathcal{H}^s(Y) = 0$.

For arbitrary $\varepsilon, \epsilon > 0$, if $Z \subseteq X$ and $n(f_{1,\infty}, Z, \varepsilon) > m$, then $f_1^k(Z) < \varepsilon$ for $k = 0, 1, \cdots, m$. By the fact that $d(f_i(x), f_i(y)) \geq \lambda d(x, y)$ for all $i \geq 1$, then

$$\lambda^m d(x, y) \leq d(f_1^m(x), f_1^m(y)) \leq \varepsilon \quad \text{for any } x, y \in Z.$$

Therefore $|Z| \leq \frac{\varepsilon}{\lambda^m}$ if $n(f_{1,\infty}, Z, \varepsilon) > m$ (we assume that $\epsilon < \frac{\varepsilon}{\lambda^m}$). This implies that

$$\lambda^{n(f_{1,\infty}, Z, \varepsilon) - 1} \leq \frac{\varepsilon}{|Z|}.$$

Hence,

$$n(f_{1,\infty}, Z, \varepsilon) \leq 1 + \frac{\log \varepsilon}{\log \lambda} - \frac{\log |Z|}{\log \lambda}.$$

Suppose $\mathcal{Z} = \{Z_i\}_{i=1}^{\infty}$ is a cover of Y satisfying

$$\sup_{i\geq 1} |Z_i| < \epsilon (< \frac{\varepsilon}{\lambda^m}).$$

By the definition of distance entropy, we have

$$\begin{aligned} D_\varepsilon(f_{1,\infty}, \mathcal{Z}, s \log \lambda) &= \sum_{i=1}^{\infty} e^{-s \log \lambda \cdot n(f_{1,\infty}, Z_i, \varepsilon)} \\ &\geq \sum_{i=1}^{\infty} e^{-s \log \lambda \left(1 + \frac{\log \varepsilon}{\log \lambda} - \frac{\log |Z_i|}{\log \lambda}\right)} \end{aligned}$$

$$= e^{-s(\log \lambda + \log \varepsilon)} \cdot \sum_{i=1}^{\infty} e^{s \log |Z_i|}$$

$$= e^{-s(\log \lambda + \log \varepsilon)} \cdot \sum_{i=1}^{\infty} |Z_i|^s.$$

Notice that the infinite summation corresponding to the Hausdorff measure, thus for this cover $\mathcal{Z}$, we are reasonable to require $D_\varepsilon(f_{1,\infty}, \mathcal{Z}, s \log \lambda) \cdot e^{s(\log \lambda + \log \varepsilon)} < \epsilon$. Let $\varepsilon \to 0$, we get that $\mathcal{H}^\delta(Y) = 0$, and then it follows that $\dim_{\mathcal{H}}(Y) \log \lambda \leq ent_H(f_{1,\infty}, Y)$. □

In the following Theorem 4.2, we consider a special subspace with some kind of uniformly Lipschitz constant. Let Y be a non-empty subset of the compact metric space (X, d). And suppose that there exists a Lipschitz constant $L_Y \geq 0$ such that for any $i \geq 1$, $j \geq 0$ and $x, y \in Y$,

$$d(f_i^{j+1}(x), f_i^{j+1}(y)) \leq L_Y d(f_i^j(x), f_i^j(y)). \tag{4.1}$$

Theorem 4.2. *Let (X, d) be a compact metric space, and let $f_{1,\infty}$ be a sequence of continuous selfmaps of X satisfying (4.1). Then for any $Y \subseteq X$,*

$$ent_H(f_{1,\infty}, Y) \leq \max\{0, \dim_{\mathcal{H}} Y \cdot \log L_Y\}.$$

Proof. This proof is based on [1,3], for the sake of completely, we add the detail here.

Let $Y \subseteq X$ satisfying (4.1). If $L_Y \leq 1$, there is nothing to do, thus, we may assume $L_Y > 1$ below.

By the definition of Hausdorff dimension of Y (2.1),

$$\mathcal{H}^s(Y) = \lim_{\delta \to 0} \mathcal{H}_\delta^s(Y) = \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } Y \right\}.$$

For any $s > \log L_Y \dim_H Y$, i.e., $s / \log L_Y > \dim_H Y$, we have

$$\mathcal{H}^{s/\log L_Y}(Y) = 0.$$

For any $\epsilon > 0$, we take $\log \varepsilon \geq \log \epsilon / (\log L_Y + 1)$, then we can construct a δ cover $\mathcal{Z}$ of Y, by taking $\mathcal{Z} = \{Z_i\}_{i=1}^{\infty}$ with $Z_i \subseteq Y$, such that

$$\frac{\varepsilon}{L_Y^{-\log \varepsilon}} \leq |Z_i| \leq \frac{\varepsilon}{L_Y^{-\log \varepsilon - 1}} \quad \text{for } i \geq 1. \tag{4.2}$$

This implies that

$$\sum_{i=1}^{\infty} |Z_i|^{\frac{s}{\log L_Y}} < \epsilon. \tag{4.3}$$

Notice that there always exists a $n \in \mathbb{N}$ such that $n \leq -\log \varepsilon < n + 1$. Then for this cover $\mathcal{Z}$, together with (4.2), we have

$$\frac{\varepsilon}{L_Y^{n+1}} \leq |Z_i| < \frac{\varepsilon}{L_Y^{n-1}},$$

and

$$\frac{\log \varepsilon - \log |Z_i|}{\log L_Y} \leq n < \frac{\log \varepsilon - \log |Z_i|}{\log L_Y} + 1.$$

From the first inequality and (4.1), it follows that

$$|f_1^k(Z_i)| < \varepsilon \ \text{ for } k = 0, 1, \cdots, n-1.$$

Hence

$$n(f_{1,\infty}, Z_i, \varepsilon) \geq n \geq \frac{\log \varepsilon - \log |Z_i|}{\log L_Y}.$$

Thus for the cover $\mathcal{Z} = \{Z_i\}_{i=1}^{\infty}$, we have $diam_\varepsilon(f_{1,\infty}, Z_i) = e^{-n(f_{1,\infty}, Z_i, \varepsilon)}$, and then

$$\begin{aligned} D_\varepsilon(f_{1,\infty}, \mathcal{Z}, s) &= \sum_{i=1}^{\infty} (diam_\varepsilon(f_{1,\infty}, Z_i))^s \\ &= \sum_{i=1}^{\infty} e^{-s \cdot n(f_{1,\infty}, Z_i, \varepsilon)} \\ &\leq \sum_{i=1}^{\infty} e^{-s \frac{\log \varepsilon - \log |Z_i|}{\log L_Y}} \\ &= e^{\frac{-s}{\log L_Y} \cdot \log \varepsilon} \sum_{i=1}^{\infty} e^{\frac{s}{\log L_Y} \cdot \log |Z_i|} \\ &= \varepsilon^{\frac{-s}{\log L_Y}} \sum_{i=1}^{\infty} |Z_i|^{\frac{s}{\log L_Y}}. \end{aligned}$$

Now, by (4.3), since ϵ and ε are arbitrary, then we take another arbitrary $\varepsilon' > 0$ such that $\varepsilon^{\frac{-s}{\log L_Y}} \epsilon < \varepsilon'$. Therefore, $D_\varepsilon(f_{1,\infty}, \mathcal{Z}, s) < \varepsilon'$. Hence, by letting $\varepsilon' \to 0$, we get that $M_\varepsilon^s(f_{1,\infty}, Y) = 0$, and therefore $ent_H(f_{1,\infty}, Y) \leq s$ for any $s > \dim_{\mathcal{H}} Y \log L_Y$. This implies that $ent_H(f_{1,\infty}, Y) \leq \dim_{\mathcal{H}} Y \log L_Y$. □

Corollary 4.3. *Under the assumption of Theorem 4.2, we have*

$$\frac{ent_H(f_{1,\infty})}{\log L_X} \leq \dim_{\mathcal{H}} X.$$

Remark 4.4. There are some basic inequalities between the Hausdorff dimension, Packing dimension and the lower and upper Box dimension, i.e., for any non-empty set $Z \subseteq X$,

$$\dim_{\mathcal{H}}(Z) \leq \dim_P(Z) \leq \overline{\dim}_B(Z), \quad \dim_{\mathcal{H}}(Z) \leq \underline{\dim}_B(Z) \leq \overline{\dim}_B(Z).$$

Note that the distance entropy, Bowen entropy and Pesin entropy are equivalent when X is compact by Corollary 3.12. It is free to discuss the relationship between Hausdorff dimension on subsets and other entropies. We would like to let readers to refer [8] for the discussion of the relationship between Box dimension and the Pesin entropy.

Let μ be the Borel probability measure on X. We define the local lower and upper μ-measure entropy at x with respect to $f_{1,\infty}$ as follows:

$$\underline{h}_\mu(f_{1,\infty}, x) = \lim_{r \to 0} \liminf_{n \to \infty} -\frac{1}{n} \log \mu(B_n(x, r));$$

$$\overline{h}_\mu(f_{1,\infty},x)=\lim_{r\to 0}\limsup_{n\to\infty}-\frac{1}{n}\log\mu(B_n(x,r)).$$

We say a set $A\subset X$ is *$f_{1,\infty}$-invariant* if $f_i^{-1}(A)=A$ for all $i=1,2,\cdots$. Let $\mu\in M(f_{1,\infty},X)$, the set of all f_i-invariant Borel probability measures on X. By [6], the set $M(f_{1,\infty},X)$ is not empty. Define the *measure-theoretical lower and upper entropies of μ* by

$$\underline{h}_\mu(f_{1,\infty})=\int\underline{h}_\mu(f_{1,\infty},x)d\mu(x)$$

and

$$\overline{h}_\mu(f_{1,\infty})=\int\overline{h}_\mu(f_{1,\infty},x)d\mu(x).$$

Let $\mu\in M(f_{1,\infty},X)$. Then $\dim_{\mathcal{H}}\mu$ defines as the infimum of the Hausdorff dimension of the sets of full measures, i.e.,

$$\dim_{\mathcal{H}}\mu=\inf\{\dim_{\mathcal{H}}(Z):\mu(Z)=1\}.$$

Lemma 4.5. *[17] Let $f_{1,\infty}$ be a sequence of continuous selfmaps on the compact space X and $\mu\in M(f_{1,\infty},X)$. If Y is a non-empty and compact subset of X with $\mu(Y)=1$, then*

$$Ph(f_{1,\infty},Y)\geq\underline{h}_\mu(f_{1,\infty}).$$

Theorem 4.6. *Let X be a compact space. If $\mu\in M(f_{1,\infty},X)$ and $f_{1,\infty}$ restricted to the support of μ is Lipschitz continuous with constant L_X (see (4.1)), then*

$$\dim_{\mathcal{H}}\mu\geq\frac{\underline{h}_\mu(f_{1,\infty})}{\log L_X}.$$

Proof. Let $Y\subseteq Z$ be a set of full measure, where Z is the support Z of μ. Since μ is invariant, Z is $f_{1,\infty}$-invariant, then by Theorem 4.2, we have

$$\dim_{\mathcal{H}}(Y)\geq\frac{ent_H(f_{1,\infty},Y)}{\log L_X}\quad\text{for } Y\subseteq X.$$

Then by noting Proposition 3.12 and Lemma 4.5, we need only to show $\dim_{\mathcal{H}}(Y)\geq\frac{\underline{h}_\mu(f_{1,\infty})}{\log L_X}$. Since $\dim_{\mathcal{H}}\mu=\inf\{\dim_{\mathcal{H}}Z:\mu(Z)=1\}$, it is enough to take the infimum over the sets Y that are contained in the support of μ. This implies that

$$\dim_{\mathcal{H}}\mu\geq\frac{\underline{h}_\mu(f_{1,\infty})}{\log L_X}.\quad\square$$

At the last of the paper, we would like to present a problem for the interesting readers as follows.

Problem: Can the weakly mixing assumption be removed in Theorem 3.13?

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