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Vector-valued Ruelle operator for weakly contractive IFS and Dini matrix potentials

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Abstract

The (scalar) Ruelle operator theory is well-known both in fractal geometry and dynamical systems. In this paper we consider vector-valued Ruelle operators for weakly contractive iterative function systems associated with Dini matrix potentials. We generalize the result in the paper (*J. Math. Anal. Appl.* **299** 341–56) to weakly contractive iterated function system. More exactly, our main theorem gives a sufficient condition for the vector-valued Ruelle operator with the Perron–Frobenius property.

Keywords: weakly contractive iterated function system, Dini potential, vector-valued Ruelle operator, Perron–Frobenius property

Mathematics Subject Classification numbers:
Primary 37C30, 37A30. Secondary 28A80.

1. Introduction

Throughout the paper we always let X be a nonempty compact subset of Euclidean space $(\mathbb{R}^d, |\cdot|)$. Let $1 < m \in \mathbb{N}$ and let $w_i : X \to X$ $(1 \leqslant i \leqslant m)$ be weakly contractive maps (see e.g. [6] or [24]). We call $(X, \{w_i\}_{i=1}^m)$ a weakly contractive iterated function system (IFS). We known from [6] that there exists a unique non-empty compact subset $E \subseteq X$ such that

$$E = \bigcup_{i=1}^{m} w_i(E),$$

which is called the *invariant set* of the IFS $(X, \{w_i\}_{i=1}^m)$. In the following we always assume that $X = E$ for simplicity. If we associate the IFS $(X, \{w_i\}_{i=1}^m)$ with a family of Dini continuous probability potentials $\{p_i\}_{i=1}^m$, we discuss the Perron–Frobenius property of the system

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$(X,\{w_i\}_{i=1}^m,\{p_i\}_{i=1}^m)$ [12]. Among the other results, we show that there exists a unique probability measure μ on E such that

$$\mu=\sum_{i=1}^m p_i(x)\mu\circ w_i^{-1},$$

provided that the system $(X,\{w_i\}_{i=1}^m,\{p_i\}_{i=1}^m)$ satisfies the following condition:

$$\sum_{i=1}^m p_i(x)\left(\sup_{\substack{y,z\in X\\ y\neq z}}\frac{|w_i(y)-w_i(z)|}{|y-z|}\right)<\sum_{i=1}^m p_i(x)=1.$$

And we call μ an *invariant measure* of the system $(X,\{w_i\}_{i=1}^m,\{p_i\}_{i=1}^m)$ [8, 12, 24]. It is one of the important topics to study the multifractal structure of measure μ. The Ruelle operator was introduced to study IFS's by Fan and Lau in [3] where contractive IFS with Dini potentials were considered. It is followed by many works in the literature (see e.g. [3–5, 9, 10, 14–17, 22] and the references there). Ruelle operator theory can be applied to the study of L^q-spectrum and multifractal structure of measure μ (see e.g. [3, 9, 21–23]). There are some studies focusing on vector-valued Ruelle operators. Leung [13] considered vector-valued subMarkov operators and recurrent IFS. The second author set up Ruelle operator theory for contractive vector-valued system [21]. The uniqueness and ergodicity of invariant measures for a Markov operator are discussed [20].

The main motivation of this paper includes the following four aspects. The first is that we have set up Ruelle operator theory for the weakly contractive IFS [12]. Can we extend the results in [12] to weakly contractive vector-valued system? That is, if we associate weakly contractive IFS $(X,\{w_i\}_{i=1}^m)$ with a family of non-negative $d\times d$ matrix potentials $\{A^{(i)}\}_{i=1}^m$ instead of positive scalar potentials $\{p_i\}_{i=1}^m$, what will happen? The second is that Leung [13] considered a $d\times d$ non-negative continuous matrix potential function $D(\cdot)=\big(d_{ij}(\cdot)\big)_{d\times d}$ and the operator $\mathcal{L}:C(X,\mathbb{R}^d)\to C(X,\mathbb{R}^d)$ defined by

$$(\mathcal{L}\mathbf{f})_i(x)=\sum_{j=1}^d d_{ij}(x)\mathbf{f}_j(w_j(x)),\quad 1\leqslant i\leqslant d.$$

He got an analogous result of Ruelle operator theorem. What will happen if we use a set of $d\times d$ continuous matrix potentials $\{A^{(i)}\}_{i=1}^m$ to replace a single matrix potential D? This is to say that we want to consider the weakly contractive IFS $(X,\{w_i\}_{i=1}^m)$ associated with $d\times d$ continuous matrix potentials $\{A^{(i)}\}_{i=1}^m$, and study the operator $\mathcal{T}:C(X,\mathbb{R}^d)\to C(X,\mathbb{R}^d)$ defined by

$$\mathcal{T}\mathbf{f}(x)=\sum_{i=1}^m A^{(i)}(x)\mathbf{f}(w_i(x)).\tag{1.1}$$

Does the analogous result of Ruelle operator theorem hold for $\mathcal{T}$? It is known that the separation property plays an important role in studying the multifractal structure of an IFS (see e.g. [3, 4, 9–11, 22] and the references there). It is difficult to study the multifractal structure for IFS with overlaps. Luckily enough, we find that, in some interesting cases, the invariant measures of a weakly contractive IFS with overlaps can be put into the vector forms, which are the vector-valued invariant measures of some weakly contractive vector-valued system without overlaps (see the following examples 3.11 and 3.12). From this, we see that there is a close relationship between weakly contractive IFS with overlaps and weakly contractive vector-valued system without overlaps. Hence, it is necessary to study vector-valued Ruelle operator. The fourth is that, in paper [21], we set up Ruelle operator theory for the operator $\mathcal{T}$ defined by contractive vector-valued system $(X,\{w_i\}_{i=1}^m,\{A^{(i)}\}_{i=1}^m)$ as in (1.1). And then in paper [22], by making use of the results in [21], we succeeded in studying multifractal structure of contractive IFS with overlaps. Recently we considered the multifractal analysis for one-dimensional weakly contractive IFS with non-overlapping [23]. However, there is few multifractal analysis works done for weakly contractive IFS with overlaps. Can we study multifractal structure of weakly contractive IFS with overlaps as we did in [21, 22]? To answer this question, we need first to generalize the work of paper [21] to weakly contractive IFS.

In this paper we consider a weakly contractive IFS $(X,\{w_i\}_{i=1}^m)$ and a set of $d\times d$ non-negative matrix potentials $\{A^{(i)}\}_{i=1}^m$. We always assume that the following two hypotheses are satisfied:

(H1) each coordinate function of $A^{(i)}$ is either positive Dini continuous or zero;
(H2) $\sum_{i=1}^m A^{(i)}$ is primitive.

The triple $(X,\{w_i\}_{i=1}^m,\{A^{(i)}\}_{i=1}^m)$ is called *weakly contractive vector-valued system*.

For any $\mathbf{x}=(x_1,x_2,\dots,x_d)^t\in\mathbb{R}^d$, define $|\mathbf{x}|=\max_{1\leqslant i\leqslant d}|x_i|$. Let $C(X,\mathbb{R}^d)$ denote the set of all continuous $\mathbb{R}^d$-valued functions on X. For any $\mathbf{f}=(f_1,f_2,\dots,f_d)^t\in C(X,\mathbb{R}^d)$, define $\|\mathbf{f}\|_\infty=\max_{x\in X}|\mathbf{f}(x)|$. It is easy to check that $C(X,\mathbb{R}^d)$ is a Banach space with the norm $\|\cdot\|_\infty$.

We can define a vector-valued Ruelle operator $T: C(X,\mathbb{R}^d)\to C(X,\mathbb{R}^d)$ by

$$T\mathbf{f}(x)=\sum_{i=1}^m A^{(i)}(x)\mathbf{f}(w_i(x)),\quad \mathbf{f}\in C(X,\mathbb{R}^d). \tag{1.2}$$

It is easy to know that the operator T is a bounded linear operator. We use $\varrho(T)$ to denote the spectral radius of the operate T, and we often denote it by ϱ for short if there is not confusion caused. Then we have

$$\varrho(T)=\lim_{n\to\infty}\|T^n\|^{\frac{1}{n}}. \tag{1.3}$$

The essential spectral radius $\varrho_{\mathrm{ess}}(T)$ of T is a well-known concept in functional analysis (see e.g. [1, 2]).

Let $M(X,\mathbb{R}^d)$ be the set of all regular Borel $\mathbb{R}^d$-valued measures on X. For any $\boldsymbol{\mu}=(\mu_1,\mu_2,\dots,\mu_d)^t\in M(X,\mathbb{R}^d)$, let $\boldsymbol{\mu}(\mathbf{f})=\sum_{i=1}^d\int f_i\,\mathrm{d}\mu_i$. We call $\boldsymbol{\mu}$ a probability measure if $\boldsymbol{\mu}(\mathbf{1})=1$, i.e. $\sum_{i=1}^d\mu_i(X)=1$. And we use T^* to denote the dual operator of T.

For any $1\leqslant i\leqslant m$, let

$$r_i=\sup_{\substack{x,y\in X\\ x\neq y}}\frac{|w_i(x)-w_i(y)|}{|x-y|}. \tag{1.4}$$

The following theorem is the main result of this paper, which is a special case of theorem 3.1.

Theorem Let $(X,\{w_i\}_{i=1}^m)$ be a weakly contractive IFS, and let matrix potentials $\{A^{(i)}\}_{i=1}^m$ satisfy the hypotheses (H1) and (H2). Assume the system $(X,\{w_i\}_{i=1}^m,\{A^{(i)}\}_{i=1}^m)$ satisfies the condition:

$$\left\|\sum_{i=1}^m r_iA^{(i)}(x)\mathbf{1}\right\|_\infty<\varrho. \tag{1.5}$$

Then there exists a unique vector-valued function $\mathbf{0}_{d\times1}<\mathbf{h}\in C(X,\mathbb{R}^d)$ and a unique vector-valued probability measure $\boldsymbol{\mu}\in M(X,\mathbb{R}^d)$ such that

(1) $T\mathbf{h}=\varrho\mathbf{h}$, $T^*\boldsymbol{\mu}=\varrho\boldsymbol{\mu}$ and $\boldsymbol{\mu}(\mathbf{h})=1$;
(2) for any $\mathbf{f}\in C(X,\mathbb{R}^d)$, $\lim_{n\to\infty}\|\varrho^{-n}T^n\mathbf{f}-\boldsymbol{\mu}(\mathbf{f})\mathbf{h}\|_\infty=0$.

The above theorem is a vector form generalization of the classical Ruelle operator theorem. It is known that the contractive IFS with Dini continuous potentials has the bounded distortion property (BDP) (see e.g. [3, 21]). By using the BDP, we can prove vector-valued Ruelle operator theorem [21]. However, the weakly contractive systems considered do not have the BDP in general. It creates difficulties for us to set up vector-valued Ruelle operator theorem. In paper [25] we consider weakly contractive IFS with Lipschitz continuous matrix potentials. We can regard vector-valued Ruelle operator T acting on the space $L(X,\mathbb{R}^d)$ of Lipschitz continuous vector-valued functions. Then we prove, by making use of Ionescu–Tulcea and Marinescu theorem, that the operator T is quasi-compact acting on $L(X,\mathbb{R}^d)$; and in this case, we have $\varrho_{\mathrm{ess}}(T) < \varrho(T)$. However when matrix potentials are Dini continuous, even if the assumption (1.5) is satisfied, the essential spectral radius $\varrho_{\mathrm{ess}}(T)$ and the spectral radius $\varrho(T)$ may be equal, which introduces difficulties in establishing the Ruelle operator theorem. To prove our main result (theorem 3.1), we set up proposition 3.3, which states that the sequence $\{\varrho^{-n}T^n\mathbf{1}\}_{n=1}^{\infty}$ is both uniformly bounded and equicontinuous. We can apply Arzela–Ascoli theorem to yield a vector-valued eigenfunction corresponding to the spectral radius $\varrho(T)$. By making use of the eigenfunction, we can define a 'normalized' vector-valued Ruelle operator, and show the main result of the paper.

We organize the paper as follow. In section 2 we present some notations and elementary facts about the weakly contractive vector-valued system. In section 3 we study the Perron–Frobenius property of the vector-valued Ruelle operator. Finally, we present two examples to illustrate the necessity to study the vector-valued Ruelle operators.

2. Preliminaries

For any $A=(a_{ij})_{m\times n}, B=(b_{ij})_{m\times n}\in\mathbb{R}^{m\times n}$, we use $A\geqslant B$ (or $A>B$) to mean that $a_{ij}\geqslant b_{ij}$ (or $a_{ij}>b_{ij}$) for any $1\leqslant i\leqslant m, 1\leqslant j\leqslant n$. And let $\mathbf{0}_{m\times n}=\big(0\big)_{m\times n}$, i.e.

$$\mathbf{0}_{m\times n}=(a_{ij})_{m\times n} \text{ with } a_{ij}=0 \text{ for all } 1\leqslant i\leqslant m, 1\leqslant j\leqslant n.$$

A non-negative $d\times d$ matrix B is called *primitive* if there exists a positive integer n such that $B^n>\mathbf{0}_{d\times d}$ [18].

Let $\big(Y_i,d_i(\cdot,\cdot)\big)$, $i=1,2$, be two metric spaces, and let $u:\big(Y_1,d_1(\cdot,\cdot)\big)\to\big(Y_2,d_2(\cdot,\cdot)\big)$ be a continuous map. The *modulus of continuity* of u is defined as

$$\alpha_u(t):=\sup_{\substack{x,y\in Y_1\\ d_1(x,y)\leqslant t}} d_2(u(x),u(y)) \text{ for all } t>0.$$

A map $w:X\to X$ is called *contraction* if there exists a constant $0<c<1$ such that

$$\alpha_w(t)\leqslant ct \text{ for all } t>0.$$

A map $w:X\to X$ is said to be *weakly contractive* if

$$\alpha_w(t)<t \text{ for all } t>0.$$

A function $p:X\to\mathbb{R}$ is said to be *Dini continuous* if there exists a $r>0$ such that

$$\int_0^r \frac{\alpha_p(t)}{t}\mathrm{d}t<\infty.$$

We would like to point out that Dini continuity is weaker than Hölder continuity.

Throughout the paper we always let $(X,\{w_i\}_{i=1}^m)$ be a weakly contractive IFS. Without loss of generality, we may assume that

$$|X|=\sup\{|x-y|:x,y\in X\}=1.$$

We always assume that the hypotheses (H1) and (H2) are satisfied. Let T be the vector-valued Ruelle operator defined as (1.2); and let ϱ be the spectral radius of T given by (1.3).

Let $\Sigma = \{1,2,\ldots,m\}^{\mathbb{N}}$ and $\Sigma_n = \{1,2,\ldots,m\}^n$. For any $I = i_1 i_2 \ldots i_n \in \Sigma_n$, we use $|I|(=n)$ to denote the length of I. Let

$$w_I(x) = w_{i_1} \circ w_{i_2} \circ \ldots \circ w_{i_n}(x).$$

Let

$$A_{w_I}(x) = A^{(i_n)}(x) A^{(i_{n-1})}(w_{i_n}(x)) \ldots A^{(i_1)}(w_{i_2} \circ w_{i_3} \circ \ldots \circ w_{i_n}(x)).$$

Denote $A_{w_I}(x) = (a_{jk}^{(I)}(x))_{d\times d}$, and then we have, in particular, $A^{(i)}(x) = \left(a_{jk}^{(i)}(x)\right)_{d\times d}$. From (1.2), we can conclude inductively that

$$T^n \mathbf{f}(x) = \sum_{|I|=n} A_{w_I}(x) \mathbf{f}(w_I(x)) \text{ for all } n \in \mathbb{N}.$$

Let

$$\alpha_0(t) = \max \alpha_{\log a_{jk}^{(i)}}(t) \text{ for any } t > 0, \tag{2.1}$$

where the maximum is taken over all entries $a_{jk}^{(i)}$ which are strictly positive. From (H1), it follows that

$$\int_0^1 \frac{\alpha_0(t)}{t} \mathrm{d}t < \infty. \tag{2.2}$$

Proposition 2.1. *Let $(X, \{w_i\}_{i=1}^m)$ be a weakly contractive IFS. Then*

$$\lim_{n\to\infty} \max_{|I|=n} |w_I(X)| = 0.$$

Proof. Let

$$\begin{cases} \text{(i)} \quad \gamma_n := \max\limits_{|I|=n} |w_I(X)| \text{ for all } n \in \mathbb{N}; \\ \text{(ii)} \quad \tau(t) := \max\limits_{1\leqslant i\leqslant m} \alpha_{w_i}(t) \text{ for all } t \geqslant 0. \end{cases} \tag{2.3}$$

Note that X is a compact subset of $\mathbb{R}^d$. From the weak contractiveness of $\{w_i\}_{i=1}^m$, we deduce that

$$\begin{cases} \text{(i)} \quad 0 \leqslant \gamma_{n+1} \leqslant \gamma_n \text{ for all } n \in \mathbb{N}; \\ \text{(ii)} \quad \tau(t) < t \text{ for all } t > 0; \\ \text{(iii)} \quad \tau(t) \text{ is right continuous on } [0,1]. \end{cases} \tag{2.4}$$

From (2.4)(i), we may let

$$a := \lim_{n\to\infty} \gamma_n \geqslant 0.$$

We claim that $a = 0$. Otherwise, suppose that $a > 0$, then we have

$$\gamma_n \geqslant a > 0.$$

From this, together with (2.3) and (2.4)(ii), we can deduce that

$$a \leqslant \gamma_{n+1} \leqslant \tau(\gamma_n) < \gamma_n \text{ for all } n \in \mathbb{N}.$$

From this, together with (2.4)(iii), we conclude that

$$0 < a \leqslant \lim_{n\to\infty} \tau(\gamma_n) = \tau(a) \leqslant \lim_{n\to\infty} \gamma_n = a.$$

This implies that $\tau(a) = a > 0$. This contradicts with (2.4)(ii). Hence the claim is proved. □

Lemma 2.2. *Let $\{A^{(i)}\}_{i=1}^m$ satisfy the hypotheses (H1) and (H2). Then*

$$T^n\mathbf{1}(x) > \mathbf{0}_{d\times 1} \text{ for all } n \in \mathbb{N} \text{ and } x \in X.$$

Proof. From (H2), it follows that there exists some $n_0 > 0$ such that

$$\sum_{|I|=n} A_{w_I}(x) > \mathbf{0}_{d\times d} \text{ for all } n \geqslant n_0. \tag{2.5}$$

Let $A^{(i)}(x) = \left(a_{jk}^{(i)}(x)\right)_{d\times d}$. Define $B^{(i)} = \left(b_{jk}^{(i)}\right)_{d\times d}$, where

$$b_{jk}^{(i)} = \min_{x\in X} a_{jk}^{(i)}(x) \text{ for all } 1 \leqslant i \leqslant m \text{ and } 1 \leqslant j,k \leqslant d.$$

From (H1), together with the compactness of X, it follows that

$$b_{jk}^{(i)} = 0 \text{ if and only if } a_{jk}^{(i)}(x) = 0 \text{ for all } x \in X.$$

From this, together with (2.5), we conclude that

$$\left(\sum_{i=1}^m B^{(i)}\right)^n > \mathbf{0}_{d\times d} \text{ for all } n \geqslant n_0.$$

Thus no row of $\sum_{i=1}^m B^{(i)}$ is identically zero vector. From this, it follows that for any $n > 0$

$$T^n\mathbf{1}(x) = \sum_{|I|=n} A_{w_I}(x)\mathbf{1} \geqslant \left(\sum_{i=1}^m B^{(i)}\right)^n \mathbf{1} > \mathbf{0}_{d\times 1}.$$

□

The following proposition 2.3 is an analogous result of [12, proposition 2.2(i)].

Proposition 2.3. *Let T be the operator defined as (1.2). Then we have*

$$\min_{x\in X} \varrho^{-n}|T^n\mathbf{1}(x)| \leqslant 1 \leqslant \max_{x\in X} \varrho^{-n}|T^n\mathbf{1}(x)| \text{ for all } n \in \mathbb{N}.$$

Proof. It can be proved similarly to the proof of [12, proposition 2.2(i)], and we omit it. □

3. Perron–Frobenius property

In this section, we will study the Perron–Frobenius property of the vector-valued Ruelle operator T defined as (1.2). For any $1 \leqslant i \leqslant m$, we let r_i be as (1.4). And for any $I = i_1 i_2 \dots i_n \in \Sigma_n$, let

$$r_I = r_{i_1} r_{i_2} \dots r_{i_n},$$

and define

$$R_I = \sup_{\substack{x,y\in X \\ x\neq y}} \frac{|w_I(x) - w_I(y)|}{|x-y|}.$$

It is the Lipschitz constant of w_I. It is obvious that

$$R_I \leqslant r_I \text{ for all } I \in \Sigma_n.$$

The following theorem 3.1 is the main result of the paper.

Theorem 3.1. *Let $(X, \{w_i\}_{i=1}^m)$ be a weakly contractive IFS, and let matrix potentials $\{A^{(i)}\}_{i=1}^m$ satisfy the hypotheses (H1) and (H2). If there exists k such that*

$$\left\| \sum_{|I|=k} R_I A_{w_I}(x)\mathbf{1} \right\|_\infty < \varrho^k, \tag{3.1}$$

then there exists a unique vector-valued function $\mathbf{0}_{d\times 1} < \boldsymbol{h} \in C(X, \mathbb{R}^d)$ and a unique vector-valued probability measure $\boldsymbol{\mu} \in M(X, \mathbb{R}^d)$ such that

(1) $T\boldsymbol{h} = \varrho\boldsymbol{h}$, $T^*\boldsymbol{\mu} = \varrho\boldsymbol{\mu}$ and $\boldsymbol{\mu}(\boldsymbol{h}) = 1$;
(2) for any $\boldsymbol{f} \in C(X, \mathbb{R}^d)$, $\lim_{n\to\infty} \|\varrho^{-n}T^n\boldsymbol{f} - \boldsymbol{\mu}(\boldsymbol{f})\boldsymbol{h}\|_\infty = 0$.

We say that the *vector-valued Ruelle operator theorem for T holds* if the assertion of theorem 3.1 holds. To prove theorem 3.1, we need some preparations.

Lemma 3.2. *If the vector-valued Ruelle operator theorem for T^q holds for some $q \geqslant 2$, then the vector-valued Ruelle operator theorem for T holds.*

Proof. The spectral radius for T^q is ϱ^q. Since the vector-valued Ruelle operator theorem for T^q holds, we have a unique vector-valued function $\mathbf{0}_{d\times 1} < \mathbf{h} \in C(X, \mathbb{R}^d)$ and a unique vector-valued probability measure $\boldsymbol{\mu} \in M(X, \mathbb{R}^d)$ such that

$$T^q\mathbf{h} = \varrho^q\mathbf{h}, \ \ (T^q)^*\boldsymbol{\mu} = \varrho^q\boldsymbol{\mu}, \ \ <\boldsymbol{\mu}, \mathbf{h}>= 1.$$

Moreover, for any $\mathbf{f} \in C(X, \mathbb{R}^d)$,

$$\lim_{n\to\infty} \|\left(\varrho^q\right)^{-n}\left(T^q\right)^n\mathbf{f} - \boldsymbol{\mu}(\mathbf{f})\mathbf{h}\|_\infty = 0.$$

It follows that for $\mathbf{f} = T\mathbf{h}$, we have

$$\lim_{n\to\infty} \varrho^{-nq}T^{nq}\left(T\mathbf{h}\right) =<\boldsymbol{\mu}, T\mathbf{h}>\mathbf{h} = \lim_{n\to\infty} T\left(\varrho^{-nq}T^{nq}\mathbf{h}\right) =<\boldsymbol{\mu}, \mathbf{h}> T\mathbf{h} = T\mathbf{h}.$$

This implies that

$$T\mathbf{h} =<\boldsymbol{\mu}, T\mathbf{h}>\mathbf{h}.$$

From this, we deduce that

$$T\mathbf{h} = \varrho\mathbf{h}.$$

Similarly, we have $T^*\boldsymbol{\mu} = \varrho\boldsymbol{\mu}$.

For the convergence, take any $\mathbf{f} \in C(X, \mathbb{R}^d)$, we have

$$\lim_{n\to\infty} \|\varrho^{-nq-j}T^{nq+j}\mathbf{f} - \varrho^{-j} <\boldsymbol{\mu}, T^j\mathbf{f}>\mathbf{h}\| = 0 \quad \text{for any } 0 \leqslant j < q.$$

Note that

$$<\boldsymbol{\mu}, T^j\mathbf{f}>=<(T^*)^j\boldsymbol{\mu}, \mathbf{f}>= \varrho^j <\boldsymbol{\mu}, \mathbf{f}> .$$

We get that

$$\lim_{n\to\infty} \|\varrho^{-n}T^n\mathbf{f}- <\boldsymbol{\mu}, \mathbf{f}>\mathbf{h}\| = 0.$$

Hence, the vector-valued Ruelle operator theorem for T holds. □

Proposition 3.3. *Let $(X,\{w_i\}_{i=1}^m)$ be a weakly contractive IFS, and let matrix potentials $\{A^{(i)}\}_{i=1}^m$ satisfy the hypotheses (H1) and (H2). If the system $(X,\{w_i\}_{i=1}^m,\{A^{(i)}\}_{i=1}^m)$ satisfies the condition:*

$$\left\|\sum_{i=1}^m r_i A^{(i)}(x)\mathbf{1}\right\|_\infty < \varrho, \tag{3.2}$$

then for any $\boldsymbol{f}\in C(X,\mathbb{R}^d)$, $\{\varrho^{-n}T^n\boldsymbol{f}\}_{n=1}^\infty$ is uniformly bounded and equicontinuous.

Proposition 3.3 plays an important role in studying the Perron–Frobenius property of the vector-valued Ruelle operator. To prove proposition 3.3, we need to set up the following lemma 3.4, proposition 3.5 and lemma 3.6 first.

For any $n\in\mathbb{N}$ and $I=i_1i_2\ldots i_n\in\Sigma_n$, we define

$$I|_k^l = i_{k+1}i_{k+2}\ldots i_l \text{ for all } 0\leqslant k<l\leqslant n.$$

We let for convenience $I|_k^k=\emptyset$ for all $0\leqslant k\leqslant n$. It is obvious that

$$A_{w_I}(x) = A_{w_{I|_k^n}}(x)A_{w_{I|_0^k}}\left(w_{I|_k^n}x\right) \text{ for all } 0\leqslant k\leqslant n.$$

Let α_0 be the function defined by (2.1). Note that for any given $\theta:\ 0<\theta<1$,

$$\sum_{k=0}^\infty \alpha_0(\theta^{k+1}) \leqslant \frac{1}{1-\theta}\sum_{k=0}^\infty\int_{\theta^{k+1}}^{\theta^k}\frac{\alpha_0(t)}{t}\mathrm{d}t = \frac{1}{1-\theta}\int_0^1\frac{\alpha_0(t)}{t}\mathrm{d}t.$$

From this, together with (2.2), it follows that

$$a := \sum_{k=0}^\infty \alpha_0(\theta^k) < \infty. \tag{3.3}$$

For any $0\leqslant t\leqslant 1$ we define

$$\phi(t) = \sum_{k=0}^\infty \alpha_0(\theta^k t).$$

We know from (3.3) that $\phi(t)$ is continuous and $0=\phi(0)\leqslant\phi(1)=a$. For any $0\leqslant t\leqslant 1$ and any $I\in\Sigma_n$, we define

$$\sigma_I(t) = \sum_{k=0}^n \alpha_0\left(\alpha_{w_{I|_k^n}}(t)\right).$$

By the definition of $A_{w_I}(\cdot)$, we have

$$A_{w_I}(x)\leqslant e^{\sigma_I(|x-y|)}A_{w_I}(y) \text{ for all } x,y\in X. \tag{3.4}$$

For any given $\theta:\ 0<\theta<1$, let

$$\begin{cases} P(n,k) = \{I\in\Sigma_n : k \text{is smallest with } r_{I|_k^n}\geqslant\theta^{n-k}\}, & 0\leqslant k<n,\\ P(n,n) = \{I\in\Sigma_n : r_{I|_k^n}<\theta^{n-k} \text{ for all } 0\leqslant k<n\}. \end{cases} \tag{3.5}$$

It is easy to see that

$$\Sigma_n = \bigcup_{k=0}^n P(n,k). \tag{3.6}$$

The system $(X,\{w_i\}_{i=1}^m,\{A^{(i)}\}_{i=1}^m)$ is said to have *the* BDP, if there exists a constant $C \geqslant 1$ such that

$$A_{w_I}(x) \leqslant CA_{w_I}(y) \ \text{ for all } I \in \bigcup_{n=1}^{\infty} \Sigma_n \text{ and } x,y \in X.$$

Although the system $(X,\{w_i\}_{i=1}^m,\{A^{(i)}\}_{i=1}^m)$ does not have BDP in general, we present a useful basic property of the system in the following lemma 3.4.

Lemma 3.4. *Let the system* $\left(X,\{w_i\}_{i=1}^m,\{A^{(i)}\}_{i=1}^m\right)$ *be as in proposition 3.3. Then for any* $I \in P(n,k)$

(i) $\sigma_I(t) \leqslant \phi(t) + (n-k)\alpha_0(t)$;
(ii) $\mathbf{0}_{d\times d} \leqslant A_{w_I}(x) \leqslant e^{\phi(|x-y|)}A_{w_{I|_k^n}}(x)A_{w_{I|_0^k}}(y)$ *for all* $x,\ y \in X$.

Proof. (i) For any $I = i_1 i_2 \ldots i_n \in P(n,k)$ and for any $0 \leqslant j < k$, we have

$$r_{I|_k^n} \geqslant \theta^{n-k} \quad \text{and} \quad r_{I|_j^n} = r_{I|_j^k} \cdot r_{I|_k^n} < \theta^{n-j}.$$

And then we have

$$R_{I|_j^k} \leqslant r_{I|_j^k} = \frac{r_{I|_j^k} \cdot r_{I|_k^n}}{r_{I|_k^n}} < \theta^{k-j}.$$

From this, it follows that for any $I = i_1 i_2 \ldots i_n \in P(n,k)$ and $t > 0$

$$\sup_{|x-y|\leqslant t} |w_{I|_j^k}(x) - w_{I|_j^k}(y)| \leqslant \theta^{k-j} t \text{ for any } 0 \leqslant j < k. \tag{3.7}$$

From this, together with the weak contractiveness of $\{w_i\}_{i=1}^m$, we get (i).

(ii) From (3.7), together with (2.1), we deduce that for any $0 \leqslant j < k$ and any $x,\ y \in X$

$$\begin{aligned}\mathbf{0}_{d\times d} &\leqslant A^{(i_j)}(w_{i_{j+1}} \circ w_{i_{j+2}} \circ \ldots \circ w_{i_k}(x)) \\ &\leqslant A^{(i_j)}(w_{i_{j+1}} \circ w_{i_{j+2}} \circ \ldots \circ w_{i_k}(y)) e^{\alpha_0(\theta^{k-j}|x-y|)}.\end{aligned}$$

It follows that for any $I \in P(n,k)$ and any $x,\ y \in X$,

$$\mathbf{0}_{d\times d} \leqslant A_{w_{I|_0^k}}(x) \leqslant A_{w_{I|_0^k}}(y) e^{\sum_{i=1}^k \alpha_0\left(\theta^{k-i}|x-y|\right)} \leqslant e^{\phi(|x-y|)} A_{w_{I|_0^k}}(y). \tag{3.8}$$

Recall that $A_{w_{I|_k^n}}(x) \geqslant \mathbf{0}_{d\times d}$. From this, we conclude that for any $I \in P(n,k)$ and any $x,\ y \in X$,

$$\begin{aligned}\mathbf{0}_{d\times d} \leqslant\ & A_{w_I}(x) = A_{w_{I|_k^n}}(x) A_{w_{I|_0^k}}(w_{I|_k^n} x) \\ \leqslant\ & e^{\phi(|x-y|)} A_{w_{I|_k^n}}(x) A_{w_{I|_0^k}}(y) \qquad (\text{by } (3.8)).\end{aligned}$$

□

Proposition 3.5. *Let the system* $(X,\{w_i\}_{i=1}^m,\{A^{(i)}\}_{i=1}^m)$ *be as in proposition 3.3. Then there exist constants* $\xi \geqslant \zeta > 0$ *such that*

$$\zeta \mathbf{1} \leqslant \varrho^{-n} T^n \mathbf{1}(x) \leqslant \xi \mathbf{1} \quad \textit{for all } n > 0 \textit{ and } x \in X. \tag{3.9}$$

Proof. We prove the existence of the upper bound of (3.9) first. Note that X is compact. From (3.2), we can find an $0 < \eta < 1$ such that

$$\sum_{i=1}^{m} r_i A^{(i)}(x)\mathbf{1} \leqslant \eta\varrho\mathbf{1} \text{ for all } x \in X.$$

Note that

$$\sum_{|J|=n+1} r_J A_{w_J}(x)\mathbf{1} = \sum_{j=1}^{m} r_j A^{(j)}(x)\left(\sum_{|I|=n} r_I A_{w_I}\big(w_j(x)\big)\mathbf{1}\right).$$

From this, we can prove, by induction, that for any $n > 0$

$$\sum_{|I|=n} r_I A_{w_I}(x)\mathbf{1} \leqslant (\eta\varrho)^n\mathbf{1} \text{ for all } x \in X. \tag{3.10}$$

Choose $\theta:\ 0 < \eta < \theta < 1$ and let $P(n,k)$ $(0 \leqslant k \leqslant n)$ be as in (3.5). Let $0 < \delta := \frac{\eta}{\theta} < 1$ and let $G = \mathbf{1}\cdot\mathbf{1}^t \in \mathbb{R}^{d\times d}$. We claim that there exists a constant $M > 0$ such that for any $n \in \mathbb{N}$

$$\varrho^{-n}\sum_{I\in P(n,k)} A_{w_I}(x) \leqslant M\delta^{n-k}G \ \text{ for all } 0 \leqslant k \leqslant n. \tag{3.11}$$

Indeed, for any $I \in P(n,0)$ we have $r_I \geqslant \theta^n > 0$. From this, together with (3.10), we conclude that

$$\sum_{I\in P(n,0)} \theta^n A_{w_I}(x)\mathbf{1} \leqslant \sum_{|I|=n} r_I A_{w_I}(x)\mathbf{1} \leqslant (\eta\varrho)^n\mathbf{1}.$$

From this, we deduce that

$$\mathbf{0}_{d\times d} \leqslant \varrho^{-n}\sum_{I\in P(n,0)} A_{w_I}(x) \leqslant \delta^n G \text{ for all } n > 0 \text{ and } x \in X. \tag{3.12}$$

By proposition 2.3, there exists some $x_k \in X$ such that

$$\mathbf{0}_{d\times d} \leqslant \varrho^{-k}\sum_{|I|=k} A_{w_I}(x_k) \leqslant G \text{ for all } k > 0. \tag{3.13}$$

Note that for any $I \in P(n,k)$, we have $I|_k^n \in P(n-k,0)$. It follows that

$$\begin{aligned}
\mathbf{0}_{d\times d} &\leqslant \varrho^{-n}\sum_{I\in P(n,k)} A_{w_I}(x)\\
&\leqslant \varrho^{-n}\sum_{I\in P(n,k)} e^a A_{w_{I|_k^n}}(x)A_{w_{I|_0^k}}(x_k) \qquad \big(\text{by lemma 3.4(ii)}\big)\\
&\leqslant e^a\left(\varrho^{-(n-k)}\sum_{I'\in P(n-k,0)} A_{w_{I'}}(x)\right)\left(\varrho^{-k}\sum_{|I''|=k} A_{w_{I''}}(x_k)\right)\\
&\leqslant e^a\delta^{n-k}G\cdot G \qquad \big(\text{by (3.12) and (3.13)}\big)\\
&= M\delta^{n-k}G \ \text{ for some } M > 0.
\end{aligned}$$

Hence the claim (3.11) is proved. It follows that

$$\begin{aligned}\varrho^{-n}\sum_{|I|=n}A_{w_I}(x) &= \sum_{k=0}^{n}\left(\varrho^{-n}\sum_{I\in P(n,k)}A_{w_I}(x)\right) \qquad (\text{by } (3.6))\\ &\leqslant \sum_{k=0}^{n}M\delta^{n-k}G \qquad \big(\text{by } (3.11)\big)\\ &\leqslant \left(M\sum_{k=0}^{\infty}\delta^k\right)G := \gamma G.\end{aligned}$$

Hence for any $n>0$ and for any $x\in X$, we have

$$\varrho^{-n}T^n\mathbf{1}(x)\leqslant \gamma d\mathbf{1} := \xi\mathbf{1}. \tag{3.14}$$

Now we try to prove the lower bound in (3.9). For this, we let

$$\sigma_I=\sum_{k=0}^{n}\alpha_0(|w_{I|_k^n}(X)|) \text{ for any } I\in\Sigma_n.$$

From this, together with lemma 3.4(i), we conclude that

$$\sigma_I=\sigma_I(1)\leqslant a+(n-k)\alpha_0(1) \text{ for all } I\in P(n,k). \tag{3.15}$$

By applying proposition 2.3 and (3.14), we get that for any $n>0$ there exists $y_n\in X$ such that

$$1\leqslant C_n := \varrho^{-n}\left|\sum_{|I|=n}A_{w_I}(y_n)\mathbf{1}\right|\leqslant \xi. \tag{3.16}$$

Note that $0<\delta<1$ and then $\sum_{k=0}^{\infty}k\cdot\delta^k<\infty$. It follows that

$$\begin{aligned}\varrho^{-n}\sum_{|I|=n}\sigma_I A_{w_I}(y_n) &= \varrho^{-n}\sum_{k=0}^{n}\sum_{I\in P(n,k)}\sigma_I A_{w_I}(y_n)\\ &\leqslant \left(M\sum_{k=0}^{n}\big(a+(n-k)\alpha_0(1)\big)\delta^{n-k}\right)G \qquad \big(\text{by } (3.11) \text{ and } (3.15)\big)\\ &\leqslant \gamma' G \quad \text{for some } \gamma'>0.\end{aligned}$$

Thus

$$\mathbf{0}_{d\times 1}\leqslant \varrho^{-n}\sum_{|I|=n}\sigma_I A_{w_I}(y_n)\mathbf{1}\leqslant \gamma' d\mathbf{1}. \tag{3.17}$$

From this, it follows that

$$\begin{aligned}\varrho^{-n}T^n\mathbf{1}(x) = \varrho^{-n}\sum_{|I|=n}A_{w_I}(x)\mathbf{1} &\geqslant \varrho^{-n}\sum_{|I|=n}e^{-\sigma_I}A_{w_I}(y_n)\mathbf{1}\\ &\geqslant \frac{\varrho^{-n}}{C_n}\sum_{|I|=n}e^{-\sigma_I}A_{w_I}(y_n)\mathbf{1} \quad (\text{by } (3.16)).\end{aligned} \tag{3.18}$$

Let $\vec{\mathbf{e}}_i$ be the d-dimensional row vector with that the ith coordinate is 1 and the rest are 0. From (3.16), for any $n > 0$ there exists $1 \leqslant i_n \leqslant d$ such that

$$C_n = \varrho^{-n}\vec{\mathbf{e}}_{i_n}\sum_{|I|=n} A_{w_I}(y_n)\mathbf{1}.$$

Then

$$\sum_{|I|=n}\frac{\varrho^{-n}}{C_n}\vec{\mathbf{e}}_{i_n}A_{w_I}(y_n)\mathbf{1} = 1.$$

Note that e^x is convex. From this, we conclude that

$$\begin{aligned}\varrho^{-n}\vec{\mathbf{e}}_{i_n}\cdot\left(\sum_{|I|=n}A_{w_I}(x)\mathbf{1}\right) &\geqslant \sum_{|I|=n}\left(\frac{\varrho^{-n}}{C_n}\vec{\mathbf{e}}_{i_n}A_{w_I}(y_n)\mathbf{1}\right)e^{-\sigma_I} \qquad (\text{by } (3.18))\\ &\geqslant \exp\left(-\frac{\varrho^{-n}}{C_n}\vec{\mathbf{e}}_{i_n}\sum_{|I|=n}\sigma_I A_{w_I}(y_n)\mathbf{1}\right)\\ &\geqslant \exp(-\gamma' d) \qquad (\text{by } (3.17)).\end{aligned} \tag{3.19}$$

Let n_0 and $B^{(i)} = (b_{jk}^{(i)})_{d\times d}$ be as in the proof of lemma 2.2. For any $I = i_1 i_2 \ldots i_n \in \Sigma_n$, let $B^{(I)} = B^{(i_n)}B^{(i_{n-1})}\ldots B^{(i_1)}$. By lemma 2.2, there exists a constant $b > 0$ such that

$$\varrho^{-n_0}\sum_{|I|=n_0} B^{(I)} \geqslant bG.$$

Note that $A^{(i)}(x) \geqslant B^{(i)} \geqslant \mathbf{0}_{d\times d}$. It follows that for any $n \geqslant n_0$

$$\begin{aligned}\varrho^{-n}\sum_{|I|=n}A_{w_I}(x)\mathbf{1} &\geqslant \left(\varrho^{-n_0}\sum_{|I'|=n_0}B^{(I')}\right)\left(\varrho^{-(n-n_0)}\sum_{|I''|=n-n_0}A_{w_{I''}}(w_{I'}(x))\right)\mathbf{1}\\ &\geqslant (b\exp(-\gamma' d))\mathbf{1} \qquad (\text{by } (3.19)).\end{aligned}$$

By applying lemma 2.2, for any $n < n_0$, we get a constant $b_n' > 0$ such that

$$\varrho^{-n}\sum_{|I|=n}A_{w_I}(x)\mathbf{1} \geqslant \varrho^{-n}\sum_{|I|=n}B^{(I)}\mathbf{1} \geqslant b_n'\mathbf{1} \text{ for all } x \in X. \tag{3.20}$$

Let $b' := \min_{1\leqslant n<n_0} b_n' > 0$. Then for any $n < n_0$,

$$\varrho^{-n}T^n\mathbf{1}(x) \geqslant b'\mathbf{1} > \mathbf{0}_{d\times 1} \text{ for any } x \in X.$$

Let $\zeta = \min\{b', b\exp(-\gamma' d)\} > 0$. From the above arguments, we deduce that

$$\varrho^{-n}T^n\mathbf{1}(x) \geqslant \zeta\mathbf{1} \text{ for all } n \in \mathbb{N} \text{ and } x \in X.$$

This completes the proof. □

Now we consider the equicontinuity of the sequence $\{\varrho^{-n}T^n\mathbf{1}\}_{n=1}^{\infty}$.

Lemma 3.6. *Under the assumption of proposition 3.3, there exists a continuous function ψ defined on $[0,1]$ such that $\psi(0) = 0$, and for any $n > 0$,*

$$\varrho^{-n}|T^n\mathbf{1}(x) - T^n\mathbf{1}(y)| \leqslant \psi(|x-y|) \textit{ for any } x, y \in X.$$

Proof. The notations are adopted from proposition 3.5. Define

$$\psi(t) = dM\sum_{k=0}^{\infty}\delta^k\left(e^{\phi(t)+k\alpha_0(t)}-1\right),\quad 0\leqslant t\leqslant 1.$$

Note that $\phi(0)=\alpha_0(0)=0$ and $0<\delta<1$. From this, together with the continuity of ϕ and α_0, we can deduce that $\psi(\cdot)$ is well-defined, and furthermore, ψ is continuous, and $\psi(0)=0$.

For any $x,y\in X$, let $t=|x-y|$. It follows that

$$\begin{aligned}
\varrho^{-n}|T^n\mathbf{1}(x)-T^n\mathbf{1}(y)| &= \varrho^{-n}\Big|\sum_{k=0}^{n}\sum_{I\in P(n,k)}\left(A_{w_I}(x)-A_{w_I}(y)\right)\mathbf{1}\Big| \qquad \big(\text{by } (3.6)\big)\\
&\leqslant \varrho^{-n}\Big|\sum_{k=0}^{n}\sum_{I\in P(n,k)}\left(e^{\sigma_I(t)}-1\right)\cdot A_{w_I}(y)\mathbf{1}\Big| \qquad \big(\text{by } (3.4)\big)\\
&\leqslant dM\sum_{k=0}^{n}\delta^{n-k}\left(e^{\phi(t)+(n-k)\alpha_0(t)}-1\right) \qquad \big(\text{by } (3.11)\text{ and lemma 3.4(i)}\big)\\
&\leqslant dM\sum_{k=0}^{\infty}\delta^{k}\left(e^{\phi(t)+k\alpha_0(t)}-1\right)=\psi(t).
\end{aligned}$$

This completes the proof. □

Proof of proposition 3.3. Let

$$C^+\left(X,\mathbb{R}^d\right)=\left\{\mathbf{f}\in C\left(X,\mathbb{R}^d\right):\ \mathbf{f}>\mathbf{0}_{d\times 1}\right\}.$$

Note that X is compact. For any $\mathbf{f}\in C^+\left(X,\mathbb{R}^d\right)$ there exist constants $d_2\geqslant d_1>0$ such that $d_1\mathbf{1}\leqslant\mathbf{f}\leqslant d_2\mathbf{1}$. From this, together with proposition 3.5, it follows that there exist constants $\xi\geqslant\zeta>0$ such that

$$\zeta d_1\mathbf{1}\leqslant\varrho^{-n}T^n\mathbf{f}(x)\leqslant\xi d_2\mathbf{1}\quad\text{for all } n>0 \text{ and } x\in X.$$

Hence for any $\mathbf{f}\in C^+(X,\mathbb{R}^d)$, $\{\varrho^{-n}T^n\mathbf{f}\}_{n=1}^{\infty}$ is uniformly bounded.

For any $\mathbf{f}\in C^+(X,\mathbb{R}^d)$ and $x,y\in X$, we have

$$\begin{aligned}
&|\varrho^{-n}T^n\mathbf{f}(y)-\varrho^{-n}T^n\mathbf{f}(x)|\\
&\quad\leqslant \varrho^{-n}\Big|\sum_{|I|=n}\left(A_{w_I}(y)-A_{w_I}(x)\right)\mathbf{f}(w_I(y))\Big|\\
&\qquad+\varrho^{-n}\Big|\sum_{|I|=n}A_{w_I}(x)\left(\mathbf{f}(w_I(y))-\mathbf{f}(w_I(x))\right)\Big|\\
&\quad\leqslant ||\mathbf{f}||_\infty\varrho^{-n}\Big|\sum_{|I|=n}\left(A_{w_I}(y)-A_{w_I}(x)\right)\mathbf{1}\Big|\\
&\qquad+\left(\Big|\varrho^{-n}\sum_{|I|=n}A_{w_I}(x)\mathbf{1}\Big|\right)\left(\max_{|I|=n}\left|\mathbf{f}(w_I(y))-\mathbf{f}(w_I(x))\right|\right)\\
&\quad\leqslant ||\mathbf{f}||_\infty\psi(|y-x|)+\xi\cdot\max_{|I|=n}\left|\mathbf{f}(w_I(y))-\mathbf{f}(w_I(x))\right| \quad \big(\text{by lemma 3.6}\big).
\end{aligned}$$

By lemma 3.6, ψ is continuous on $[0,1]$, and $\psi(0)=0$. This, together with the continuity of $\mathbf{f}$ and the weak contractivity of $\{w_i\}_{i=1}^m$, implies that for any $\mathbf{f}\in C^+\left(X,\mathbb{R}^d\right)$, $\{\varrho^{-n}T^n\mathbf{f}\}_{n=1}^{\infty}$ is equicontinuous.

For any $\mathbf{f} \in C\left(X, \mathbb{R}^d\right)$, we can choose $\mathbf{a} > \mathbf{0}_{d\times 1}$ such that $\mathbf{f} + \mathbf{a} > \mathbf{0}_{d\times 1}$. Then both $\{\varrho^{-n}T^n(\mathbf{f}+\mathbf{a})\}_{n=1}^{\infty}$ and $\{\varrho^{-n}T^n\mathbf{a}\}_{n=1}^{\infty}$ are uniformly bounded and equicontinuous subsets of $C^+\left(X, \mathbb{R}^d\right)$. Hence for any $\mathbf{f} \in C\left(X, \mathbb{R}^d\right)$, $\{\varrho^{-n}T^n\mathbf{f}\}_{n=1}^{\infty}$ is uniformly bounded and equicontinuous. □

Now we are able to prove the existence of the eigen-function of the vector-valued Ruelle operator, which is important for us to study the Perron–Frobenius property of the vector-valued Ruelle operator.

Proposition 3.7. *Let the system $\left(X, \{w_i\}_{i=1}^m, \{A^{(i)}\}_{i=1}^m\right)$ be as in proposition 3.3. Then there exists a $\mathbf{0}_{d\times 1} < \boldsymbol{h} \in C\left(X, \mathbb{R}^d\right)$ such that $T\boldsymbol{h} = \varrho\boldsymbol{h}$.*

Proof. Let

$$\mathbf{f}_n(x) = \frac{1}{n}\sum_{i=0}^{n-1}\varrho^{-i}T^i\mathbf{1}(x),\ n \in \mathbb{N}.$$

From proposition 3.3, we deduce that the sequence $\{\mathbf{f}_n\}_{n=1}^{\infty}$ is uniformly bounded and equicontinuous. From this, we conclude, by applying Arzela–Ascoli theorem, that there exists a $\mathbf{0}_{d\times 1} < \mathbf{h} \in C\left(X, \mathbb{R}^d\right)$ and a subsequence $\{\mathbf{f}_{n_i}\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty}||\mathbf{f}_{n_i} - \mathbf{h}||_\infty = 0$. Then

$$\begin{aligned}
||T\mathbf{h} - \varrho\mathbf{h}||_\infty &= \lim_{i\to\infty}||T\mathbf{f}_{n_i} - \varrho\mathbf{f}_{n_i}||_\infty \\
&= \lim_{i\to\infty}\frac{\varrho}{n_i}||\varrho^{-n_i}T^{n_i}\mathbf{1} - \mathbf{1}||_\infty \\
&\leqslant \lim_{i\to\infty}\frac{\varrho(\xi+1)}{n_i} = 0.
\end{aligned}$$

This implies that $||T\mathbf{h} - \varrho\mathbf{h}||_\infty = 0$, and then $T\mathbf{h} = \varrho\mathbf{h}$. □

Let $\mathbf{0}_{d\times 1} < \mathbf{h} \in C\left(X, \mathbb{R}^d\right)$ be the eigen-function determined by proposition 3.7, i.e. $T\mathbf{h} = \varrho\mathbf{h}$. We denote $\mathbf{h} = (h_1, h_2, \ldots, h_d)^t$. For any $1 \leqslant i \leqslant m$, let

$$q_{jk}^{(i)}(x) = \frac{h_k(w_i(x))}{\varrho h_j(x)}a_{jk}^{(i)}(x),\ \text{for all } 1 \leqslant j,k \leqslant d,$$

and define

$$Q^{(i)}(x) = \left(q_{jk}^{(i)}(x)\right)_{d\times d}. \tag{3.21}$$

For any $I = i_1 i_2 \ldots i_n \in \Sigma_n$, we define

$$Q_{w_I}(x) = Q^{(i_n)}(x)Q^{(i_{n-1})}(w_{i_n}(x))\ldots Q^{(i_1)}(w_{i_2}\circ\ldots\circ w_{i_n}(x)).$$

Denote $Q_{w_I}(x) = (q_{jk}^{(I)}(x))_{d\times d}$. From $T\mathbf{h} = \varrho\mathbf{h}$, together with the definition of $Q^{(i)}$'s, we conclude that

$$\sum_{|I|=n}Q_{w_I}(x)\mathbf{1} = \mathbf{1}\ \text{ for all } n \in \mathbb{N}. \tag{3.22}$$

Define a 'normalized' operator $L : C\left(X, \mathbb{R}^d\right) \to C\left(X, \mathbb{R}^d\right)$ by

$$L\mathbf{f}(x) = \sum_{i=1}^m Q^{(i)}(x)\mathbf{f}\big(w_i(x)\big). \tag{3.23}$$

Let

$$H(x)=\begin{pmatrix} h_1(x) & 0 & \dots & 0\\ 0 & h_2(x) & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & h_d(x)\end{pmatrix}.$$

Then H is invertible, i.e. $H\cdot H^{-1}=H^{-1}\cdot H=\mathbb{E}$, where $\mathbb{E}$ is identity matrix. It can be checked that

$$\varrho^{-1}T(H\mathbf{f})(x)=H(x)L(\mathbf{f})(x).$$

And by induction, we have

$$\varrho^{-n}T^n(H\mathbf{f})(x)=H(x)L^n(\mathbf{f})(x)\ \text{ for all } n\in\mathbb{N}. \tag{3.24}$$

Proposition 3.8. *Let the system $(X,\{w_i\}_{i=1}^m,\{A^{(i)}\}_{i=1}^m)$ be as in proposition 3.3, and let $Q^{(i)}$ $(1\leqslant i\leqslant m)$ be defined as (3.21). Then $\{Q^{(i)}\}_{i=1}^m$ satisfy the following two conditions:*

(i) each coordinate function of $Q^{(i)}$ is either positive continuous or zero;
(ii) $\sum_{i=1}^m Q^{(i)}$ is primitive.

Proof. From (3.21), together with (H1) and $\mathbf{0}_{d\times 1}<\mathbf{h}\in C\left(X,\mathbb{R}^d\right)$, it follows that for any $1\leqslant i\leqslant m$ and $1\leqslant j,k\leqslant d$, $q_{jk}^{(i)}(\cdot)$ is either positive continuous or zero on X.

We let

$$b=\min_{1\leqslant i\leqslant m}\min_{1\leqslant j,k\leqslant d}\min_{x\in X}\frac{h_k(w_i(x))}{h_j(x)}\quad\text{and}\quad c=\max_{1\leqslant i\leqslant m}\max_{1\leqslant j,k\leqslant d}\max_{x\in X}\frac{h_k(w_i(x))}{h_j(x)}.$$

From the compactness of X, together with $\mathbf{0}_{d\times 1}<\mathbf{h}\in C\left(X,\mathbb{R}^d\right)$, we deduce that $0<b\leqslant c<\infty$. And from (3.21), it follows that for any $1\leqslant j,k\leqslant d$ and $I\in\Sigma_n$

$$\varrho^{-n}b\cdot a_{jk}^{(I)}(x)\leqslant q_{jk}^{(I)}(x)\leqslant\varrho^{-n}c\cdot a_{jk}^{(I)}(x).$$

This implies that

$$\varrho^{-n}bA_{w_I}(x)\leqslant Q_{w_I}(x)\leqslant\varrho^{-n}cA_{w_I}(x)\text{ for all } I\in\Sigma_n.$$

From this, together with (H2), we conclude that there exists n_0 such that

$$\sum_{|I|=n}Q_{w_I}(x)\geqslant\varrho^{-n}b\sum_{|I|=n}A_{w_I}(x)>\mathbf{0}_{d\times d}\text{ for all } n\geqslant n_0. \tag{3.25}$$

This implies that $\sum_{i=1}^m Q^{(i)}(x)$ is primitive. □

Proposition 3.9. *Let the system $(X,\{w_i\}_{i=1}^m,\{A^{(i)}\}_{i=1}^m)$ be as in proposition 3.3, and let L be the operator defined as (3.23). Then there exists a unique vector-valued invariant probability measure $\boldsymbol{\nu}$ such that*

$$\lim_{n\to\infty}||L^n\boldsymbol{f}-\boldsymbol{\nu}(\boldsymbol{f})\cdot\mathbf{1}||_\infty=0\,\textit{for any}\,\boldsymbol{f}\in C\left(X,\mathbb{R}^d\right).$$

Proof. For any $\mathbf{f}\in C(X,\mathbb{R}^d)$, we have $H\mathbf{f}\in C(X,\mathbb{R}^d)$. From proposition 3.3, we know that $\{\varrho^{-n}T^n(H\mathbf{f})\}_{n=1}^\infty$ is uniformly bounded and equicontinuous. From this, together with $\mathbf{0}_{d\times 1}<\mathbf{h}\in C(X,\mathbb{R}^d)$, we deduce that $\{\varrho^{-n}H^{-1}T^n(H\mathbf{f})\}_{n=1}^\infty$ is uniformly bounded and equicontinuous. This, combined with (3.24), implies that for any $\mathbf{f}\in C(X,\mathbb{R}^d)$, the sequence $\{L^n\mathbf{f}\}_{n=1}^\infty$ is uniformly bounded and equicontinuous. From this, we can get, by applying Arzela–Ascoli theorem, a $\tilde{\mathbf{f}} \in C(X,\mathbb{R}^d)$ and a subsequence $\{L^{n_i}\mathbf{f}\}_{i=1}^{\infty}$ such that

$$\lim_{i\to\infty} \|L^{n_i}\mathbf{f} - \tilde{\mathbf{f}}\|_\infty = 0. \tag{3.26}$$

We claim that for any $\mathbf{f} \in C(X,\mathbb{R}^d)$ there exists a constant $b_{\mathbf{f}}$ such that $\tilde{\mathbf{f}} = b_{\mathbf{f}} \cdot \mathbf{1}$. Indeed, for any $\mathbf{f} = (f_1,f_2,\ldots,f_d)^t \in C(X,\mathbb{R}^d)$, let

$$\lambda_0(f_k) = \max_{x\in X} f_k(x), \text{ and } \lambda(\mathbf{f}) = \max_{1\leqslant k\leqslant d} \lambda_0(f_k).$$

From (3.22), it follows that for any $\mathbf{f} \in C(X,\mathbb{R}^d)$, $\{\lambda(L^n\mathbf{f})\}_{n=1}^{\infty}$ is a decreasing sequence, i.e.

$$\lambda(L^{n+1}\mathbf{f}) \leqslant \lambda(L^n\mathbf{f}) \text{ for all } n > 0. \tag{3.27}$$

From this, together with (3.26), we deduce that for any $k \in \mathbb{N}$,

$$\begin{cases} \text{(i)} \quad \lambda(L^k\tilde{\mathbf{f}}) \leqslant \lambda(\tilde{\mathbf{f}}); \\ \text{(ii)} \quad \lambda(\tilde{\mathbf{f}}) \leqslant \lambda(L^k\mathbf{f}). \end{cases} \tag{3.28}$$

Note that the operator L is continuous. And from (3.26), we conclude that for any $k \in \mathbb{N}$

$$\lim_{i\to\infty} \|L^{k+n_i}\mathbf{f} - L^k\tilde{\mathbf{f}}\|_\infty = \lim_{i\to\infty} \|L^k\big(L^{n_i}\mathbf{f} - \tilde{\mathbf{f}}\big)\|_\infty = 0.$$

From this, together with (3.28)(ii), we deduce that

$$\lambda(\tilde{\mathbf{f}}) \leqslant \lambda(L^k\tilde{\mathbf{f}}) \text{ for all } k \in \mathbb{N}.$$

This, combined with (3.28)(i), implies that

$$\lambda(\tilde{\mathbf{f}}) = \lambda(L^k\tilde{\mathbf{f}}) \text{ for all } k \in \mathbb{N}. \tag{3.29}$$

We denote $L^n\tilde{\mathbf{f}} = \big((L^n\tilde{\mathbf{f}})_1,(L^n\tilde{\mathbf{f}})_2,\ldots,(L^n\tilde{\mathbf{f}})_d\big)^t$. Note that X is compact, and $\tilde{\mathbf{f}}$ is continuous. From this, it follows that for any $n \in \mathbb{N}$ there exists a $1 \leqslant j_n \leqslant d$ and a $x_{n,j_n} \in X$ such that

$$\lambda(\tilde{\mathbf{f}}) = \lambda(L^n\tilde{\mathbf{f}}) = \lambda_0\big((L^n\tilde{\mathbf{f}})_{j_n}\big) = (L^n\tilde{\mathbf{f}})_{j_n}(x_{n,j_n}). \tag{3.30}$$

Note that $Q_{w_I}(x) = \big(q_{jk}^{(I)}(x)\big)_{d\times d}$ and

$$L^n\tilde{\mathbf{f}}(x) = \sum_{|I|=n} Q_{w_I}(x)\tilde{\mathbf{f}}\big(w_I(x)\big). \tag{3.31}$$

From this, together with (3.30), it follows that

$$\lambda(\tilde{\mathbf{f}}) = \lambda(L^n\tilde{\mathbf{f}}) = \sum_{|I|=n}\sum_{k=1}^{d} q_{j_nk}^{(I)}(x_{n,j_n})\tilde{f}_k\big(w_I(x_{n,j_n})\big).$$

From this, together with (3.22), we deduce that for any $1 \leqslant k \leqslant d$ and $I \in \Sigma_n$

$$\lambda(\tilde{\mathbf{f}}) = \tilde{f}_k\big(w_I(x_{n,j_n})\big) \text{ if } q_{j_nk}^{(I)}(x_{n,j_n}) > 0.$$

By proposition 3.8(ii), for any $1 \leqslant j,k \leqslant d$ and $n \geqslant n_0$ there exists an $I_{jk}^{(n)} \in \Sigma_n$ such that

$$q_{jk}^{\left(I_{jk}^{(n)}\right)}(x) > 0 \text{ for all } x \in X. \tag{3.32}$$

Hence for any $1 \leqslant k \leqslant d$ and $n \geqslant n_0$ there exists an $I^{(n)}_{j_n k} \in \Sigma_n$ such that

$$\lambda(\tilde{\mathbf{f}}) = \lambda_0(\tilde{f}_k) = \tilde{f}_k\left(w_{I^{(n)}_{j_n k}}(x_{n,j_n})\right).$$

Note that

$$L^{\ell+n}\tilde{\mathbf{f}} = L^{\ell}(L^n\tilde{\mathbf{f}}).$$

From this, together with (3.29), we deduce similarly further that for any $1 \leqslant k \leqslant d$ and any $n \geqslant 1$

$$\lambda_0(\tilde{f}_k) = \lambda(\tilde{\mathbf{f}}) = \lambda(L^n\tilde{\mathbf{f}}) = \lambda_0\left((L^n\tilde{\mathbf{f}})_k\right). \tag{3.33}$$

This implies that we may choose any one $1 \leqslant k \leqslant d$ to replace j_n in (3.30).

For any $\mathbf{f} = (f_1, f_2, \ldots, f_d)^t \in C\left(X, \mathbb{R}^d\right)$, let

$$\tau_0(f_k) = \min_{x\in X} f_k(x), \text{and } \tau(\mathbf{f}) = \min_{1\leqslant k\leqslant d} \tau_0(f_k).$$

Similar to the proof of (3.33), we can prove that for any $1 \leqslant k \leqslant d$ and any $n \geqslant 1$

$$\tau_0(\tilde{f}_k) = \tau(\tilde{\mathbf{f}}) = \tau(L^n\tilde{\mathbf{f}}) = \tau_0\left((L^n\tilde{\mathbf{f}})_k\right).$$

From this, together with (3.33), we deduce that, by applying (3.31) and (3.32), for any $1 \leqslant k \leqslant d$ and $n \geqslant n_0$ there exist $x^{(k)}_n, y^{(k)}_n \in X$ and $I^{(k)}_n \in \Sigma_n$ such that

$$\begin{cases} \lambda(\tilde{\mathbf{f}}) = \lambda_0(\tilde{f}_k) = \tilde{f}_k\left(w_{I^{(k)}_n}(x^{(k)}_n)\right); \\ \tau(\tilde{\mathbf{f}}) = \tau_0(\tilde{f}_k) = \tilde{f}_k\left(w_{I^{(k)}_n}(y^{(k)}_n)\right). \end{cases} \tag{3.34}$$

Let x_0 be a fixed point in X. Then for any $1 \leqslant k \leqslant d$,

$$w_{I^{(k)}_n}(x_0) \in X \quad \text{for all } n \in \mathbb{N}.$$

This implies that the sequence $\{w_{I^{(k)}_n}(x_0)\}_{n=n_0}^{\infty}$ contains a convergent subsequence $\{w_{I^{(k)}_{n_\ell}}(x_0)\}_{\ell=1}^{\infty}$, and we let

$$z_k = \lim_{\ell\to\infty} w_{I^{(k)}_{n_\ell}}(x_0).$$

Then $z_k \in X$. Let

$$a_n = \max_{|I|=n} |w_I(X)| \quad \text{for all } n \in \mathbb{N}.$$

By proposition 2.1, we have

$$\lim_{n\to\infty} a_n = 0.$$

From this, we conclude that

$$\lim_{\ell\to\infty} w_{I^{(k)}_{n_\ell}}(x^{(k)}_{n_\ell}) = \lim_{\ell\to\infty} w_{I^{(k)}_{n_\ell}}(y^{(k)}_{n_\ell}) = \lim_{\ell\to\infty} w_{I^{(k)}_{n_\ell}}(x_0) = z_k \in X.$$

From this, together with (3.34), we deduce that for any $1 \leqslant k \leqslant d$

$$\begin{aligned}\lambda(\tilde{\mathbf{f}}) &= \lambda_0(\tilde{f}_k) = \lim_{\ell\to\infty} \tilde{f}_k\left(w_{I^{(k)}_{n_\ell}}(x^{(k)}_{n_\ell})\right) = \tilde{f}_k(z_k)\\ &= \lim_{\ell\to\infty} \tilde{f}_k\left(w_{I^{(k)}_{n_\ell}}(y^{(k)}_{n_\ell})\right) = \tau_0(\tilde{f}_k) = \tau(\tilde{\mathbf{f}}).\end{aligned}$$

This implies that

$$\tilde{\mathbf{f}} = \lambda(\tilde{\mathbf{f}}) = b_{\mathbf{f}} \cdot \mathbf{1} \text{ for some constant } b_{\mathbf{f}},$$

and the claim is proved. Furthermore, from the claim, together with (3.27) and (3.28), we can deduce that

$$\lim_{n\to\infty} ||L^n\mathbf{f} - b_{\mathbf{f}} \cdot \mathbf{1}||_\infty = 0.$$

Define $\boldsymbol{\nu} : C(X, \mathbb{R}^d) \to \mathbb{R}$ by

$$\boldsymbol{\nu}(\mathbf{f}) = b_{\mathbf{f}}, \quad \mathbf{f} \in C(X, \mathbb{R}^d).$$

Then $\boldsymbol{\nu}(\mathbf{1}) = 1$. And we can check that

$$\boldsymbol{\nu}(\alpha\mathbf{f} + \beta\mathbf{g}) = \alpha\boldsymbol{\nu}(\mathbf{f}) + \beta\boldsymbol{\nu}(\mathbf{g}) \text{ for all } \alpha,\ \beta \in \mathbb{R} \text{ and } \mathbf{f}, \mathbf{g} \in C\left(X, \mathbb{R}^d\right).$$

Hence, $\boldsymbol{\nu}$ is a vector-valued probability measure on X. Note that

$$\boldsymbol{\nu}(L(\mathbf{f})) = b_{L(\mathbf{f})} = b_{\mathbf{f}} = \boldsymbol{\nu}(\mathbf{f}).$$

We have $L^*\boldsymbol{\nu} = \boldsymbol{\nu}$. Suppose that there exists another vector-valued probability measure $\boldsymbol{\upsilon}$ such that $L^*\boldsymbol{\upsilon} = \boldsymbol{\upsilon}$. Then for any $\mathbf{f} \in C\left(K, \mathbb{R}^d\right)$

$$\boldsymbol{\upsilon}(\mathbf{f}) = \lim_{n\to\infty} L^{*n}\boldsymbol{\upsilon}(\mathbf{f}) = \lim_{n\to\infty} \boldsymbol{\upsilon}(L^n\mathbf{f}) = \boldsymbol{\upsilon}(\boldsymbol{\nu}(\mathbf{f}) \cdot \mathbf{1}) = \boldsymbol{\nu}(\mathbf{f}).$$

Thus $\boldsymbol{\upsilon} = \boldsymbol{\nu}$.

□

Proof of theorem 3.1. From lemma 3.2, we know that if the vector-valued Ruelle operator theorem for T^q holds for some $q \geqslant 2$, then the vector-valued Ruelle operator theorem for T holds. From this, we may assume $k = 1$ in the condition (3.1) so that it is reduced to (3.2).

By proposition 3.7, we let $\mathbf{0}_{d\times 1} < \mathbf{h} \in C\left(X, \mathbb{R}^d\right)$ satisfy the equation: $T\mathbf{h} = \varrho\mathbf{h}$. Let L be the 'normalized' operator defined as (3.23). For any $\mathbf{f} \in C\left(X, \mathbb{R}^d\right)$, we have $H^{-1}\mathbf{f} \in C\left(X, \mathbb{R}^d\right)$. This, combined with proposition 3.9, implies that there exists a unique vector-valued invariant probability measure $\boldsymbol{\nu}$ such that

$$\lim_{n\to\infty} ||L^n(H^{-1}\mathbf{f}) - \boldsymbol{\nu}(H^{-1}\mathbf{f}) \cdot \mathbf{1}||_\infty = 0.$$

Define $\boldsymbol{\mu} \in M(X, \mathbb{R}^d)$ by

$$\boldsymbol{\mu}(\mathbf{f}) = \boldsymbol{\nu}(H^{-1}\mathbf{f}),\ \mathbf{f} \in C(X, \mathbb{R}^d).$$

From this, together with (3.24), it follows that

$$\lim_{n\to\infty} ||\varrho^{-n}T^n(\mathbf{f}) - \boldsymbol{\mu}(\mathbf{f}) \cdot \mathbf{h}||_\infty = 0. \tag{3.35}$$

Let $\mathbf{f}_n$ and $\mathbf{f}_{n_i}$ be the vector-valued functions as in the proof of proposition 3.7. Then we have

$$\lim_{i\to\infty} ||\mathbf{f}_{n_i} - \mathbf{h}||_\infty = 0.$$

From this, together with (3.35), we conclude that

$$\lim_{n\to\infty} ||\varrho^{-n}T^n\mathbf{1}-\mathbf{h}||_\infty = 0.$$

From this, we can deduce that $\boldsymbol{\mu}(\mathbf{1}) = 1$, i.e. $\boldsymbol{\mu}$ is a vector-valued probability measure.

From the definition of $\boldsymbol{\mu}$, it follows that

$$\boldsymbol{\mu}(\mathbf{h}) = \boldsymbol{\nu}(H^{-1}\mathbf{h}) = \boldsymbol{\nu}(\mathbf{1}) = 1.$$

Note that $L^*\boldsymbol{\nu} = \boldsymbol{\nu}$. From this, together with (3.24), we deduce that for any $\mathbf{f}\in C\left(X,\mathbb{R}^d\right)$

$$\boldsymbol{\mu}(T\mathbf{f}) = \varrho\boldsymbol{\mu}(H\cdot L(H^{-1}\mathbf{f})) = \varrho\boldsymbol{\nu}(L(H^{-1}\mathbf{f})) = \varrho\boldsymbol{\nu}(H^{-1}\mathbf{f}) = \varrho\boldsymbol{\mu}(\mathbf{f}).$$

From this, we conclude that $T^*\boldsymbol{\mu} = \varrho\boldsymbol{\mu}$. The uniqueness of $\mathbf{h}$ and $\boldsymbol{\mu}$ can be deduced easily from (3.35).

□

Note that a lower bound of the spectral radius can be obtained as

$$\min_{x\in X}\left|\sum_{i=1}^m A^{(i)}(x)\mathbf{1}\right| \leqslant \varrho.$$

From this, together with theorem 3.1, we get the following corollary immediately.

Corollary 3.10. *Let $(X,\{w_i\}_{i=1}^m)$ be a weakly contractive IFS, and let matrix potentials $\{A^{(i)}\}_{i=1}^m$ satisfy the hypotheses (H1) and (H2). Assume the system $(X,\{w_i\}_{i=1}^m,\{A^{(i)}\}_{i=1}^m)$ satisfies the condition:*

$$\left\|\sum_{i=1}^m r_i A^{(i)}(x)\mathbf{1}\right\|_\infty < \min_{x\in X}\left|\sum_{i=1}^m A^{(i)}(x)\mathbf{1}\right|. \tag{3.36}$$

Then the vector-valued Ruelle theorem holds.

We would like to point out that the condition (3.36) is checkable. At the end of the paper, we present examples to show that, in some cases, invariant measure of a weakly contractive IFS with overlaps can be put into vector form, which is the vector-valued invariant measure of some newly defined weakly contractive non-overlapping IFS associated with matrix potentials. The first example is almost copied from paper [25].

Example 3.11. Let $X=[0,2]$, and let $s_1(x)=\frac{x}{1+2x}$, $s_2(x)=\frac{x+2}{5}$, $s_3(x)=\frac{x+3}{5}$, $s_{i+3}(x) = s_i(x)+1$ $(1\leqslant i\leqslant 3)$, $x\in X$.

It is easy to see that both s_1 and s_4 are weakly contractive maps; and s_i $(i\neq 1,4)$ are all contractive maps. Hence $(X,\ \{s_i\}_{i=1}^6)$ is a weakly contractive IFS. It is easy to see that

$$X=\bigcup_{i=1}^6 s_i(X),$$

i.e. X is the invariant set of the IFS $\left(X,\ \{s_i\}_{i=1}^6\right)$.

Let $\{p_i\}_{i=1}^6$ be a set of Dini continuous positive potentials on X with

$$\sum_{i=1}^6 p_i\left(s_i(x)\right) = 1.$$

We know from paper [24] that there exists a unique probability measure ν on X such that

$$\nu = \sum_{i=1}^{6} p_i(x)\nu \circ s_i^{-1}. \tag{3.37}$$

We, however, find that

$$s_2 \circ s_6(x) = s_3 \circ s_3(x) = \frac{x}{5^2} + \frac{3}{5^2} + \frac{3}{5}.$$

This implies that the IFS $(X,\ \{s_i\}_{i=1}^{6})$ has overlaps. It creates difficulty for us to study the measure ν, such as the multi-fractal structure of the measure ν. Luckily enough, we can transform the measure ν into a vector-valued measure $\boldsymbol{\mu}$ on $\mathbb{R}$:

$$\boldsymbol{\mu}(D) = \begin{pmatrix} \nu(D \cap [0,1]) \\ \nu((D \cap [0,1]) + 1) \end{pmatrix},$$

for any Borel subset $D \subseteq \mathbb{R}$. In fact, let $K = [0,1]$, and define for any $x \in K$

$$\begin{cases} w_1(x) = s_1(x) = \frac{x}{1+2x}; \\ w_2(x) = s_1(x+1) = \frac{x+1}{3+2x}; \\ w_3(x) = s_2(x) = \frac{x+2}{5}; \\ w_4(x) = s_2(x+1) = s_3(x) = \frac{x+3}{5}; \\ w_5(x) = s_3(x+1) = \frac{x+4}{5}. \end{cases} \tag{3.38}$$

Then we have $\bigcup_{i=1}^{5} w_i(K) = K$, and moreover

$$w_i(K^\circ) \bigcap w_j(K^\circ) = \emptyset \text{ for any } i \neq j.$$

This implies that the IFS $(K, \{w_i\}_{i=1}^{5})$ has non-overlapping.

We can check directly that $\mathrm{supp}(\boldsymbol{\mu}) \subseteq K$. From (3.38), we can deduce that (3.37) is equivalent to the following:

$$\boldsymbol{\mu} = \sum_{i=1}^{5} B_i(x)\boldsymbol{\mu} \circ w_i^{-1}. \tag{3.39}$$

Where

$$B_1 = \begin{pmatrix} p_1 & 0 \\ p_4 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & p_1 \\ 0 & p_4 \end{pmatrix}, \quad B_3 = \begin{pmatrix} p_2 & 0 \\ p_5 & 0 \end{pmatrix},$$
$$B_4 = \begin{pmatrix} p_3 & p_2 \\ p_6 & p_5 \end{pmatrix}, \quad B_5 = \begin{pmatrix} 0 & p_3 \\ 0 & p_6 \end{pmatrix}.$$

Let $A^{(i)} = B_i^t \circ w_i$. Then a non-overlapping system $(K, \{w_i\}_{i=1}^{5}, \{A^{(i)}\}_{i=1}^{5})$ is set up. It is obvious that the hypotheses (H1) and (H2) are satisfied. Now we consider a vector-valued Ruelle operator $T : C\left(K, \mathbb{R}^d\right) \to C\left(K, \mathbb{R}^d\right)$ defined by

$$T\mathbf{f}(x) = \sum_{i=1}^{5} A^{(i)}(x)\mathbf{f}(w_i(x)), \quad \mathbf{f} \in C\left(K, \mathbb{R}^d\right).$$

From (3.39), it follows that $T^*\boldsymbol{\mu} = \boldsymbol{\mu}$, i.e. $\boldsymbol{\mu}$ is a vector-valued invariant probability measure of the system $(K, \{w_i\}_{i=1}^{5}, \{A^{(i)}\}_{i=1}^{5})$.

The following example 3.12 is more general.

Example 3.12. Let S be a C^1 function defined on an open interval containing $[0,1]$ and assume that S satisfies the following condition:

$$\begin{cases} S(0)=0,\ S(1)=\frac{1}{2}; \\ 0<S'(x)<1=S'(0)=S'(1) \text{ for all } 0<x<1. \end{cases} \tag{3.40}$$

Let $X:=[0,2]$. For any $0\leqslant i\leqslant 2$, we define $S_i : X\to X$ by

$$S_i(x)=S(x-[x])+\frac{[x]}{2}+\frac{i}{2}. \tag{3.41}$$

(Recall that $[x]$ is the integer part of x.) From (3.40), we conclude that the IFS $\left(X,\ \{S_i\}_{i=0}^2\right)$ is weakly contractive. It is easy to see that

$$X=\bigcup_{i=0}^{2}S_i(X),$$

i.e. X is the invariant set of IFS $(X,\ \{S_i\}_{i=0}^2)$. Moreover, we can deduce from (3.41) that

$$S_0\circ S_2=S_1\circ S_0 \text{ and } S_1\circ S_2=S_2\circ S_0.$$

This implies that the IFS $(X,\ \{S_i\}_{i=0}^2)$ has overlaps.

Let $\{p_i\}_{i=0}^2$ be a family of Dini continuous positive potentials on X satisfying the condition:

$$\sum_{i=0}^{2}p_i(S_i(x))=1. \tag{3.42}$$

Note that $S_0\circ S_1(X)\subset(0,1)$. This, combined with (3.40), implies that $S_0\circ S_0\circ S_1(x)$ is contractive. From this, it follows that the weakly contractive system $\left(X,\ \{S_i\}_{i=0}^2,\ \{p_i\}_{i=0}^2\right)$ satisfies the condition of [24, theorem 4.3] for $k=3$. Hence from this, we know that there exists a unique probability measure ν on X such that

$$\nu=\sum_{i=0}^{2}p_i\,\nu\circ S_i^{-1}. \tag{3.43}$$

It is difficult to study the measure ν, such as the L^q-spectrum of the measure ν, as the IFS $(X,\ \{S_i\}_{i=0}^2)$ has overlaps. Luckily enough again, we can split the measure ν into a vector-valued measure $\boldsymbol{\mu}$ on $\mathbb{R}$ defined by

$$\boldsymbol{\mu}(D)=\begin{pmatrix}\nu(D\cap[0,1])\\ \nu(D\cap[0,1]+1)\end{pmatrix},$$

for any Borel subset $D\subset\mathbb{R}$. In fact, let $K=[0,1]$, and from (3.41), we define for any $0\leqslant i\leqslant 1$

$$w_i(x)=S(x)+\frac{i}{2},\ x\in K. \tag{3.44}$$

Then we have $w_0(K)\bigcup w_1(K)=K$, and moreover

$$w_0(K^\circ)\bigcap w_1(K^\circ)=\emptyset.$$

This implies that the IFS $(K,\{w_i\}_{i=0}^1)$ has non-overlapping.

It follows directly that $\mathrm{supp}(\boldsymbol{\mu})\subseteq[0,1]$. From (3.44), we can deduce that (3.43) is equivalent to the following:

$$\boldsymbol{\mu}=B_0\boldsymbol{\mu}\circ w_0^{-1}+B_1\boldsymbol{\mu}\circ w_1^{-1}, \tag{3.45}$$

where

$$B_0 = \begin{pmatrix} p_0 & 0 \\ p_2 & p_1 \end{pmatrix}, \; B_1 = \begin{pmatrix} p_1 & p_0 \\ 0 & p_2 \end{pmatrix}.$$

Let $A^{(i)} = B_i^t \circ w_i$. Then a non-overlapping system $(K, \{w_i\}_{i=0}^1, \{A^{(i)}\}_{i=0}^1)$ is set up. It is obvious that the hypotheses (H1) and (H2) are satisfied. Now we consider vector-valued Ruelle operator $T : C(K, \mathbb{R}^d) \to C(K, \mathbb{R}^d)$ defined by

$$T\mathbf{f}(x) = \sum_{i=0}^{1} A^{(i)}(x)\mathbf{f}(w_i(x)), \quad \mathbf{f} \in C(K, \mathbb{R}^d).$$

From (3.45), we see that $\boldsymbol{\mu}$ is a vector-valued invariant probability measure of the system $(K, \{w_i\}_{i=0}^1, \{A^{(i)}\}_{i=0}^1)$, i.e. $T^*\boldsymbol{\mu} = \boldsymbol{\mu}$.

We would like to point out that the vector-valued measures in the above examples are the invariant measures of newly defined IFSs with non-overlapping. They have a close relationship with invariant probability measures of the systems with overlaps. This hints us that it is necessary to set up vector-valued Ruelle operator theory.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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