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ENTIRE SOLUTIONS OF HIGHER ORDER DIFFERENTIAL EQUATIONS WITH ENTIRE COEFFICIENTS HAVING THE SAME ORDER*

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Abstract In this paper, we consider entire solutions of higher order homogeneous differential equations with the entire coefficients having the same order, and prove that the entire solutions are of infinite lower order. The properties on the radial distribution, the limit direction of the Julia set and the existence of a Baker wandering domain of the entire solutions are also discussed.

Key words entire solutions; radial order; Julia limiting direction; Baker wandering domain; transcendental direction

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1 Introduction

We assume that the reader is familiar with the fundamental results and the standard notations of Nevanlinna's value distribution of meromorphic functions (see [9, 14, 24, 30]). For a meromorphic function $f(z)$ in the complex plane $\mathbb{C}$, the order $\rho(f)$ and the lower order $\mu(f)$ are defined by, respectively,

$$\rho(f)=\limsup_{r\to\infty}\frac{\log^+ T(r,f)}{\log r} \text{ and } \mu(f)=\liminf_{r\to\infty}\frac{\log^+ T(r,f)}{\log r}.$$

If f is entire function, then the Nevanlinna characteristic $T(r,f)$ can be replaced with $\log M(r,f)$, where $M(r,f)=\max\{|f(z)|:|z|\le r\}$. Let $a\in\mathbb{C}$ and $n(r,f=a)$ denote the numbers of $f(z)-a=0$ in disk $\{z:|z|\le r\}$. If

$$\limsup_{r\to\infty}\frac{\log n(r,f=a)}{\log r}<\rho(f),$$

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then a is called the Borel exceptional value of f.

This paper is devoted to considering the properties, such as the growth order, the radial oscillation and limiting direction of Julia sets, and the existence of a Baker wandering domain, of solutions to higher order linear differential equations

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_0(z)f=0, \tag{1.1}$$

where $A_j(z)$ $(j=0,1,2,\cdots,k-1)$ are entire functions. Due to the classical result by Wittich [23], all solutions to (1.1) are entire functions with finite order if and only if all coefficients are polynomials. If $\max\{\rho(A_j), j=1,2,\cdots,k-1\}<\rho(A_0)$, then every non-trivial solution to (1.1) is of infinite order. Furthermore, if the coefficients have the properties on the Phragmén-Lindelöf indicator function, every non-trivial solution to (1.1) is also of infinite order [10]. In this paper, we concentrate on looking at the situation when the coefficients of (1.1) are exponential polynomials with the same degree, that is, all coefficients have the same order.

2 Radial Distribution of Entire Solutions

We first recall Nevanlinna's Characteristic in an angle (see [29]). Assumeing that $0<\alpha<\beta<2\pi$, we denote that

$$\Omega(\alpha,\beta)=\{z\in\mathbb{C}:\arg z\in(\alpha,\beta)\} \text{ and } \Omega(r,\alpha,\beta)=\Omega(\alpha,\beta)\cap\{z:|z|<r\},$$

and use $\overline{\Omega}(\alpha,\beta)$ and $\overline{\Omega}(r,\alpha,\beta)$ to denote the closure of $\Omega(\alpha,\beta)$ and $\Omega(r,\alpha,\beta)$, respectively. For the function $g(z)$, analytic in $\Omega(\alpha,\beta)$, we define that

$$A_{\alpha,\beta}(r,g)=\frac{\omega}{\pi}\int_1^r\left(\frac{1}{t^\omega}-\frac{t^\omega}{r^{2\omega}}\right)\{\log^+|g(re^{i\alpha})|+\log^+|g(re^{i\beta})|\}\frac{dt}{t},$$
$$B_{\alpha,\beta}(r,g)=\frac{2\omega}{\pi r^\omega}\int_\alpha^\beta\log^+|g(re^{i\theta})|\sin\omega(\theta-\alpha)d\theta,$$
$$C_{\alpha,\beta}(r,g)=2\sum_{1<|b_\nu|<r}\left(\frac{1}{|b_\nu|^\omega}-\frac{|b_\nu|^\omega}{r^{2\omega}}\right)\sin\omega(\beta_\nu-\alpha),$$

where $\omega=\frac{\pi}{\beta-\alpha}$, $b_\nu=|b_\nu|re^{i\beta_\nu}$ are poles (counting multiplicities) of $g(z)$ in $\Omega(\alpha,\beta)$. Nevanlinna's angular characteristic of g is defined by

$$S_{\alpha,\beta}(r,g)=A_{\alpha,\beta}(r,g)+B_{\alpha,\beta}(r,g)+C_{\alpha,\beta}(r,g),$$

and the order $\rho_{\alpha,\beta}(g)$ of g on $\Omega(\alpha,\beta)$ is defined by

$$\rho_{\alpha,\beta}(g)=\limsup_{r\to\infty}\frac{\log^+S_{\alpha,\beta}(r,g)}{\log r}=\limsup_{r\to\infty}\frac{\log^+\log^+M(r,\Omega(\alpha,\beta),g)}{\log r},$$

where $M(r,\Omega(\alpha,\beta),g):=\max\{|g(z)|:z\in\overline{\Omega}(r,\alpha,\beta)\}$. If g is analytic on $\mathbb{C}$, $\rho(g)\geq\rho_{\alpha,\beta}(g)$. If $\rho_{\alpha,\beta}(g)=\infty$, then $\rho(g)=\infty$. Otherwise, the above may not be true. For example, for $g(z)=\exp\{e^z\}$, we have that $\rho_{-\pi/2,\pi/2}(g)=\rho(g)=\infty$, but $\rho_{\pi/2,3\pi/2}(g)=0$.

Moreover, the sectorial order $\rho_{\theta,\varepsilon}(g)$ and the radial order $\rho_\theta(g)$ are defined by

$$\rho_{\theta,\varepsilon}(g)=\limsup_{r\to\infty}\frac{\log^+\log^+M(r,\Omega(\theta-\epsilon,\theta+\epsilon),g)}{\log r} \text{ and } \rho_\theta(g)=\lim_{\epsilon\to0}\rho_{\theta,\varepsilon}(g).$$

Define that

$$I(g):=\{\theta\in[0,2\pi):\rho_\theta(g)=\infty\}.$$

Clearly, $I(g)$ is closed, so it is measurable. We use mes $I(g)$ for the linear measure of $I(g)$. For instance, mes $I(g) = \pi$ when $g(z) = \exp\{e^z\}$.

A natural question that arises is: what is the lower boundary of mes $I(g)$ when the entire function $g(z)$ is of infinite order? The radial distribution of transcendental entire solutions has been well studied, for instance, see [13, 16, 18, 25]. We now recall Huang and Wang's result on the differential equation.

Theorem 2.1 ([13, Theorem 1.3]) Suppose that $A(z)$ and $B(z)$ are entire functions with $\mu(B) > \rho(A)$. If $g(z)$ is a non-trivial solution of the equation

$$g'' + A(z)' + B(z)g = 0, \tag{2.1}$$

then mes $I(g) \geq \min\{2\pi, \pi/\mu(B)\}$.

Theorem 2.1 tells us that mes $I(g) = 2\pi$ when $\mu(B) \leq 1/2$. Furthermore, we also note that equation (2.1) and all other previous results have dominated coefficients. Now, we consider that all entire coefficients have the same order and obtain

Theorem 2.2 Suppose that $g_j(z) = \omega_j(z)e^{P_j(z)} + \alpha_j$ $(j = 0, 1, \cdots, k-1)$, where $\alpha_j \in \mathbb{C}$, $P_j(z) = a_n^j z^n + \cdots + a_0^j$ are polynomials with degree n (≥ 1) and $a_n^j = |a_n^j|e^{i\varphi_n^j} \neq 0$, $\varphi_n^j \in [0, 2\pi)$ and $\omega_j(z) \not\equiv 0$ are meromorphic functions with $\rho(\omega_j) < n$. If there exists $\phi \in [0, \pi)$ such that either

(1) $\pi - \phi < \varphi_n^j - \varphi_n^0 < \pi + \phi$, or

(2) $\pi - \phi < \varphi_n^j - \varphi_n^0 + 2\pi < \pi + \phi$, or

(3) $a_n^j = c_j a_n^0$ $(0 < c_j < 1)$,

then every non-trivial solution $f(z)$ of equation

$$f^{(k)} + g_{k-1}(z)f^{(k-1)} + \cdots + g_1(z)f^{'} + g_0(z)f = 0 \tag{2.2}$$

satisfies that $\mu(f) = \infty$ and that mes $I(f) \geq \frac{\lambda - 1}{\lambda}\pi$ for some $\lambda > 1$.

Before proceeding to the actual proof of Theorem 2.2, we introduce some lemmas.

Lemma 2.3 ([8, Theorem 2]) Let $f(z)$ be a transcendental meromorphic function and let $\alpha > 1$ be a real constant. Then there exists a set $E \subset [0, 2\pi)$ that has linear measure of zero, and there exists a constant $B > 0$ such that if $\theta \in [0, 2\pi)\backslash E$, then there exists a constant $R_0 = R_0(\theta) > 1$ such that, for all z satisfying $\arg z = \theta$ and $|z| = r > R_0$, we have that

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq B[T(\alpha r, f)\log T(\alpha r, f)]^{j-i}, \quad (0 \leq i < j).$$

Lemma 2.4 ([15]) Suppose that $P(z) = (\alpha + i\beta)z^n + \cdots$ is a non-constant polynomial with degree $n \geq 1$, that α, β are real constants, and that $\omega(z) \not\equiv 0$ is a meromorphic function with $\rho(\omega) < n$. Set $g(z) = \omega(z)e^{P(z)}, z = re^{i\theta}$, and $\delta(P, \theta) = \alpha\cos n\theta - \beta\sin n\theta$. Then, for any given $\varepsilon > 0$, there exists a set $H_1 \subset [0, 2\pi)$ of linear measure of zero such that, for any $\theta \in [0, 2\pi)\backslash(H_1 \cup H_2)$ and $|z| = r > r_0(\theta, \varepsilon)$, we have that

(i) if $\delta(P, \theta) > 0$, then $\exp\{(1 - \varepsilon)\delta(P, \theta)r^n\} < |g(re^{i\theta})| < \exp\{(1 + \varepsilon)\delta(P, \theta)r^n\}$;

(ii) if $\delta(P, \theta) < 0$, then $\exp\{(1 + \varepsilon)\delta(P, \theta)r^n\} < |g(re^{i\theta})| < \exp\{(1 - \varepsilon)\delta(P, \theta)r^n\}$,

where $H_2 = \{\theta \in [0, 2\pi) : \delta(P, \theta) = 0\}$.

Remark 2.5 As described in Lemma 2.4,

(i) if we set that $\alpha + i\beta = a_n = |a_n|e^{i\varphi_n}$, then we have that $\delta(P, \theta) = |a_n|\cos(\varphi_n + n\theta)$;

(ii) for every given ε $(0<\varepsilon<\frac{\pi}{2\lambda n})$ when $\lambda>1$, we define a $2n$ open angular domain

$$S_j(P,\theta)=\left\{\theta: -\frac{\varphi_n}{n}+\frac{(2j-1)}{2n}\pi+\varepsilon<\theta<-\frac{\varphi_n}{n}+\frac{(2j+1)}{2n}\pi-\varepsilon\right\},\ j=0,1,\cdots,2n-1.$$

Obviously, if $\theta\in S_j(P,\theta)$, then $\delta(P,\theta)>0$ for even j, and $\delta(P,\theta)<0$ for odd j.

Lemma 2.6 Suppose that $g_j(z)=\omega_j(z)\mathrm{e}^{P_j(z)}+\alpha_j$ $(j=0,1,\cdots,k-1)$, where $\alpha_j\in\mathbb{C}$, $P_j(z)=a_n^j z^n+\cdots+a_0^j$ are polynomials with degree n (≥ 1) and $a_n^j=|a_n^j|\mathrm{e}^{\mathrm{i}\varphi_n^j}\neq 0$, $\varphi_n^j\in[0,2\pi)$, and $\omega_j(z)\not\equiv 0$ are meromorphic functions with $\rho(\omega_j)<n$. If there exists $\phi\in[0,\pi)$ such that either

(1) $\phi<\varphi_n^j-\varphi_n^0<\pi+\phi$, or

(2) $\phi<\varphi_n^j-\varphi_n^0+2\pi<\pi+\phi$, or

(3) $a_n^j=c_j a_n^0$ $(0<c_j<1)$,

then every non-trivial solution $f(z)$ of equation (2.2) satisfies that $\mu(f)=\infty$.

Proof By Lemma 2.3, for all z satisfying that $\arg z=\theta\in[0,2\pi)\backslash E_1$ and that $|z|=r\geq R>R(\theta)>1$, we obtain that

$$\left|\frac{f^{(j)}(z)}{f(z)}\right|\leq BT(2r,f)^{2k},\quad j=1,2,\cdots,k, \tag{2.3}$$

where E_1 is a set of linear measure zero and B is a positive constant.

By Lemma 2.4 and Remark 2.5, there exists a set $H_0=\{\theta\in[0,2\pi):\delta(P_0,\theta)>0\}$ such that, for all z satisfying that $\arg z=\theta\in H=H_0\backslash E_1$, one of the following statements holds:

(a) $\delta(P_j,\theta)<0$ $(j=1,2,\cdots,k-1)$ for some proper ϕ and $\theta\in H_3$ when one of the conditions (1) or (2) holds, where H_3 is a subset of H with a positive linear measure;

(b) $\delta(P_j,\theta)>0$ $(j=1,2,\cdots,k-1)$ for $\theta\in H_0$ when condition (3) holds.

Set $c=\max\{c_j: j=1,2,\cdots,k-1\}$. Then $0<c<1$. Furthermore, by Lemma 2.4, for any given ε $(0<\varepsilon<\frac{1-c}{1+c})$ and a sufficiently large $|z|=r$, we have that

$$|g_0(z)-\alpha_0|\geq\exp\{(1-\varepsilon)\delta(P_0,\theta)r^n\} \tag{2.4}$$

and

$$|g_j(z)-\alpha_j|\leq\exp\{(1-\varepsilon)\delta(P_j,\theta)r^n\}<1,\ (j=1,2,\cdots,k-1) \tag{2.5}$$

in case (a), and

$$|g_j(z)-\alpha_j|\leq\exp\{(1+\varepsilon)c\delta(P_0,\theta)r^n\},\ (j=1,2,\cdots,k-1), \tag{2.6}$$

in case (b). Therefore, for all z satisfying that $\arg z=\theta\in H_3$, for any given ε $\left(0<\varepsilon<\frac{1-c}{1+c}\right)$ and a sufficiently large $|z|=r$, we obtain from (2.2)–(2.6) that

$$\begin{aligned}&\exp\{(1-\varepsilon)\delta(P_0,\theta)r^n\}\leq|g_0-\alpha_0|\\ \leq\ &|\alpha_0|+\left|\frac{f^{(k)}}{f}\right|+\left(|g_{k-1}-\alpha_{k-1}|+|\alpha_{k-1}|\right)\left|\frac{f^{(k-1)}}{f}\right|+\cdots+\left(|g_1-\alpha_1|+|\alpha_1|\right)\left|\frac{f'}{f}\right|\\ \leq\ &BT(2r,f)^{2k}\exp\{(1+\varepsilon)c\delta(P_0,\theta)r^n\},\end{aligned} \tag{2.7}$$

when one of the conditions (1), (2) or (3) holds. We further obtain that $\mu(f)=\infty$ from (2.7) and the fact that $0<\varepsilon<\frac{1-c}{1+c}$. □

Lemma 2.7 ([11, Lemma 7]) Let $z = re^{i\theta}$, $r_0 + 1 < r$ and $\alpha \le \theta \le \beta$, where $0 < \beta - \alpha \le 2\pi$. Suppose that $n(\ge 2)$ is an integer, and that $g(z)$ is analytic in $\Omega(\alpha, \beta)$ with $\rho_{(\alpha,\beta)} < \infty$. Then, for every $\varepsilon_j \in \left(0, \frac{\beta_j - \alpha_j}{2}\right) \backslash E$ $(j = 1, 2, \cdots, n-1)$ outside a set E of linear measure zero with $\alpha_j = \alpha + \sum_{s=1}^{j-1} \varepsilon_s$ and $\beta_j = \beta - \sum_{s=1}^{j-1} \varepsilon_s$, there exist $K > 0$ and $M > 0$ such that

$$\left|\frac{g^{(n)}(z)}{g(z)}\right| \le Kr^M \left(\sin k(\theta - \alpha) \prod_{j=1}^{n-1} \sin k_{\varepsilon_j}(\theta - \alpha_j)\right)^{-2}$$

for all $z \in \Omega(\alpha_{n-1}, \beta_{n-1})$ outside an R-set D, where $k = \frac{\pi}{\beta - \alpha}$ and $k_{\varepsilon_j} = \dfrac{\pi}{\beta_j - \alpha_j}$ $(j = 1, 2, \cdots, n-1)$.

Remark 2.8 ([14]) Define that $D(z_n, r_n) = \{z : |z - z_n| < r_n\}$, and the set of form $D = \bigcup_{n=1}^{\infty} D(z_n, r_n)$ is called the R-set if $\sum_{n=1}^{\infty} r_n < \infty$ and $z_n \to \infty$ $(n \to \infty)$.

Lemma 2.9 Suppose that $g(z) = \omega(z)e^{P(z)}$, where $P(z) = (\alpha + i\beta)z^n + \cdots$ is a polynomial with degree n, $\alpha, \beta \in \mathbb{R}$, and $\omega(z) \not\equiv 0$ is a meromorophic function with $\rho(\omega) < n$. Set $z = re^{i\theta}$ and $\delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$. Then, for some constant $\lambda > 1$,

$$\text{mes } H^+(\theta) := \text{mes}\{\theta : \delta(P, \theta) > 0\} > \frac{\lambda - 1}{\lambda}\pi.$$

Proof of Theorem 2.2 By Lemma 2.4 and Remark 2.5, for all given ε $(0 < \varepsilon < \frac{\pi}{2\lambda n})$ when $\lambda > 1$, we have that

$$\delta(P, \theta) > 0 \text{ when } \theta \in S_j(P, \theta) \ (j = 0, 2, \cdots, 2n-2).$$

We note that

$$\text{mes } S_j(P, \theta) = \frac{\pi}{n} - 2\varepsilon > \frac{\lambda - 1}{\lambda n}\pi \ (j = 0, 2, \cdots, 2n-2),$$

and so

$$\text{mes } H^+(\theta) := \text{mes}\{\theta : \delta(P, \theta) > 0\} = n \cdot \text{mes } S_j(P, \theta) > \frac{\lambda - 1}{\lambda}\pi.$$

□

We now proceed to the actual proof of Theorem 2.2.

Proof By Lemma 2.6, we easily obtain that every non-trivial solution $f(z)$ of equation (2.2) satisfies that $\mu(f) = \infty$. We now just estimate the measure of $I(f)$. We first assume that mes $I(f) < \frac{\lambda-1}{\lambda}\pi$, and so $\eta := \frac{\lambda-1}{\lambda}\pi - \text{mes } I(f) > 0$.

Since $I(f)$ is closed, $\Phi := (0, 2\pi) \backslash I(f)$ is open. Thus it can be covered by at most countably many open intervals. We can choose finitely many open intervals $I_i = (\alpha_i, \beta_i)$ $(i = 1, 2, \cdots, m)$ satisfying $[\alpha_i, \beta_i] \subset \Phi$ and $\text{mes}\left(\Phi \backslash \bigcup_{i=1}^{m} I_i\right) < \frac{\eta}{4}$. For the definitions of $I(f)$ and I_i, we have that

$$I_i \cap I(f) = \emptyset \quad \text{and} \quad \rho_{\alpha_i, \beta_i}(f) < \infty \quad (i = 1, 2, \cdots, m).$$

By Lemma 2.7, for sufficiently small $\gamma > 0$, there exist two constants $K > 0$ and $M > 0$ such that

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \le Kr^M \quad (j = 1, 2, \cdots, k) \tag{2.8}$$

for all $z \in \Omega(\alpha_i + 2\gamma, \beta_i - 2\gamma)$ outside an R-set D.

Denote that

$$H_0^+(\theta) := \{\theta : \delta(P_0, \theta) > 0\}. \tag{2.9}$$

Then we obtain from Lemma 2.9 that, for some constant $\lambda > 1$,

$$\text{mes } H_0^+(\theta) > \frac{\lambda - 1}{\lambda}\pi.$$

Similarly to the proof in Lemma 2.6, we obtain that

$$\delta(P_j, \theta) < 0, \quad j = 1, 2, \cdots, k-1 \tag{2.10}$$

for some proper ϕ and $\theta \in H_0^+(\theta)\backslash H_1$ when one of the conditions (1) or (2) holds, and

$$\delta(P_j, \theta) > 0, \quad j = 1, 2, \cdots, k-1 \tag{2.11}$$

for $\theta \in H_0^+(\theta)$ when condition (3) holds, where H_1 is a set with a linear measure of zero.

Since

$$\text{mes}(H_0^+(\theta) \cap \Phi) = \text{mes}(H_0^+(\theta)\backslash(I(f) \cap H_0^+(\theta))) \geq \text{mes } H_0^+(\theta) - \text{mes } I(f) > \eta > \frac{3\eta}{4},$$

and then

$$\text{mes}\left(H_0^+(\theta) \cap \bigcup_{i=1}^{m} I_i\right) = \text{mes}(H_0^+(\theta) \cap \Phi) - \text{mes}\left(H_0^+(\theta) \cap \left(\Phi\backslash \bigcup_{i=1}^{m} I_i\right)\right) > \frac{3\eta}{4} - \frac{\eta}{4} = \frac{\eta}{2},$$

there exists at least an open interval $I_i = (\alpha_i, \beta_i)$ of $\bigcup\limits_{i=1}^{m} I_i$ such that

$$\text{mes}\left(H_0^+(\theta) \cap I_i\right) > \frac{\eta}{2m} > 0,$$

and so $\widetilde{H} := H_0^+(\theta) \cap (\alpha_i + 2\gamma, \beta_i - 2\gamma) \neq \emptyset$. Therefore, we obtain from (2.2), (2.4)–(2.6) and (2.8)–(2.11), for each $\theta \in \widetilde{H}$, that there exists a sequence $\{z_s = r_s\mathrm{e}^{\mathrm{i}\theta}\}$ with $r_s \to \infty$ $(s \to \infty)$ such that

$$\begin{aligned}
&\exp\{(1-\varepsilon)\delta(P_0,\theta)r_s^n\} \leq |g_0(r_s\mathrm{e}^{\mathrm{i}\theta}) - \alpha_0| \\
\leq\ & |\alpha_0| + \left|\frac{f^{(k)}}{f}\right| + \left(|g_{k-1} - \alpha_{k-1}| + |\alpha_{k-1}|\right)\left|\frac{f^{(k-1)}}{f}\right| + \cdots + \left(|g_1 - \alpha_1| + |\alpha_1|\right)\left|\frac{f'}{f}\right| \\
\leq\ & Kr_s^M \exp\{(1+\varepsilon)c\delta(P_0,\theta)r_s^n\}.
\end{aligned} \tag{2.12}$$

A contradiction arrives from (2.12), and so mes $I(f) \geq \frac{\lambda-1}{\lambda}\pi$. □

3 Limiting Direction on Julia Sets of Entire Solutions

Before stating our main results, we first recall some definitions. Let $f : \mathbb{C} \to \overline{\mathbb{C}} = \mathbb{C}\bigcup\{\infty\}$ be a transcendental meromorphic function, and let f^n $(n \in \mathbb{N})$ denote the nth iterate of f, that is, $f^1 = f, f^2 = f \circ f, \cdots, f^n = f \circ (f^{n-1})$. Define the Fatou set of f by $\mathcal{F}(f)$, which is the set of those points in $\overline{\mathbb{C}}$ such that f^n is defined and normal in some neighborhood of z, and the Julia set of f by $\mathcal{J}(f)$, the complement of $\mathcal{F}(f)$. It is well known that $\mathcal{F}(f)$ is open and completely invariant and that $\mathcal{J}(f)$ is closed and non-empty.

Given $\theta \in [0, 2\pi)$, if $\Omega(\theta - \varepsilon, \theta + \varepsilon)\bigcap \mathcal{J}(f)$ is unbounded for any $\varepsilon > 0$, then we call the ray $\arg z = \theta$ the limiting direction of $\mathcal{J}(f)$. Denote that

$$\Delta(f) := \{\theta \in [0, 2\pi) : \arg z = \theta \text{ is the limiting direction of } \mathcal{J}(f)\}.$$

Obviously, $\Delta(f)$ is closed, and so it is measurable. We use mes $\Delta(f)$ for the linear measure of $\Delta(f)$. The limiting direction of $\mathcal{J}(f)$ of the transcendental meromorphic functions has been well studied; see, for instance, [1, 11, 12, 16, 17, 19–22, 26]. For the transcendental entire function f, Qiao [16] proved that mes $\Delta(f) = 2\pi$ when $\mu(f) < \frac{1}{2}$, and that mes $\Delta(f) \geq \pi/\mu(f)$ when $\mu(f) \geq \frac{1}{2}$.

For the transcendental meromorphic function f, a value $\theta \in [0, 2\pi)$ is said to be a transcendental direction of f if there exists an unbounded sequence of $\{z_n\}$ such that

$$\lim_{n\to\infty} \arg z_n = \theta \text{ and } \lim_{n\to\infty} \frac{\log|f(z_n)|}{\log|z_n|} = +\infty.$$

We use $TD(f)$ to denote the union of all transcendental directions, and so $TD(f)$ is a non-empty compact set in $[0, 2\pi)$ and $TD(f) \subseteq \Delta(f)$ [21].

We secondly recall the differential monomials and differential polynomials of f. By differential monomial, we mean an expression of type

$$\prod_{s=0}^{k} (f^{(s)})^{n_{sj}} = f^{n_{0j}} (f^{'})^{n_{1j}} \cdots (f^{(k)})^{n_{kj}},$$

where $n_{0j}, n_{1j}, \cdots, n_{kj}$ are non-negative integers. A differential polynomial $P(z, f)$ is a finite sum of differential monomials, that is, an expression of the form

$$P(z, f) = \sum_{j=1}^{l} \prod_{s=0}^{k} a_j (f^{(s)})^{n_{sj}},$$

where a_j are meromorphic. γ_P is defined by

$$\gamma_P := \min_{1\leq j\leq l} \left(\sum_{s=0}^{k} n_{sj} \right).$$

Recently, Wang et al. [21] investigated the limiting direction and transcendental direction of transcendental entire solutions of complex differential equations, and obtained

Theorem 3.1 ([21, Theorem 1.3]) Suppose that s and m are integers, $F(z)$ is a transcendental entire function of finite lower order, and that $P(z, f)$ is a differential polynomial in f with $\gamma_P \geq s$, where all coefficients $a_j(z)$ $(j = 1, 2, \cdots, l)$ are polynomials if $\mu(F) = 0$, or all $a_j(z)$ $(j = 1, 2, \cdots, l)$ are entire functions with $\rho(a_j) < \mu(F)$ if $\mu(F) > 0$. Then, for every non-zero transcendental entire solution f of the differential equation

$$P(z, f) + F(z) f^s = 0, \tag{3.1}$$

we have that $TD(f^{(m)} \cap TD(F)) \subseteq \Delta(f^{(m)})$ and

$$\mathrm{mes}(\Delta(f^{(m)})) \geq \mathrm{mes}(TD(f^{(m)}) \cap TD(F)) \geq \min\left\{2\pi, \frac{\pi}{\mu(F)}\right\}.$$

We now consider the higher order homogeneous differential equations (2.2) with entire coefficients having the same order, and obtain

Theorem 3.2 Let $m \in \mathbb{Z}$. Suppose that the entire coefficients $g_j(z)$ $(j = 0, 1, \cdots, k-1)$ of equation (2.2) satisfy the conditions given in Theorem 2.2. Then every non-trivial solution $f(z)$ of equation (2.2) satisfies that

$$TD(f^{(m)}) \cap TD(g_0) \subseteq \Delta(f^{(m)}), \quad \mathrm{mes}(\Delta(f^{(m)})) \geq \mathrm{mes}(TD(f^{(m)}) \cap TD(g_0)) \geq \frac{\pi}{n}.$$

Before proving Theorem 3.2, we introduce some preliminary lemmas.

Lemma 3.3 ([5]) Let $f(z)$ be a transcendental meormorphic function with finite lower order μ and have a positive deficiency

$$\delta(\infty,f):=1-\limsup_{r\to\infty}\frac{N(r,f)}{T(r,f)}>0.$$

Let $\Lambda(r)$ be a positive function such that $\Lambda(r)=o(T(r,f))$ as $r\to\infty$, and let $D_\Lambda(r)=\{\theta\in[0,2\pi):|f(re^{i\theta})|>e^{\Lambda(r)}\}$. Then, for any fixed sequence of Pólya peaks $\{r_n\}$ of order μ, we have that

$$\liminf_{n\to\infty}\operatorname{mes}(D_\Lambda(r_n))\geq\min\left\{2\pi,\frac{4}{\mu}\arcsin\sqrt{\frac{\delta(\infty,f)}{2}}\right\}.$$

Lemma 3.4 ([21]) Let $f(z)$ be a transcendental meromorphic function with finite lower order μ and $\delta(\infty,f)>0$, and let $\Lambda(r)$ be a positive function such that $\Lambda(r)=o(T(r,f))$ and $\Lambda(r)/\log r\to\infty$ as $r\to\infty$. Then

$$\min\left\{2\pi,\frac{4}{\mu}\arcsin\sqrt{\frac{\delta(\infty,f)}{2}}\right\}\leq\operatorname{mes}(E_\Lambda(f))\leq\operatorname{mes}(\Delta(f)),$$

where $E_\Lambda(f):=\bigcap\limits_{n=1}^{\infty}B_n$ and $B_n:=\bigcup\limits_{j=n}^{\infty}D_\Lambda(r_j)$.

Lemma 3.5 ([30, Theorem 2.5.1]) Let $f(z)$ be a meromorphic function on $\Omega(\alpha-\epsilon,\beta+\epsilon)$ for $\epsilon>0$ and $0<\alpha<\beta<2\pi$. Then

$$S_{\alpha,\beta}\left(r,\frac{f'}{f}\right)\leq K(\log^+S_{\alpha-\epsilon,\beta+\epsilon}(r,f)+\log r+1),$$

where $K>0$ and $r>1$, possibly except for a set with a finite linear measure.

Lemma 3.6 Let $f(z)$ be transcendental entire function, and let $m\in\mathbb{Z}$. Then $TD(f)\subseteq TD(f^{(m)})$.

Proof By Lemma 2.9 in [21], we just need to prove that the conclusion holds when $m<0$. For any given $\theta\notin TD(f^{(m)})$, it follows from the definition of the transcendental direction that there exist $\epsilon>0$ and $K_0>0$ such that, for all $z=re^{i\theta}\in\Omega(\theta-\epsilon,\theta+\epsilon)$,

$$\frac{\log|f^{(m)}(re^{i\theta})|}{\log r}\leq K_0,$$

and so

$$S_{\theta-\epsilon,\theta+\epsilon}(r,f^{(m)})=O(\log r). \tag{3.2}$$

By Lemma 3.5 and (3.2), there exists a set E of linear measure zero such that, for all $r\in[1,\infty)\backslash E$,

$$S_{\theta-\epsilon+\epsilon_1,\theta+\epsilon-\epsilon_1}\left(r,\frac{f^{(m+1)}}{f^{(m)}}\right)\leq K(\log^+S_{\theta-\epsilon,\theta+\epsilon}(r,f^{(m)})+\log r+1)=O(\log r), \tag{3.3}$$

and so

$$\begin{aligned}S_{\theta-\epsilon+\epsilon_1,\theta+\epsilon-\epsilon_1}(r,f^{(m+1)})&\leq S_{\theta-\epsilon+\epsilon_1,\theta+\epsilon-\epsilon_1}\left(r,\frac{f^{(m+1)}}{f^{(m)}}\right)+S_{\theta-\epsilon+\epsilon_1,\theta+\epsilon-\epsilon_1}(r,f^{(m)})\\&=O(\log r),\end{aligned}$$

where $0 < |m|\epsilon_1 < \frac{\epsilon}{2}$.

Repeating the above processes $|m|$ times, we have that

$$S_{\theta-\frac{\epsilon}{2},\theta+\frac{\epsilon}{2}}(r,f) = O(\log r). \tag{3.4}$$

Thus, it follows from (3.4) that, for all $z = re^{i\theta} \in \Omega(r; \theta - \frac{\epsilon}{2}, \theta + \frac{\epsilon}{2})$,

$$\lim_{r\to\infty} \frac{\log|f(re^{i\theta})|}{\log r} < +\infty,$$

which implies that $\theta \notin TD(f)$. Hence $TD(f) \subset TD(f^{(m)})$. □

We now proceed to the proof of Theorem 3.2.

Proof of Theorem 3.2 Since $m \in \mathbb{Z}$, we will spilt our proof into two cases.

Case 1 $m = 0$.

Since $g_0(z) = \omega_0(z)e^{P_0(z)} + \alpha_0$, we deduce from Lemmas 3.3 and 3.4 that $|g_0(re^{i\theta})| \to \infty$ as $r \to \infty$ for all $\theta \in E_\Lambda(g_0)$. Therefore, we deduce from Lemma 2.4 that $\theta \in H_0^+(\theta)$, and then $E_\Lambda(g_0) \subseteq H_0^+(\theta)$. We assert that $\theta \in TD(f)$. Otherwise, if $\theta \notin TD(f)$, there exist $\epsilon > 0$ and $K_1 > 0$ such that, for all $z = re^{i\theta} \in \Omega(\theta - \epsilon, \theta + \epsilon)$,

$$\frac{\log|f(re^{i\theta})|}{\log r} \le K_1.$$

Therefore, by Lemma 2.7, for sufficiently small ϵ_1 $(0 < \epsilon_1 < \epsilon)$, there exist two constants $K_1 > 0$ and $M_1 > 0$ such that

$$\left|\frac{f^{(s)}(re^{i\theta})}{f(re^{i\theta})}\right| \le M_1 r^{K_2} \quad (s = 1, 2, 3, \cdots, k) \tag{3.5}$$

for all $z = re^{i\theta} \in \Omega(\theta - \epsilon_1, \theta + \epsilon_1)$, outside an R-set D.

Thus, we obtain from (2.2), (2.4)–(2.6), (2.10), (2.11) and (3.5) that, for any given ε $(0 < \varepsilon < \frac{1-c}{1+c})$,

$$\begin{aligned}
&\exp\{(1-\varepsilon)\delta(P_0,\theta)r^n\} \le |g_0 - \alpha_0| \\
&\le |\alpha_0| + \left|\frac{f^{(k)}}{f}\right| + \big(|g_{k-1} - \alpha_{k-1}| + |\alpha_{k-1}|\big)\left|\frac{f^{(k-1)}}{f}\right| + \cdots + \big(|g_1 - \alpha_1| + |\alpha_1|\big)\left|\frac{f'}{f}\right| \\
&\le Br^{K_2}\exp\{(1+\varepsilon)c\delta(P_0,\theta)r^n\},
\end{aligned} \tag{3.6}$$

when $\theta \in H_0^+(\theta)$ and $z = re^{i\theta} \in \Omega(\theta - \epsilon_1, \theta + \epsilon_1)\backslash D$. A contradiction arrives from (3.6). Thus, $\theta \in TD(f)$ and

$$E_\Lambda(g_0) \subseteq H_0^+(\theta) \subseteq TD(f) \subseteq \Delta(f). \tag{3.7}$$

Since $g_0(z) = \omega_0(z)e^{P_0(z)} + \alpha_0$, it follows from Lemma 3.4 that $E_\Lambda(g_0) \subseteq TD(g_0)$ and

$$\mathrm{mes}(E_\Lambda(g_0)) \ge \min\left\{2\pi, \frac{\pi}{n}\right\} = \frac{\pi}{n}. \tag{3.8}$$

Thus, (3.7) and (3.8) yield that

$$\begin{aligned}
&E_\Lambda(g_0) \subseteq TD(f) \cap TD(g_0) \subseteq \Delta(f), \\
&\mathrm{mes}(\Delta(f)) \ge \mathrm{mes}(TD(f) \cap TD(g_0)) \ge \frac{\pi}{n}.
\end{aligned}$$

Case 2 $m \neq 0$.

It follows from Lemma 3.6 and the proof of Case 1 that $TD(f) \cap TD(g_0) \subseteq TD(f^{(m)}) \cap TD(g_0)$. Thus

$$E_\Lambda(g_0) \subseteq TD(f^{(m)}) \cap TD(g_0) \subseteq \Delta(f^{(m)}),$$
$$\text{mes}(\Delta(f^{(m)})) \geq \text{mes}(TD(f^{(m)}) \cap TD(g_0)) \geq \frac{\pi}{n},$$

since $E_\Lambda(g_0) \subseteq TD(f)$. □

4 Baker Wandering Domain of Entire Solutions

Let U be a connected component of $\mathcal{F}(f)$. Then $f^n(U)$ is contained in a component of $\mathcal{F}(f)$, denoted by U_n. If, for some integer $p \geq 1$, $f^p(U) \subset U_p = U$, then U is called a periodic component of $\mathcal{F}(f)$, such the smallest integer p is the period of the periodic component U. If, for some n, U_n is periodic but U is not periodic, then U is called pre-periodic. U is called a wandering domain if it is neither periodic nor pre-periodic, that is, $U_n \neq U_m$ for all $n \neq m$. If U is wandering and all U_n are multiply-connected and surround 0, and the Euclidean distance is $\text{dist}(0, U_n) \to \infty$ as $n \to \infty$, then U is called the Baker wandering domain. By Sullivan's famous theorem, rational functions have no wandering domains. For a transcendental entire function, it has been shown by Baker [3] that such domains may exist; each multiply-connected component of $\mathcal{F}(f)$ must be a Baker wandering domain (see [2, 7, 28, 29]). There are some criteria of non-existence for the Baker wandering domains [4, 7], which also determine whether there exists only a simply connected Fatou component for given entire functions.

As is well all known, the properties of solutions of differential equations are always controlled by the behavior of coefficients. When there is a dominated coefficient g_0 in the sense that $T(r, g_j) = o(T(r, g_0))$ $(j = 1, 2, \cdots, k-1)$ as $r \to \infty$, the dynamical properties of differential equations (3.2) have been investigated in [11, 12, 20]. However, we are interested in the dynamical properties of solutions of differential equations (2.2) without the dominated coefficient, that is, where all coefficients are of the same growth order. Wang and Chen [20] considered the second order differential equation and obtained

Theorem 4.1 ([20, Theorem 1.2]) Suppose that B_j $(j = 1, 2)$ are constants and that $A_j(z)$ $(j = 1, 2)$ are entire functions, and that $P_j(z) = a_j z^{k_j} + \cdots$ $(j = 1, 2)$ are two polynomials of degree $k_j \geq 0$. Suppose that any one of the following two conditions holds:

(1) $k_1 < k_2$;

(2) $k_1 = k_2$ and $\frac{a_1}{a_2} = b \notin \mathbb{R}$ or $b \in (0, 1)$.

Then, for every solution $f(\not\equiv 0)$ of

$$f'' + (A_1(z)\mathrm{e}^{P_1(z)} + B_1)f' + (A_2(z)\mathrm{e}^{P_2(z)} + B_2)f = 0, \tag{4.1}$$

all $f^{(n)}(n \in \mathbb{Z})$ have no Baker wandering domain, that is, they only have a simply connected Fatou component.

We focus our interest on the higher differential equations (2.2) with coefficients having the same order and obtian

Theorem 4.2 Suppose that the entire coefficients $g_j(z)$ $(j = 0, 1, \cdots, k-1)$ of equation (2.2) satisfy the conditions given in Theorem 2.2. Then, for every non-trivial solution $f(z)$ of equation (2.2), $f^{(m)}(z)(m \in \mathbb{Z})$ have no Baker wandering domain, that is, they only have a simply connected Fatou component.

We now present some Lemmas.

Lemma 4.3 ([28, Corollary 1]) Let $f(z)$ be a transcendental meromorphic function with at most finitely many poles. If $\mathcal{J}(f)$ has only bounded components, then for any complex number $a \in \mathbb{C}$, there exists a constant $0 < d < 1$ and two sequences $\{r_n\}$ and $\{R_n\}$ of positive numbers with $r_n \to \infty$ and $R_n/r_n \to \infty (n \to \infty)$ such that

$$M(r, a, f)^d \leq L(r, a, f),\ r \in G,$$

where $M(r, a, f) = \max\{|f(z)| : |z - a| = r\}$, $L(r, a, f) = \min\{|f(z)| : |z - a| = r\}$ and $G = \bigcup\limits_{n=1}^{\infty} \{r : r_n < r < R_n\}$, which has an infinite logarithmic measure.

Lemma 4.4 ([6]) Let $p_j(x)$ $(j = 1, 2, \cdots, n)$ and $f(x)$ be a continuous complex value functions on the interval $[a, b]$, and let $P_j(x) (j = 1, 2, \cdots, n)$ and $F(x)$ be non-negative continuous functions with $|p_j(x)| \leq P_j(x)$ and $f(x) \leq F(x)$. Suppose that $v(x)$ and $V(x)$ are the solutions of the differential equations

$$v^{(n)} - \sum_{j=1}^{n} p_j(x) v^{(n-j)} = f(x)$$

and

$$V^{(n)} - \sum_{j=1}^{n} P_j(x) V^{(n-j)} = F(x),$$

respectively. Then, if $V^{(k)}(a) \geq |v^{(k)}(a)|$ $(k = 0, 1, \cdots, n-1)$, we have that

$$|v^{(k)}(x)| \leq V^{(k)}(x), \quad x \in [a, b].$$

We now proceed to the actual proof of Theorem 4.2.

Proof We now assume that $u(z) = f^{(m)}(z)$ $(m \in \mathbb{Z})$ has a Baker wandering domain, and complete the proof by reduction to absurdity. Zheng [27] shows that the Julia set of a transcendental meromorphic function with at most finitely many poles has only bounded components if and only if it has a Baker wandering domain. Since u is a transcendental entire function, $\mathcal{J}(u)$ has only bounded components. Thus, it follows from Lemma 4.3 that there exists $0 < d < 1$ such that

$$|u(z)| \geq M(r, u)^d,\ r \in G, \tag{4.2}$$

where G is a set with infinite logarithmic measure.

Set $H_j^+(\theta) = \{\theta : \delta(P_j, \theta) > 0\}$ and $H_j^-(\theta) = \{\theta : \delta(P_j, \theta) < 0\}$ $(j = 0, 1, \cdots, k-1)$. If one of the conditions (1) or (2) holds, we can choose a proper ϕ such that $\text{mes} H_0^+(\theta) \cap (\bigcap\limits_{j=1}^{k-1} H_j^-(\theta)) > 0$. Therefore, we further obtain from Remark 2.5(ii) that there exist odd integers $l_0, l_1, \cdots, l_{k-1}$ such that $\bigcap\limits_{j=0}^{k-1} S_{l_j}(P_j, \theta) \neq \emptyset$. If condition (3) holds, we have that $H_0^-(\theta) = H_j^-(\theta)$ $(j = 1, 2, \cdots, k-1)$. Therefore, we again obtain from Remark 2.5(ii) that $S_l(P_j, \theta) = S_l(P_0, \theta)$ $(l = 0, 1, \cdots, 2n-1,\ j = 1, 2, \cdots, k-1)$. Thus, there exist $\theta_1, \theta_2 \in \bigcap\limits_{j=0}^{k-1} S_{l_j}(P_j, \theta)$ with $\theta_1 < \theta_2$ such that

$$\delta(P_j,\theta_i)<0,\quad j=0,1,\cdots,k-1,\ i=1,2.$$

By Phragmén-Lindelöf Theorem and Lemma 2.4, there exists a positive constant M_0 such that

$$\max\{|g_j(z)|:j=0,1,\cdots,k-1\}\le M_0,\quad z\in\overline{\Omega}(\theta_1,\theta_2). \tag{4.3}$$

We now split our proof into two cases.

Case a $m\le 0$.

It follows from (2.2) that $u(z)=f^{(m)}(z)$ satisfies the differential equation

$$u^{(n)}(z)+g_{k-1}(z)u^{(n-1)}(z)+\cdots+g_1(z)g^{(n-k+1)}(z)+g_0(z)u^{(n-k)}(z)=0, \tag{4.4}$$

where $n=-m+k$.

Set $v(r)=u(r\mathrm{e}^{\mathrm{i}\theta}),\theta\in[\theta_1,\theta_2]$. Then $v^{(j)}(r)=\mathrm{e}^{\mathrm{i}j\theta}u^{(j)}(r\mathrm{e}^{\mathrm{i}\theta})$ $(j\in\mathbb{N})$, and equation (4.4) turns into

$$v^{(n)}+g_{k-1}(r\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}\theta}v^{(n-1)}+\cdots+g_1(r\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}(k-1)\theta}v^{(n-k+1)}+g_0(r\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}k\theta}v^{(n-k)}=0. \tag{4.5}$$

Set $M=\max\{M_0,M(r_0,u^{(j)}),j=0,1,\cdots,k-1\}$ and $l\ge\omega=\frac{\pi}{\theta_2-\theta_1}$. We note that

$$\left[\exp(r^l)\right]^{(s)}=P_{s(l-1)}(r)\exp(r^l),\ s\in\mathbb{N},$$

where $P_{s(l-1)}(r)$ are polynomials in r with degree $s(l-1)$. Therefore $V(r)=M\exp(r^l)$ satisfies the differential equation

$$V^{(n)}-\sum_{j=1}^{k}\frac{1}{k}\Bigg(\prod_{s=n-j+1}^{n}P_{s(l-1)}(r)\Bigg)V^{(n-j)}=0. \tag{4.6}$$

Clearly, $|v^{(j)}(r_0)|=|\mathrm{e}^{\mathrm{i}j\theta}u^{(j)}(r\mathrm{e}^{\mathrm{i}\theta})|\le V^{(j)}(r_0),\ j\in\mathbb{N}$. Thus, we conclude from Lemma 4.4, (4.5) and (4.6) that, for sufficiently large r_0 and $r\ge r_0$,

$$|f^{(m)}(r\mathrm{e}^{\mathrm{i}\theta})|=|v(r)|\le V(r)=M\exp(r^l)\text{ for all }z=r\mathrm{e}^{\mathrm{i}\theta}\in\Omega(r;\theta_1,\theta_2). \tag{4.7}$$

If $m=0$, (4.2) and (4.7) yield that

$$M(r,f)^d\le M\exp(r^l),$$

which implies that $\mu(f)\le l$, which contradicts to Lemma 2.6.

Since f is entire function, we have that $C_{\theta_1,\theta_2}(r,f^m)=0=C_{\theta_1,\theta_2}(r,u)=0$. Thus, we obtain from (4.2) and (4.7) that, for all $r\ge r_0$ and $m\le 0$,

$$S_{\theta_1,\theta_2}(r,f^{(m)})=A_{\theta_1,\theta_2}(r,f^{(m)})+B_{\theta_1,\theta_2}(r,f^{(m)})=O(r^{l-\omega}), \tag{4.8}$$

and

$$\begin{aligned}S_{\theta_1,\theta_2}(r,f^{(m)})\ge B_{\alpha,\beta}(r,f^{(m)})&=\frac{2\omega}{\pi r^{\omega}}\int_{\theta_1}^{\theta_2}\log^+|f^{(m)}(r\mathrm{e}^{\mathrm{i}\theta})|\sin\omega(\theta-\theta_1)\mathrm{d}\theta\\&\ge\frac{2\omega}{\pi r^{\omega}}\int_{\theta_1}^{\theta_2}d\log^+M(r,f^{(m)})\frac{2\omega}{\pi}(\theta-\theta_1)\mathrm{d}\theta\\&=\frac{2d}{r^{\omega}}\log M(r,f^{(m)}),\ r\in G.\end{aligned} \tag{4.9}$$

Clearly, (4.8)and (4.9) imply that $\mu(f)=\mu(f^{(m)})<\infty$, which again contradicts to Lemma 2.6.

Case b $m > 0$.

Lemma 3.5 gives that

$$\begin{aligned} S_{\theta_1+\varepsilon,\theta_2-\varepsilon}\left(r,\frac{f^{(m)}}{f}\right) &\le \sum_{j=0}^{m-1} S_{\theta_1+\varepsilon,\theta_2-\varepsilon}\left(r,\frac{f^{(j+1)}}{f^{(j)}}\right) \\ &\le K\left(\sum_{j=0}^{m-1}\log^+ S_{\theta_1,\theta_2}(r,f^{(j)})+\log r+1\right),\ r\notin F, \end{aligned} \tag{4.10}$$

where $K > 0$ and F is a set with a finite linear measure.

When $m = 1$, we deduce from (4.8) and (4.10) that

$$S_{\theta_1+\varepsilon,\theta_2-\varepsilon}\left(r,\frac{f'}{f}\right) = K\left(\log^+ S_{\theta_1,\theta_2}(r,f)+\log r+1\right) = O(\log r),\ r\notin F \tag{4.11}$$

and

$$S_{\theta_1+\varepsilon,\theta_2-\varepsilon}(r,f') \le S_{\theta_1+\varepsilon,\theta_2-\varepsilon}\left(r,\frac{f'}{f}\right) + S_{\theta_1+\varepsilon,\theta_2-\varepsilon}(r,f) = O(r^{l-\omega}),\ r\in G\backslash F. \tag{4.12}$$

By mathematical induction, we obtain from (4.11) and (4.12) that

$$S_{\theta_1+\varepsilon,\theta_2-\varepsilon}\left(r,\frac{f^{(m)}}{f}\right) = O(\log r),\ S_{\theta_1+\varepsilon,\theta_2-\varepsilon}(r,f^{(m)}) = O(r^{l-\omega}),\ r\in G\backslash F. \tag{4.13}$$

Thus, we deduce from (4.2) and (4.13) that, for $r \ge r_0$ and $m > 0$,

$$\begin{aligned} S_{\theta_1+\varepsilon,\theta_2-\varepsilon}(r,f^{(m)}) &\ge B_{\theta_1+\varepsilon,\theta_2-\varepsilon}(r,f^{(m)}) \\ &= \frac{2\omega}{\pi r^{\omega}}\int_{\theta_1+\varepsilon}^{\theta_2-\varepsilon}\log^+|f^{(m)}(re^{i\theta})|\sin\omega(\theta-\theta_1-\varepsilon)d\theta \\ &\ge \frac{2\omega}{\pi r^{\omega}}\int_{\theta_1+\varepsilon}^{\theta_2-\varepsilon} d\log^+ M(r,f^{(m)})\frac{2\omega}{\pi}(\theta-\theta_1-\varepsilon)d\theta \\ &= \frac{2d}{r^{\omega}}\log M(r,f^{(m)}),\ r\in G\backslash F. \end{aligned} \tag{4.14}$$

Obviously, (4.13) and (4.14) yield that $\mu(f) = \mu(f^{(m)}) < \infty$, which contradicts to Lemma 2.6.

Thus, Cases a and b imply that, for every non-trivial solution $f(z)$ of equation (2.2), $f^{(m)}(z)(m\in\mathbb{Z})$ have no Baker wandering domain. That is, they only have a simply connected Fatou component. □

Conflict of Interest The authors declare no conflict of interest.

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