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Lower order and limiting directions of Julia sets of solutions to second order differential equations

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A B S T R A C T

In this paper, we consider the properties of entire solutions to second order differential equation

$$f'' + Af' + Bf = 0, \tag{$*$}$$

where $A(z)$ and $B(z) \not\equiv 0$ are entire functions. Under certain assumptions on $A(z)$ and $B(z)$, we prove that every non-trivial solution f of equation (*) is of infinite lower order, and then obtain the measure estimation of the limiting directions of Julia sets for those infinite lower order entire solutions. The existence of Baker domain for $f^{(n)}$ is also discussed.

1. Introduction and main results

This paper is devoted to considering the properties of solutions to second order differential equations

$$f''(z) + A(z)f'(z) + B(z)f(z) = 0, \tag{1.1}$$

where $A(z)$ and $B(z)$ are entire functions. It's well known that every non-trivial solution of equation (1.1) is entire function. Furthermore, every non-trivial solution of equation (1.1) is of infinite order, whenever either $A(z)$ and $B(z)$ are entire functions with $\rho(A) < \rho(B)$, or $A(z)$ is a polynomial and $B(z)$ is transcendental, or $\rho(B) < \rho(A) \le \frac{1}{2}$, see Gundersen [7], Hellerstein, Miles and Rossi [11], Korhonen et al. [14], and Ozawa [20].

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We assume that reader is familiar with the fundamental results and standard notations of the Nevanlinna value distribution theory of meromorphic functions (see [10,30]). In particular, we use $\rho(f)$, resp. $\mu(f)$, to denote the order, resp. the lower order, of an entire function $f(z)$, $\lambda(f)$, resp. $\overline{\lambda}(f)$, to denote the exponent of convergence of zeros, resp. of distinct zeros, of $f(z)$ (see [30]) frequently in what follows.

Recently, a number of papers appear to proving that, under certain conditions upon $B(z)$, every non-trivial solution to equation (1.1) is of infinite order, whenever the coefficient $A(z)$ in equation (1.1) is a non-trivial solution to equation

$$w'' + P(z)w = 0, \tag{1.2}$$

where $P(z) = a_n z^n + \cdots + a_0$ is a polynomial of degree $n \geq 1$, see e.g. [16,17,28,29,31]. It is well-known that every non-trivial solution to equation (1.2) is of order $(n+2)/2$. We first recall a result of this type, see [28]:

Theorem 1.1. *Let $A(z)$ be a non-trivial solution to equation (1.2), and let $B(z)$ be a transcendental entire function with $\rho(B) < 1/2$. Then every non-trivial solution to equation (1.1) is of infinite order.*

Let $f : \mathbb{C} \to \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be a transcendental meromorphic function, and let $f^n(z)$ $(n \in \mathbb{N})$ denote the n-th iteration of f, that is, $f^1 = f, f^2 = f \circ f, \cdots, f^n = f \circ f^{n-1}$. The Fatou set $\mathcal{F}(f)$ of f is the subset of $\mathbb{C}$ where the iteration $f^n(z)$ $(n \in \mathbb{N})$ is well defined and $\{f^n(z)\}$ forms a normal family. The complement of $\mathcal{F}(f)$ is called the Julia set $\mathcal{J}(f)$ of f. It is well known that $\mathcal{F}(f)$ is open, $\mathcal{J}(f)$ is closed and non-empty. In general, the Julia set is very complicated. Some basic knowledge of complex dynamics of meromorphic functions can be found in Bergweiler's paper [6] and Zheng's book [33].

For transcendental entire function f, Baker [4] first observed that $\mathcal{J}(f)$ cannot lie in finitely many rays emanating from the origin. Qiao [22] introduced the definition of limiting direction of $\mathcal{J}(f)$, and proved that the $\mathcal{J}(f)$ of a transcendental entire function f of finite order has infinitely many limiting directions. Here, a limiting direction of $\mathcal{J}(f)$ means a limit of the set $\{\arg z_n | z_n \in \mathcal{J}(f)$ is an unbound sequence$\}$. Set

$$\Delta(f) = \{\theta \in [0, 2\pi) : \arg z = \theta \text{ is a limiting direction of } \mathcal{J}(f)\}$$

Clearly, $\Delta(f)$ is closed. We use $mes\Delta(f)$ for the linear measure of $\Delta(f)$.

If f is a transcendental entire function of finite lower order $\mu(f)$, Qiao [22] proved that $\text{mes}\Delta(f) \geq \min\{2\pi, \pi/\mu(f)\}$. Later some observations for a transcendental meromorphic function f were made by Qiu and Wu [23] and Zheng [35]: if $\mu(f) < \infty$ and $\delta(\infty, f) > 0$, then

$$\text{mes}\Delta(f) \geq \min\left\{2\pi, \frac{4}{\mu(f)} \arcsin\sqrt{\frac{\delta(\infty, f)}{2}}\right\}.$$

By using the spread relation, there are some profound results on limiting directions of entire solutions to differential equations, see e.g. [13,14,22,23,25,26,31]. We now recall a result obtained by Wang and Chen [25] as follows

Theorem 1.2. *[25, Theorem 1.2] Suppose that $A(z)$ and $B(z)$ are entire functions such that $B(z)$ is transcendental and $T(r, B) \sim \log M(r, B)$ as $r \to \infty$ outside a set of finite logarithmic measure, $A(z)$ has a finite deficient value a i.e., $\delta(a, A) > 0$. For every non-trivial solution f to equation (1.1), we have*

$$\text{mes}E(f) \geq \min\left\{2\pi, \frac{4}{\mu(A)} \arcsin\sqrt{\frac{\delta(a, A)}{2}}\right\},$$

where $E(f) = \bigcap_{n\in\mathbb{Z}} \Delta(f^{(n)})$.

In this paper, we are mainly treating to the second order differential equation (1.1). We are trying to consider the following two questions:

Question 1.3. Under what assumptions on coefficients $A(z)$ and $B(z)$, can every non-trivial solution f to equation (1.1) be of infinite lower order?

Question 1.4. What is the measure estimation of limiting directions of Julia sets for every infinite lower order entire solution f to equation (1.1)?

We are now ready to provide a positive answer to Question 1.3 and Question 1.4, and state our main results as follows.

Theorem 1.5. *Suppose that $A(z)$ is a non-trivial solution to equation (1.2) such that the number of accumulation lines of zero sequence of $A(z)$ is strictly less than $n+2$, and let $B(z)$ be a transcendental entire function satisfying $T(r,B) \sim \log M(r,B)$ as $r \to \infty$ outside a set of finite logarithmic measure. Then, every non-trivial solution f to equation (1.1) is of infinite lower order and* $\operatorname{mes}E(f) \geq \frac{2\pi}{n+2}$.

Remark 1.6. $B(z) = \sum_{n=1}^{\infty} a_n z^{\lambda_n}$ is said Fejér gaps if $\sum_{n=1}^{\infty} \lambda_n^{-1} < \infty$. Murai [19] pointed that $T(r,B) \sim \log M(r,B)$ as $r \to \infty$ outside a set of finite logarithmic measure, which shows that there really exists an entire function $B(z)$ satisfying the hypothesis in Theorem 1.5.

Remark 1.7. Let $\gamma = re^{i\theta}$ be a ray from origin. For each $\varepsilon > 0$, the exponent of convergence of the zero sequence of $g(z)$ at the ray $\gamma = re^{i\theta}$ is denoted by $\lambda_\theta(g) = \lim_{\varepsilon\to 0^+} \lambda_{\theta,\varepsilon}(g)$, where

$$\lambda_{\theta,\varepsilon}(g) = \limsup_{r\to\infty} \frac{\log^+ n(\Omega(r,\theta-\varepsilon,\theta+\varepsilon),1/g)}{\log r},$$

where $n(\Omega(r,\theta-\varepsilon,\theta+\varepsilon),1/g)$ counts the number of zeros of $g(z)$ with multiplicities in the angular sector $\Omega(r,\theta-\varepsilon,\theta+\varepsilon)$. The ray $\gamma = re^{i\theta}$ is now called an *accumulation ray* of the zero sequence of $g(z)$ if $\lambda_\theta(g) = \rho(g)$, see e.g. [17,24,27].

A natural related question is now to find different conditions that ensuring every non-trivial solution to equation (1.1) is of infinite lower order, whenever the number of accumulation rays of the zero sequence of solutions to equation (1.2) equals to $n+2$. Indeed, it follows from Lemma 2.6 below that the number of accumulation rays of the zero sequence of every non-trivial solution to equation (1.2) is not more than $n+2$, and the set of the accumulation rays of the zero sequence of every non-trivial solution to equation (1.2) is a subset of $\{\theta_j : 0 \leq j \leq n+1\}$, where $\theta_j = \frac{2j\pi - \arg a_n}{n+2}, j = 0, 1, \cdots, n+1$ mentioned in Lemma 2.6.

We now state other results of this type as follows.

Theorem 1.8. *Suppose that $A(z)$ and $B(z)$ are two linearly independent solutions to equation (1.2). If the number of accumulation rays of the zero sequence of $A(z)$ is strictly less than $n+2$, then every non-trivial solution f to equation (1.1) is of infinite lower order and* $\operatorname{mes}E(f) \geq \frac{2\pi}{n+2}$.

Theorem 1.9. *Suppose $A(z)$ is a non-trivial solution to equation (1.2) such that the number of accumulation rays of the zero sequence of $A(z)$ is strictly less than $n+2$, and let $B(z)$ be a non-trivial solution to*

$$w'' + Q(z)w = 0, \tag{1.3}$$

where $Q(z) = b_m z^m + \cdots + b_0$ *is a polynomial of degree* $m \geq 1$*, then every non-trivial solution to equation (1.1) is of infinite lower order and* $\mathrm{mes}E(f) \geq \frac{2\pi}{n+2}$.

Theorem 1.10. *Suppose* $A(z)$ *is a non-trivial solution to equation (1.2) such that the number of accumulation rays of the zero sequence of* $A(z)$ *is strictly less than* $n+2$*, and let* $B(z)$ *be a transcendental entire function with a multiply-connected Fatou component, then every non-trivial solution to equation (1.1) is of infinite lower order and* $\mathrm{mes}E(f) \geq \frac{2\pi}{n+2}$.

Theorem 1.11. *Suppose* $B(z)$ *is a non-trivial solution to equation (1.2) such that the number of accumulation rays of the zero sequence of* $B(z)$ *equals to* $n+2$ *and that* $A(z)$ *is an entire function, then every non-trivial solution* f *to equation (1.1) is of infinite lower order. Furthermore,*

(1) *if* $A(z)$ *has a finite Borel exception value, then* $\mathrm{mes}E(f) \geq \pi$*;*
(2) *if* $A(z)$ *has a finite deficient value* a*, i.e.,* $\delta(a, A) > 0$*, then*

$$\mathrm{mes}E(f) \geq \min\left\{2\pi, \frac{4}{\mu(A)} \arcsin\sqrt{\frac{\delta(a,A)}{2}}\right\}.$$

Remark 1.12. Let $A(z)$ be a non-trivial solution to equation (1.2). We denote by $p(A)$ the number of rays $\arg z = \theta_j$, which are not accumulation rays of the zero sequences of $A(z)$, where $\theta_j = \frac{2j\pi - \arg a_n}{n+2}, j = 0, 1, \ldots, n+1$ [9]. It is easy to deduce that $p(A)$ must be an even integer from Lemma 2.6. From the Hille's asymptotic theory [12], if there is an infinite number of zeros clustering around a critical ray, then the exponent of convergence of these clustering zeros near that one ray must be $\frac{n+2}{2}$. Therefore, the condition $\lambda(A) < \rho(A)$ implies that $p(A) = n+2$ by Lemma 2.6. In other words, the number of accumulation rays of the zero sequence of $A(z)$ is zero. Therefore, Theorem 1.8 yields

Corollary 1.13. *Suppose that* $A(z)$ *and* $B(z)$ *are two linearly independent solutions to equation (1.2). If* $\lambda(A) < \rho(A)$*, then every non-trivial solution* f *to equation (1.1) is of infinite lower order and* $\mathrm{mes}E(f) \geq \frac{2\pi}{n+2}$.

Theorem 1.14. *Suppose that* $A(z)$ *is a non-trivial solution to (1.2) such that the number of accumulation rays of the zero sequence of* $A(z)$ *is strictly less than* $n+2$ *and let* $B(z)$ *be a finite Borel exception value* b*, i.e.,* $B(z) - b = h(z)e^{Q(z)}$ *with* $\rho(h) < \deg Q(z)$ *and* $Q(z) = b_m z^m + \cdots + b_0, b_m \neq 0$*. If one of the following two conditions holds:*

(1) $n + 2 < 2m$*;*
(2) $n + 2 = 2m$ *and* $\arg a_n - 2\arg b_m \neq (2s+1)\pi, s \in \mathbb{Z}$*,*

then for every non-trivial solution to equation (1.1), all $f^{(n)} (n \in \mathbb{Z})$ *have no Baker wandering domain, that is, they only have simply connected Fatou component.*

2. Preliminary lemmas

We first recall Nevanlinna's Characteristic in an angle (see [33]). Assuming that $0 < \alpha < \beta < 2\pi$, we denote that

$$\Omega(\alpha, \beta) = \{z \in \mathbb{C} : \arg z \in (\alpha, \beta)\} \text{ and } \Omega(r, \alpha, \beta) = \Omega(\alpha, \beta) \cap \{z : |z| < r\},$$

and use $\overline{\Omega}(\alpha,\beta)$ and $\overline{\Omega}(r,\alpha,\beta)$ to denote the closure of $\Omega(\alpha,\beta)$ and $\Omega(r,\alpha,\beta)$, respectively. For the function $g(z)$, analytic in $\Omega(\alpha,\beta)$, we define that

$$A_{\alpha,\beta}(r,g) = \frac{\omega}{\pi}\int_1^r \left(\frac{1}{t^\omega} - \frac{t^\omega}{r^{2\omega}}\right)\{\log^+|g(re^{i\alpha})| + \log^+|g(re^{i\beta})|\}\frac{dt}{t},$$
$$B_{\alpha,\beta}(r,g) = \frac{2\omega}{\pi r^\omega}\int_\alpha^\beta \log^+|g(re^{i\theta})|\sin\omega(\theta-\alpha)d\theta,$$
$$C_{\alpha,\beta}(r,g) = 2\sum_{1<|b_\nu|<r}\left(\frac{1}{|b_\nu|^\omega} - \frac{|b_\nu|^\omega}{r^{2\omega}}\right)\sin\omega(\beta_\nu-\alpha),$$

where $\omega = \frac{\pi}{\beta-\alpha}$, $b_\nu = |b_\nu|re^{i\beta_\nu}$ are poles (counting multiplicities) of $g(z)$ in $\Omega(\alpha,\beta)$. Nevanlinna's angular characteristic of g is defined by

$$S_{\alpha,\beta}(r,g) = A_{\alpha,\beta}(r,g) + B_{\alpha,\beta}(r,g) + C_{\alpha,\beta}(r,g),$$

and the order $\rho_{\alpha,\beta}(g)$ of entire function g on $\Omega(\alpha,\beta)$ is defined by

$$\rho_{\alpha,\beta}(g) = \limsup_{r\to\infty}\frac{\log^+ S_{\alpha,\beta}(r,g)}{\log r} = \limsup_{r\to\infty}\frac{\log^+\log^+ M(r,\Omega(\alpha,\beta),g)}{\log r},$$

where $M(r,\Omega(\alpha,\beta),g) := \max\{|g(z)| : z\in\overline{\Omega}(r,\alpha,\beta)\}$.

Before proceeding to prove our theorems, we need the following lemmas.

Lemma 2.1. *[3, Theorem 1] If f is a transcendental entire function, then the Fatou set of f has no unbounded multiply connected component.*

Lemma 2.2. *[35, Lemma 2.2] Let $f(z)$ be analytic in $\Omega(r_0,\theta_1,\theta_2)$, U is a hyperbolic domain and $f : \Omega(r_0,\theta_1,\theta_2)\to U$. If there exists a point $a\in\partial U\setminus\{\infty\}$, such that $C_U(a) > 0$, then there exists a constant $d > 0$ such that for sufficiently small $\varepsilon > 0$, we have*

$$|f(z)| = O(|z|^d), z\to\infty, z\in\Omega(r_0;\theta_1+\varepsilon,\theta_2-\varepsilon).$$

Remark 2.3. [35, p. 4] The open set W is hyperbolic if $\overline{\mathbb{C}}\setminus W$ has at least three points. For any $a\in\mathbb{C}\setminus W$, we define

$$C_W(a) = \inf\{\lambda_W(z)|z-a| : \forall z\in W\},$$

where $\lambda_W(z)$ is the hyperbolic density on W. Note that $|z-a|\geq\delta_W(z)$ where $\delta_W(z)$ is the Euclidean distance of $z\in W$ to ∂W. It is well known that if every component of W is simply connected, then $C_W(a)\geq\frac{1}{2}$.

Lemma 2.4. *[32, Theorem 2.5.1] Let $f(z)$ be a meromorphic function on $\Omega(\alpha-\varepsilon,\beta+\varepsilon)$ for $\varepsilon > 0$ and $0 < \alpha < \beta < 2\pi$. Then*

$$A_{\alpha,\beta}\left(r,\frac{f'}{f}\right) + B_{\alpha,\beta}\left(r,\frac{f'}{f}\right) \leq K(\log^+ S_{\alpha-\varepsilon,\beta+\varepsilon}(r,f) + \log r + 1)$$

for $r > 1$ possibly except a set with finite linear measure.

Lemma 2.5. *[13, Lemma 2.2] Let $z = re^{i\varsigma}, r > r_0+1$ and $\alpha\leq\varsigma\leq\beta$, where $0 < \beta-\alpha\leq 2\pi$. Suppose that $g(z)$ is analytic in $\overline{\Omega}(r,\alpha,\beta)$ with $\rho_{\alpha,\beta}(g) < \infty$. Choose two real numbers, α_1 and β_1, satisfying that*

$\alpha < \alpha_1 < \beta_1 < \beta$. Then, for every $\varepsilon_j \in \left(0, \frac{\beta_j-\alpha_j}{2}\right)$ $(j = 1, 2, \cdots, n-1)$ outside a set of zero linear measure, where $n \geq 2$ is an integer, with

$$\alpha_j = \alpha + \sum_{s=1}^{j-1} \varepsilon_s, \quad \beta_j = \beta - \sum_{s=1}^{j-1} \varepsilon_s, \quad j = 2, 3, \cdots, n-1,$$

there exist $K > 0$ and $M > 0$ depending only on $g(z), \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{n-1}$ and $\Omega(\alpha_{n-1}, \beta_{n-1})$, and not depending on z, such that

$$\left|\frac{g'(z)}{g(z)}\right| \leq K r^M (\sin k(\varsigma - \alpha))^{-2}$$

and

$$\left|\frac{g^{(n)}(z)}{g(z)}\right| \leq K r^M \left(\sin k(\varsigma - \alpha) \prod_{j=1}^{n-1} \sin k_j(\varsigma - \alpha_j)\right)^{-2}$$

for all $z \in \Omega(\alpha_{n-1}, \beta_{n-1})$ outside an R-set H, where $k = \frac{\pi}{\beta-\alpha}$ and $k_j = \frac{\pi}{\beta_j-\alpha_j}, (j = 1, \cdots, n-1)$.

Furthermore, some auxiliary results of equation (1.2) are also needed. Let $A(z)$ be an entire function with finite positive order $\rho(A)$. We say that $A(z)$ blows up exponentially, resp. $A(z)$ decays to zero exponentially, in $\overline{\Omega}(\alpha, \beta)$ if, for any $\theta \in (\alpha, \beta)$,

$$\lim_{r\to\infty} \frac{\log\log|A(re^{i\theta})|}{\log r} = \rho(A), \quad \text{resp.} \quad \lim_{r\to\infty} \frac{\log\log|A(re^{i\theta})|^{-1}}{\log r} = \rho(A).$$

Lemma 2.6. *[12, Chapter 7.4] Let $A(z)$ be a non-trivial solution to equation (1.2). Set $\theta_j = \frac{2j\pi - \arg a_n}{n+2}$ and $S_j = \Omega(\theta_j, \theta_{j+1})$, where $j = 0, 1, \cdots, n+1$ and $\theta_{n+2} = \theta_0 + 2\pi$. Then $A(z)$ has the following properties:*

(1) *In each sector S_j, $A(z)$ either blows up or decays to zero exponentially.*
(2) *If, for some j, $A(z)$ decays to zero in S_j, then it must blow up in S_{j-1} and S_{j+1}. However, it is possible for $A(z)$ to blow up in several adjacent sectors.*
(3) *If $A(z)$ decays to zero in S_j, then $A(z)$ has at most finitely many zeros in any closed sub-sector within $S_{j-1} \cup \overline{S_j} \cup S_{j+1}$.*
(4) *If $A(z)$ blows up in S_{j-1} and S_j, then for each $\varepsilon > 0$, $A(z)$ has infinitely many zeros in each sector $\overline{\Omega}(\theta_j - \varepsilon, \theta_j + \varepsilon)$, and furthermore, as $r \to \infty$,*

$$n(\overline{\Omega}(r, \theta_j - \varepsilon, \theta_j + \varepsilon), 0, A) = (1 + o(1)) \frac{2\sqrt{|a_n|}}{\pi(n+2)} r^{\frac{n+2}{2}},$$

where $n(\overline{\Omega}(r, \theta_j - \varepsilon, \theta_j + \varepsilon), 0, A)$ is the numbers of zeros of $A(z)$ counting multiplicity in $\overline{\Omega}(r, \theta_j - \varepsilon, \theta_j + \varepsilon)$.

Remark 2.7. If the number of accumulation rays of zeros sequence of $A(z)$ is exactly $n + 2$, then we know $A(z)$ blows up exponentially in each sector $S_j = \Omega(\theta_j, \theta_{j+1})$ by the condition (3) of Lemma 2.6, also see [21, Lemma 2.7].

Lemma 2.8. *Suppose that $A(z)$ and $B(z)$ satisfy the hypothesis of Theorem 1.5. Then, every non-trivial solution f to equation (1.1) satisfies $\mu(f) = \infty$.*

Proof. Since the number of accumulation lines of zero sequence of $A(z)$ is strictly less than $n+2$, we obtain from Remark 1.7 that there exists at least a $j_0 \in \{0, 1, \dots, n+1\}$ such that the ray $\arg z = \theta_{j_0}$ is not the accumulation line of the zero sequence of $A(z)$. This implies that $A(z)$ decays to zero exponentially in S_{j_0-1} or S_{j_0}. Otherwise, if $A(z)$ blows up in S_{j_0-1} and S_{j_0}, we have from (4) of Lemma 2.6 that

$$\lambda_{\theta_{j_0}}(A) = \lim_{\varepsilon\to 0} \limsup_{r\to\infty} \frac{\log^+ n(\Omega(r, \theta_{j_0} - \varepsilon, \theta_{j_0} + \varepsilon), 0, A)}{\log r} = \frac{n+2}{2} = \rho(A),$$

a contradiction. Thus, without loss of generality, we assume that $A(z)$ decays to zero exponentially in sector $S_{j_0} = \Omega(\theta_{j_0}, \theta_{j_0+1}), 0 \le j_0 \le n+1$. Therefore, for any $\theta \in D_{j_0} = \{\arg z | z \in S_{j_0}\}$, we have

$$\lim_{r\to\infty} \frac{\log\log |A(re^{i\theta})|^{-1}}{\log r} = \rho(A) = \frac{n+2}{2} \tag{2.1}$$

and $\mathrm{mes}D_{j_0} = \frac{2\pi}{n+2}$. So, there exists an arbitrarily small $\varepsilon > 0$, and for all sufficiently large $|z| = r \ \ (z \in S_{j_0})$, we have

$$|A(re^{i\theta})| \le \exp(-r^{\rho(A)-\varepsilon}). \tag{2.2}$$

Set, for some constant $k \in (0, 1)$,

$$G_k(r) = \{\theta \in [0, 2\pi) : \log^+ |B(re^{i\theta})| \le k \log M(r, B)\}. \tag{2.3}$$

Since $B(z)$ is an entire function satisfying $T(r, B) \sim \log M(r, B)$ as $r \to \infty$ outside a set E_1 of finite logarithmic measure, we have from (2.3) that

$$\begin{aligned} 2\pi \log M(r, B) \sim & 2\pi m(r, B) \\ = & \int_{G_k(r)} \log^+ |B(re^{i\theta})| d\theta + \int_{[0,2\pi)\setminus G_k(r)} \log^+ |B(re^{i\theta})| d\theta \\ \le & k \mathrm{mes}G_k \log M(r, B) + (2\pi - \mathrm{mes}G_k(r)) \log M(r, B) \end{aligned} \tag{2.4}$$

as $r(\notin E_1) \to \infty$. It is not hard to see that $\mathrm{mes}G_k(r) \to 0$ as $r(\notin E_1) \to \infty$. Set

$$F_{j_0}(r) = \left\{ \theta \in D_{j_0} \setminus G_k(r) \middle| \begin{array}{l} |A(re^{i\theta})| \le \exp(-r^{\rho(A)-\varepsilon}), \\ {[M(r, B)]^k} < |B(re^{i\theta})| \end{array} \right\} \tag{2.5}$$

as $r \notin E_1$. We deduce from (2.2)-(2.5) that $\mathrm{mes}F_{j_0}(r) = \frac{2\pi}{n+2} > 0$. Set

$$F(r) = \bigcup_{j_0 \in \{0,1,\dots,n+1\}} F_{j_0}(r). \tag{2.6}$$

Then

$$F(r) = \left\{ \theta \in [0, 2\pi) \middle| \begin{array}{l} |A(re^{i\theta})| \le \exp(-r^{\rho(A)-\varepsilon}), \\ {[M(r, B)]^k} < |B(re^{i\theta})| \end{array} \right\} \tag{2.7}$$

as $r \notin E_1$.

We now have from the estimation of the logarithmic derivative given by Gundersen [8, Theorem 3] that

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq C\left(\frac{T(\alpha r, f)}{r}\log^{\alpha} r \log T(\alpha r, f)\right)^{j}, \quad j = 1, 2 \tag{2.8}$$

for all z satisfying $|z| \notin E_2 \cup [0, 1]$, where $E_2 \subset (1, \infty)$ is a set of finite linear measure, $C > 0$ and $\alpha > 1$ are constants.

Thus, it follows from (1.1), (2.7) and (2.8) that there exists an sequence $z = re^{i\theta}$ such that for all sufficient large $r \notin E_1 \cup E_2 \cup [0, 1]$ and for $\theta = \arg z \in F(r)$, we have

$$\begin{aligned}(M(r, B))^{k} < |B(z)| &\leq C(T(2r, f))^{4}(1 + \exp(-r^{\rho(A)-\varepsilon})) \\ &\leq C(T(2r, f))^{4}(1 + o(1)),\end{aligned} \tag{2.9}$$

where $C > 0$ is a constant. Since $B(z)$ is a transcendental entire function, we know that

$$\liminf_{r\to\infty} \frac{\log M(r, B)}{\log r} = +\infty. \tag{2.10}$$

Therefore, we obtain from (2.9) and (2.10) that $\mu(f) = \infty$. □

Lemma 2.9. *[18] Suppose that $P(z) = a_n z^n + \cdots + a_0 (n \in \mathbb{N}^+)$ is a non-constant polynomial, and that $g(z)(\not\equiv 0)$ is an entire function with $\rho(g) < n$. Set $A(z) = g(z)e^{P(z)}$, $z = re^{i\theta}$, and $\delta(P, \theta) = \Re(a_n e^{i\theta})$. Then for any given $\varepsilon > 0$, there exists a set $H_1 \subset [0, 2\pi)$ of linear measure zero such that for any $\theta \in [0, 2\pi) \setminus (H_1 \cup H_2)$, there is $R > 0$ such that for $|z| = r > R$, we have*

(1) *if $\delta(P, \theta) > 0$, then*

$$\exp\{(1 - \varepsilon)\delta(P, \theta)r^n\} < |A(re^{i\theta})| < \exp\{(1 + \varepsilon)\delta(P, \theta)r^n\};$$

(2) *if $\delta(P, \theta) < 0$, then*

$$\exp\{(1 + \varepsilon)\delta(P, \theta)r^n\} < |A(re^{i\theta})| < \exp\{(1 - \varepsilon)\delta(P, \theta)r^n\},$$

where $H_2 = \{\theta \in [0, 2\pi) : \delta(P, \theta) = 0\}$.

Remark 2.10. For the polynomial $P(z)$, we define

$$S_j(P, \theta) = \left\{\theta : -\frac{\arg a_n}{n} + (2j - 1)\frac{\pi}{2n} < \theta < -\frac{\arg a_n}{n} + (2j + 1)\frac{\pi}{2n}\right\}$$

for $j = 0, 1, \cdots, 2n - 1$. From the basic property of polynomials [18], if $\theta \in S_j(P, \theta)$, then $\delta(P, \theta) > 0$ for even j, and $\delta(P, \delta) < 0$ for odd j.

Lemma 2.11. *[1] Let $f(z)$ be a meromorphic function of finite lower order $\mu := \mu(f)$, and have one deficient value a. Let $\Lambda(r)$ be a positive function with $\Lambda(r) = o(T(r, f))$ as $r \to \infty$. Then for any fixed sequence of Pólya peaks $\{r_n\}$ of order μ, we have*

$$\liminf_{r\to\infty} \operatorname{mes} D_{\Lambda}(r_n, a) \geq \min\left\{2\pi, \frac{4}{\mu}\arcsin\sqrt{\frac{\delta(a, f)}{2}}\right\},$$

where $D_\Lambda(r, a)$ is defined by

$$D_\Lambda(r,\infty)=\{\theta\in[-\pi,\pi):|f(re^{i\theta})|>e^{\Lambda(r)}\}$$

and for finite a,

$$D_\Lambda(r,a)=\{\theta\in[-\pi,\pi):|f(re^{i\theta}-a)|<e^{-\Lambda(r)}\}.$$

Baker [2] showed that for a transcendental meromorphic function, every multiply-connected Fatou component has a Baker wandering domain. From [34], $\mathcal{J}(f)$ has only bounded components if a transcendental meromorphic function f has a Baker wandering domain. Thus, every multiply-connected Fatou component of a transcendental meromorphic function f has only bounded Julia components. The following Lemma 2.12 can be applied to a transcendental meromorphic function having a multiply-connected Fatou component.

Lemma 2.12. *[34, Corollary 1] Suppose f is a transcendental meromorphic function having at most finite poles. If $\mathcal{J}(f)$ has only bounded components, then for any complex number, there exists a constant $0<\beta<1$ and two sequences of positive numbers $\{r_n\}$ and $\{R_n\}$ with $r_n\to\infty$ and $R_n/r_n\to\infty(n\to\infty)$ such that*

$$M(r,f)^\beta\le L(r,f)\quad \textit{for}\quad r\in H,$$

where $H=\bigcup_{n=1}^{\infty}\{r:r_n<r<R_n\}$.

3. Proof of Theorem 1.5

Proof. Lemma 2.8 shows that every non-trivial solution f to equation (1.1) satisfies $\mu(f)=\infty$. Thus, we then estimate the measure of $E(f)$. Suppose, contrary to the assertion, that $\text{mes}E(f)<\frac{2\pi}{n+2}:=\sigma$, and so $t:=\sigma-\text{mes}E(f)>0$.

Since $E(f)$ is a closed, we have $\Phi:=(0,2\pi)\setminus E(f)$ is open and Φ can be covered by at most countably many open intervals. Thus, we can choose finitely many open intervals $I_i=(\alpha_i,\beta_i)$ $(i=1,2,\cdots,m)$ in Φ such that

$$\text{mes}\left(\Phi\setminus\bigcup_{i=1}^{m}I_i\right)<\frac{t}{4}. \tag{3.1}$$

Furthermore, it is easy to see that

$$(\alpha_i,\beta_i)\cap E(f)=\emptyset\ \text{ and }\ \Omega(r;\alpha_i,\beta_i)\cap\mathcal{J}(f^{(n_i)})=\emptyset \tag{3.2}$$

for sufficiently large r. It follows from Lemma 2.1 and (3.2) that, for each $i=1,2,\cdots,m$, there exist the corresponding r_i and an unbounded Fatou component U_i of $\mathcal{F}(f^{(n_i)})$ such that $\Omega(r_i,\alpha_i,\beta_i)\subset U_i$. Therefore, we take a unbounded and connected closed section Γ_i on boundary ∂U_i such that $\mathbb{C}\setminus\Gamma_i$ is simply connected. Clearly, $\mathbb{C}\setminus\Gamma_i$ is hyperbolic and open. By Remark 2.3, we have $C_{\mathbb{C}\setminus\Gamma_i}(a)\ge\frac{1}{2}(a\in\Gamma_i)$. Since the mapping $f^{(n_i)}:\Omega(r_i;\alpha_i,\beta_i)\to\mathbb{C}\setminus\Gamma_i$ is analytic for all i, it follows from Lemma 2.2 that there exists a positive constant d such that

$$|f^{(n_i)}(z)|=O(|z|^d)\quad\text{as}\quad|z|\to\infty \tag{3.3}$$

for $z\in\bigcup_{i=1}^{m}\Omega(r_i,\alpha_i+\varepsilon,\beta_i-\varepsilon)$.

Case 3.1. $n_i > 0$. We note that

$$f^{(n_i-1)}(z) = \int_0^z f^{(n_i)}(\zeta)d\zeta + c,$$

where c is a constant, and the integral path is the segment of a straight line from 0 to z. From this and (3.3), it is easy to deduce $|f^{(n_i-1)}(z)| = O(|z|^{d+1})$ for $z \in \bigcup_{i=1}^{m} \Omega(r_i, \alpha_i + \varepsilon, \beta_i - \varepsilon)$. Repeating the discussion n_i times, we can obtain

$$|f(z)| = O(|z|^{d+n_i}) \quad \text{for } z \in \bigcup_{i=1}^{m} \Omega(r_i, \alpha_i + \varepsilon, \beta_i - \varepsilon).$$

Thus, we immediately have

$$S_{\alpha_i+\varepsilon,\beta_i-\varepsilon}(r, f) = O(\log r), \quad i = 1, 2, \cdots, m. \tag{3.4}$$

Case 3.2. $n_i < 0$. For any angular domain $\Omega(\alpha, \beta)$, we have

$$S_{\alpha,\beta}(r, f^{(n_i+1)}) \leq S_{\alpha,\beta}\left(r, \frac{f^{(n_i+1)}}{f^{(n_i)}}\right) + S_{\alpha,\beta}(r, f^{(n_i)}).$$

Thus, we obtain from (3.3) and Lemma 2.4 that

$$S_{\alpha_i+\varepsilon',\beta_i-\varepsilon'}(r, f^{(n_i+1)}) = O(\log r)$$

for $|n_i|\varepsilon' = \varepsilon$. Repeating the discussion $|n_i|$ times, we also obtain

$$S_{\alpha_i+\varepsilon,\beta_i-\varepsilon}(r, f) = O(\log r) \tag{3.5}$$

By Lemma 2.5, there exists two constants $M > 0$ and $K > 0$ such that

$$\left|\frac{f^{(s)}(z)}{f(z)}\right| \leq Kr^M \ \ (s = 1, 2, \cdots, n) \tag{3.6}$$

for all $z \in \bigcup_{i=1}^{m} \Omega(\alpha_i + 2\varepsilon, \beta_i - 2\varepsilon)$ outside a R-set H.

It follows from (2.5) and (2.6) that there exists a subsequence $\{r_n\}(r_n \notin E_1)$ with $\lim_{n\to\infty} r_n = \infty$ satisfying

$$F(r_n) = \left\{\theta \in [0, 2\pi) \middle| \begin{array}{l} |A(r_n e^{i\theta})| \leq \exp(-r_n^{\rho(A)-\varepsilon}), \\ [M(r_n, B)]^k < |B(r_n e^{i\theta})| \end{array}\right\},$$

and $\text{mes}F(r_n) = \text{mes}F(r) \geq \text{mes}F_{j_0}(r) = \frac{2\pi}{n+2} > 0$, which means that

$$\text{mes}F(r_n) = \text{mes}\left\{\theta \in [0, 2\pi) \middle| \begin{array}{l} |A(r_n e^{i\theta}) \leq \exp(-r_n^{\rho(A)-\varepsilon}), \\ [M(r_n, B)]^k < |B(r_n e^{i\theta})| \end{array}\right\} \geq \frac{2\pi}{n+2} = \sigma. \tag{3.7}$$

Next, we assert that the intersection of $F(r_n)$ and $\bigcup_{i=1}^m I_i^*$ is non-empty, where $I_i^* = (\alpha_i + 2\varepsilon, \beta_i - 2\varepsilon)$. By $\bigcup_{i=1}^m I_i \subset \Phi$, it is easy to have that

$$\begin{aligned}\operatorname{mes}\left(F(r_n)\bigcap\left(\bigcup_{i=1}^m I_i\right)\right) &= \operatorname{mes}\left(\Phi\bigcap F(r_n)\right) - \operatorname{mes}\left(\left(\Phi\setminus\bigcup_{i=1}^m I_i\right)\bigcap F(r_n)\right)\\ &\geq \operatorname{mes}\left(F(r_n)\setminus\left(E(f)\bigcap F(r_n)\right)\right) - \operatorname{mes}\left(\Phi\setminus\bigcup_{i=1}^m I_i\right).\end{aligned}$$

(3.1) and (3.7) yield that

$$\begin{aligned}\operatorname{mes}\left(F(r_n)\bigcap\left(\bigcup_{i=1}^m I_i\right)\right) &\geq \operatorname{mes}F(r_n) - \operatorname{mes}E(f) - \operatorname{mes}\left(\Phi\setminus\bigcup_{i=1}^m I_i\right)\\ &= \sigma - \operatorname{mes}E(f) - \operatorname{mes}\left(\Phi\setminus\bigcup_{i=1}^m I_i\right) \geq \frac{3}{4}t > 0.\end{aligned}$$

On the other hand,

$$\operatorname{mes}\left(\bigcup_{i=1}^m I_i^*\right) \geq \operatorname{mes}\left(\bigcup_{i=1}^m I_i\right) - 2\varepsilon m. \tag{3.8}$$

If we take ε sufficiently small, we can conclude that

$$\operatorname{mes}\left(F(r_n)\cap\bigcup_{i=1}^m I_i^*\right) \geq \frac{3}{8}t.$$

Thus, there must exist an open interval I_k^* of all I_i^* such that $F(r_n)\cap I_k^* \neq \emptyset$ as $\varepsilon \to 0$ and for infinitely many n,

$$\operatorname{mes}(F(r_n)\cap I_k^*) > \frac{3t}{8m} > 0.$$

According to (1.1), (3.6) and (3.7), for any $\theta \in F(r_n)\cap I_k^*$, we have

$$[M(r_n, B)]^k < |B(r_n e^{i\theta})| \leq O(r_n^M)\left(1 + \exp(-r_n^{\rho(A)-\varepsilon})\right)$$

as $r_n(\notin (E_1 \cup H)) \to \infty$. This contradicts the assumption that $B(z)$ is a transcendental entire function. Thus the proof of Theorem 1.5 is completed. □

4. Proof of Theorem 1.8

Proof. We firstly prove that every non-trivial solution f to equation (1.1) satisfies $\mu(f) = \infty$. By the assumptions of Theorem 1.8 and Remark 1.12, we obtain that $p(A) \geq 2$. Similar to the proof of Lemma 2.8, there exists at least a sector of the $n+2$ sectors, say $S_{j_0}, 0 \leq j_0 \leq n+1$ such that, for any $\theta \in D_{j_0} = \{\arg z | z \in S_{j_0}\}$, $\operatorname{mes}D_{j_0} = \frac{2\pi}{n+2}$. Thus, (2.1) and (2.2) hold for an arbitrarily small $\varepsilon > 0$ and $\theta \in D_{j_0}$.

By the Proof of [17, Theorem 1.8], it is impossible that both $A(z)$ and $B(z)$ decay to zero exponentially in a common sector. Hence, $B(z)$ blows up exponentially in S_{j_0}, that is, for any $\theta \in D_{j_0}$,

$$\lim_{r\to\infty}\frac{\log\log|B(re^{i\theta})|}{\log r}=\rho(B)=\frac{n+2}{2}. \tag{4.1}$$

Set

$$F_0(r)=\left\{\theta\in[0,2\pi)\left|\begin{array}{c}|A(re^{i\theta})|\le\exp(-r^{\rho(A)-\varepsilon}),\\ |B(re^{i\theta})|\ge\exp(r^{\rho(B)-\varepsilon})\end{array}\right.\right\}, \tag{4.2}$$

and so $\mathrm{mes}F_0(r)=\mathrm{mes}D_{j_0}=\frac{2\pi}{n+2}>0$.

Thus, we obtain from (1.1), (2.8) and (4.2) that there exists a sequence of points $z=re^{i\theta}$ such that, for all sufficient large $r\notin E_2\cup[0,1]$ and for $\theta=\arg z\in F_0(r)$,

$$\begin{aligned}\exp(r^{\rho(B)-\varepsilon})\le|B(re^{i\theta})|&\le C(T(2r,f))^4(1+\exp(-r^{\rho(A)-\varepsilon}))\\&\le C(T(2r,f))^4(1+o(1))\end{aligned}$$

where $C>0$ is a constant. Thus, we get $\mu(f)=\infty$.

We secondly prove $\mathrm{mes}E(f)\ge\frac{2\pi}{n+2}$. Suppose, contrary to the assertion, that $\mathrm{mes}E(f)<\frac{2\pi}{n+2}:=\sigma$, and so $t:=\sigma-\mathrm{mes}E(f)>0$. Choose a sequence $\{r_n\}$ with $\lim\limits_{n\to\infty}r_n=\infty$ satisfying

$$F_0(r_n)=\left\{\theta\in[0,2\pi)\left|\begin{array}{c}|A(r_ne^{i\theta})|\le\exp(-r_n^{\rho(A)-\varepsilon}),\\ |B(r_ne^{i\theta})|\ge\exp(r_n^{\rho(B)-\varepsilon})\end{array}\right.\right\} \tag{4.3}$$

and so $\mathrm{mes}F_0(r_n)=\mathrm{mes}F_0(r)\ge\frac{2\pi}{n+2}$.

Similar to the proof of Theorem 1.5, we get that

$$\mathrm{mes}\left(F_0(r_n)\cap\bigcup_{i=1}^{m}I_i^*\right)\ge\frac{3}{8}t,$$

for all sufficiently small ε. Thus, we obtain from (1.1), (3.6) and (4.3) that, for $\theta\in F_0(r_n)\cap I_i^*$,

$$\exp(r_n^{\rho(B)-\varepsilon})\le|B(r_ne^{i\theta})|\le O(r_n^M)\left(1+\exp(-r_n^{\rho(A)-\varepsilon})\right)$$

as $r_n(\notin H)\to\infty$, a contradiction. Therefore, we have $\mathrm{mes}E(f)\ge\sigma$. □

5. Proof of Theorem 1.9

Proof. By Lemma 2.6, we obtain that

$$\theta_j(A)=\frac{2j\pi-\arg a_n}{n+2}\quad\text{and}\quad\theta_k(B)=\frac{2k\pi-\arg b_m}{m+2}.$$

Suppose that $S_j(A)=\Omega(\theta_j(A),\theta_{j+1}(A))$ and $S_k(B)=\Omega(\theta_k(B),\theta_{k+1}(B))$, where $j=0,\cdots,n+1;k=0,\cdots,m+1$. Since the number of accumulation rays of the zero sequence of $A(z)$ is strictly less than $n+2$, there exists a $j_0\in\{0,\cdots,n+1\}$ such that $A(z)$ decays to zero exponentially in $S_{j_0}(A)$.

We now discuss the following three cases.

Case 1. $m=n$.

Case 1.1. $\arg a_n=\arg b_m$.

Obviously, $\theta_j(A)=\theta_k(B)$. Then for $\theta\in(\theta_{j_0}(A),\theta_{j_0+1}(A))$, $A(z)$ and $B(z)$ have two possible growth types on the ray $\arg z=\theta$:

Type a. $A(re^{i\theta})$ satisfies (2.1) and $B(re^{i\theta})$ satisfies (4.1).
Type b. $A(re^{i\theta})$ satisfies (2.1) and $B(re^{i\theta})$ satisfies

$$\lim_{r\to\infty}\frac{\log\log|B(re^{i\theta})|^{-1}}{\log r}=\rho(B)=\frac{n+2}{2}. \tag{5.1}$$

We now assert that $A(re^{i\theta})$ and $B(re^{i\theta})$ just satisfy Type a in $S_{j_0}(A)$. Otherwise, suppose that $|f''(z)|$ is unbounded on the ray $\arg z=\theta$. Using [15, Lemma 3.1], there exists an infinite sequence of points $z_l=r_le^{i\theta}$ tending to infinity such that $f''(z_l)\to\infty$ and

$$\left|\frac{f^{(s)}(z_l)}{f''(z_l)}\right|\le\frac{1}{(2-s)!}(1+o(1))|z_l|^{2-s},\quad s=0,1,$$

as $l\to\infty$. It follows from (1.1) and Type b that

$$\begin{aligned}1&\le|A(z_l)|\left|\frac{f'(z_l)}{f''(z_l)}\right|+|B(z_l)|\left|\frac{f(z_l)}{f''(z_l)}\right|\\&\le(1+o(1))|z_l|^2\exp\{-r_l^{\frac{n+2}{2}-\varepsilon}\}\to0,\quad\text{as}\quad l\to\infty.\end{aligned}$$

This contradiction implies that $|f''(z)|$ is bounded on the ray $\arg z=\theta$. Therefore, $|f(z)|\le M|z|^2$ on the ray $\arg z=\theta$, where M is a positive constant. Furthermore, $|f(z)|\le M|z|^2$ for $z\in\mathbb{C}$ by the Phragmén-Lindelöf principle, contradicting to the fact that f is transcendental.

Based on Type a, we set

$$F_0(r)=\left\{\theta\in[0,2\pi)\left|\begin{array}{c}|A(re^{i\theta})|\le\exp(-r^{\rho(A)-\varepsilon}),\\|B(re^{i\theta})|\ge\exp(r^{\rho(B)-\varepsilon})\end{array}\right.\right\}, \tag{5.2}$$

and so $\text{mes}F_0(r)\ge\text{mes}D_{j_0}=\frac{2\pi}{n+2}$. It follows from (1.1), (2.8) and (5.2) that there exists a sequence of points $z=re^{i\theta}$ such that for $\theta\in F_0(r)$ and for all sufficient large $|z|=r\notin E_2\cup[0,1]$, we have

$$\begin{aligned}\exp(r^{\frac{n+2}{2}-\varepsilon})\le|B(z)|&\le C(T(2r,f))^4(1+\exp(-r^{\frac{n+2}{2}-\varepsilon}))\\&\le C(T(2r,f))^4(1+o(1))\end{aligned}$$

where $C>0$ is a constant. Thus, we obtain $\mu(f)=\infty$.

The remainder is trivial by similar reasoning as in the proof of Theorem 1.8.

Subcase 1.2. $\arg a_n\neq\arg b_m$.

Without loss of generality, we assume that $\arg a_n>\arg b_m$. For $z\in S_{j_0}(A)$, we set

$$\Omega_1=S_{j_0}(A)\cap S_{j_0}(B)=\{z:\theta_{j_0}(B)<\arg z<\theta_{j_0+1}(A)\},$$

and

$$\Omega_2=S_{j_0}(A)\setminus S_{j_0}(B)=\{z:\theta_{j_0}(A)<\arg z<\theta_{j_0}(B)\}.$$

Obviously, $A(z)$ and $B(z)$ satisfy one of Type a and Type b on the ray $\arg z=\theta\in(\theta_{j_0}(B),\theta_{j_0+1}(A))$.

If $A(re^{i\theta})$ and $B(re^{i\theta})$ satisfy Type a in Ω_1, it means that $B(re^{i\theta})$ blows up exponentially in $S_{j_0}(B)$. According to Lemma 2.6, $A(z)$ and $B(z)$ also have two possible growth types in Ω_2. One is that $A(z)$ and

$B(z)$ satisfy Type a in Ω_2, another is that $A(z)$ and $B(z)$ satisfy Type b in Ω_2. However, from the proof of Subcase 1.1, we know that $A(z)$ and $B(z)$ only satisfy Type a in Ω_2.

If $A(re^{i\theta})$ and $B(re^{i\theta})$ satisfy the growth Type b in Ω_1, it is impossible by the proof of Subcase 1.1.

Hence, $A(re^{i\theta})$ and $B(re^{i\theta})$ satisfy Type a in $S_{j_0}(A)$. Using the method of the proof of Subcase 1.1, we again obtain $\mu(f) = \infty$ and $\text{mes}E(f) \geq \frac{n+2}{2}$.

Case 2. $m < n$.

For $z \in S_{j_0}(A)$, we split our proof into two subcases.

Subcase 2.1. For j_0, there exists a $k_0 \ \ (k_0 = 0, \cdots, m+1)$ such that $S_{j_0}(A) \subset S_{k_0}(B)$. Similar to Subcase 1.1, $A(re^{i\theta})$ and $B(re^{i\theta})$ satisfy Type a in $S_{j_0}(A)$.

Subcase 2.2. For j_0, there exists a $k_0 \ \ (k_0 = 0, \cdots, m+1)$ such that $S_{j_0}(A)$ is not a subset of $S_{k_0}(B)$ and $S_{j_0}(A) \cap S_{k_0}(B) \neq \emptyset$. Let

$$\Omega_1 = S_{j_0}(A) \cap S_{k_0}(B) \quad \text{and} \quad \Omega_2 = S_{j_0}(A) \setminus S_{k_0}(B).$$

We now divide $S_{j_0}(A)$ into Ω_1 and Ω_2. Similar to Subcase 1.2, we obtain $A(re^{i\theta})$ and $B(re^{i\theta})$ satisfy Type a in $S_{j_0}(A)$.

Similar to Case 1, we also have $\mu(f) = \infty$ and $\text{mes}E(f) \geq \frac{n+2}{2}$.

Case 3. $m > n$.

For $z \in S_{j_0}(A)$, we again split our proof into two subcases.

Subcase 3.1. For j_0, there exists a $k_0 \ \ (k_0 = 0, \cdots, m+1)$ such that $S_{j_0}(A) \supset S_{k_0}(B)$. We divide $S_{j_0}(A)$ into $S_{k_0}(B)$ and $S_{j_0}(A) \setminus S_{k_0}(B)$. In $S_{k_0}(B)$, either $A(z)$ and $B(z)$ both decay to zero exponentially or $A(z)$ decays to zero exponentially and $B(z)$ blows up. It is easy to know that $A(z)$ decays to zero exponentially and $B(z)$ blows up in $S_{k_0}(B)$. Similar to the above, we get $A(z)$ decays to zero exponentially and $B(z)$ blows up in $S_{j_0}(A) \setminus S_{k_0}(B)$.

Subcase 3.2. For j_0, there exists a $k_0 \ \ (k_0 = 0, \cdots, m+1)$ such that $S_{k_0}(B)$ is not a subset of $S_{j_0}(A)$ and $S_{j_0}(A) \cap S_{k_0}(B) \neq \emptyset$. Similarly, we divide $S_{j_0}(A)$ into two sectors. Then $A(z)$ decays to zero exponentially and $B(z)$ blows up in $S_{j_0}(A)$.

Similar to Case 1, we again have $\mu(f) = \infty$ and $\text{mes}E(f) \geq \frac{n+2}{2}$. $\square$

6. Proof of Theorem 1.10

Proof. Let f be a non-trivial solution to equation (1.1). Since the number of accumulation lines of zero sequence of $A(z)$ is strictly less than $n+2$, there exists at least a sector $S_{j_0}(0 \leq j_0 \leq n+1)$ such that, for any $\theta \in D_{j_0} = \{\arg z | z \in S_{j_0}\}$, $\text{mes}D_{j_0} = \frac{2\pi}{n+2}$. Thus, (2.1) and (2.2) hold for an arbitrarily small $\varepsilon > 0$ and $\theta \in D_{j_0}$.

Since $B(z)$ is a transcendental entire function with a multiply-connected Fatou component, we obtain from Lemma 2.12 that, for $0 < \beta < 1$ and $r \in H_1 = \bigcup_{n=1}^{\infty}\{r : r_n < r < R_n\}$,

$$M(r,B)^{\beta} \leq L(r,B) \leq |B(re^{i\theta})| \tag{6.1}$$

Thus, it follows from (1.1), (2.2), (2.8) and (6.1) that

$$M(r,B)^{\beta} < |B(re^{i\theta})| \leq C(T(2r,f))^4 \left(1 + \exp(-r^{\rho(A)-\varepsilon})\right) \tag{6.2}$$

for large $r \in H_2 \setminus (E_1 \cup [0,1])$ and $\theta \in D_{j_0}$. Thus, we obtain from (2.10) and (6.2) that $\mu(f) = \infty$.

Set

$$F_{j_0}(r) = \left\{ \theta \in D_{j_0} \middle| \begin{array}{c} |A(re^{i\theta})| \le \exp(-r^{\rho(A)-\varepsilon}), \\ [M(r,B)]^\beta < |B(re^{i\theta})| \end{array} \right\}$$

as $r(\in H_1) \to \infty$, and

$$\begin{aligned} F(r) &= \bigcup_{j_0 \in \{0,1,\ldots,n+1\}} F_{j_0}(r) \\ &= \left\{ \theta \in [0,2\pi) \middle| \begin{array}{c} |A(re^{i\theta})| \le \exp(-r^{\rho(A)-\varepsilon}), \\ [M(r,B)]^\beta < |B(re^{i\theta})| \end{array} \right\} \end{aligned} \tag{6.3}$$

as $r(\in H_1) \to \infty$. Then we get that mes$F(r) \ge$ mes$F_{j_0}(r) = \frac{2\pi}{n+2}$. The remainder is similar to the proof of Theorem 1.5, for $\theta \in F(r) \cap I_i^*$, we obtain from (1.1), (3.6) and (6.3) that

$$[M(r,B)]^\beta < |B(re^{i\theta})| \le O(r^M)\left(1 + \exp(-r^{\rho(A)-\varepsilon})\right)$$

as $r(\in H_1 \setminus H) \to \infty$, contradicting to the assumption that $B(z)$ is a transcendental entire function. Hence, Theorem 1.10 is arrived. □

7. Proof of Theorem 1.11

Proof. Since the number of accumulation lines of zero sequence of $B(z)$ equals to $n+2$, we know that $B(z)$ blows up exponentially in every sector $S_j (0 \le j \le n+1)$ by Remark 2.7, and (4.1) holds for any $\theta \in S = \left\{ \arg z | z \in \bigcup_{j=0}^{n+1} S_j \right\}$. Furthermore, there exists an arbitrarily small $\varepsilon > 0$ such that, for $z \in \bigcup_{j=0}^{n+1} S_j$,

$$|B(re^{i\theta})| \ge \exp(r^{\rho(B)-\varepsilon}). \tag{7.1}$$

(1) If $c \in \mathbb{C}$ is a Borel exceptional value of $A(z)$, then

$$A(z) - c = g(z)e^{Q(z)}, \tag{7.2}$$

with $Q(z) = b_m z^m + \cdots + b_0 \quad (b_m \ne 0)$ and $\rho(g) < \rho(A) = \deg Q(z)$. By Lemma 2.9 and Remark 2.10, we set, for $q = 0, 1, \ldots, 2m-1$,

$$D_q(Q,\theta) = \left\{ \theta : -\frac{\arg b_m}{m} + \frac{(2q-1)\pi}{2m} < \theta < -\frac{\arg b_m}{m} + \frac{(2q+1)\pi}{2m} \right\}.$$

Obviously,

$$\text{mes}D_q(Q,\theta) = \frac{\pi}{m},$$

and, for any $0 \le q_1 \ne q_2 \le 2m-1$,

$$D_{q_1}(Q,\theta) \cap D_{q_2}(Q,\theta) = \emptyset.$$

Since $\rho(g) < \rho(A) = m$, it follows from (7.2) and Lemma 2.9 that

$$|A(z) - c| \le \exp\{(1-\varepsilon)\delta(Q,\theta)r^m\} \tag{7.3}$$

as $|z| \to \infty$ for $\theta \in D_q(Q,\theta) \setminus H_2$ with odd q and zero linear measure set $H_2 \subset [0, 2\pi)$.

According to Remark 2.10, $D_q(Q,\theta)$ with odd q have m open intervals. Thus, there exists a sector S_k such that $\theta \in S_k \cap D_q(Q,\theta) \setminus H_2$ with odd q, we still have (7.1) holds. It follows from (1.1), (2.8), (7.1) and (7.3) that there exists a sequence $z = re^{i\theta}$ such that for $\theta \in S_k \cap D_q(Q,\theta) \setminus H_2$ with odd q, and for all sufficient large $r \notin E_2 \cup [0, 1]$, we have

$$\begin{aligned}\exp(r^{\rho(B)-\varepsilon}) &\le \left|\frac{f''(z)}{f(z)}\right| + |(A(z)-c)+c|\left|\frac{f'(z)}{f(z)}\right| \\ &\le C(T(2r,f))^4\left(\exp\{(1-\varepsilon)\delta(Q,\theta)r^d\}+c+1\right) \\ &\le C(T(2r,f))^4(1+c+o(1)).\end{aligned}$$

Therefore, every non-trivial solution to (1.1) satisfies $\mu(f) = \infty$.

We then affirm that the union of such $D_q(Q,\theta)$ is contained in $E(f)$ and $\text{mes}E(f) \ge \pi$. Otherwise, there must exists a $D_{q_0}(Q,\theta) \not\subseteq E(f)$ with odd q_0. By [26, Lemma 2.5], there exists an interval $(\alpha, \beta) \subseteq D_{q_0}(Q,\theta)$ such that

$$\left|\frac{f^{(s)}(z)}{f(z)}\right| \le Kr^M \quad (s = 1, 2) \tag{7.4}$$

for all $z \in \Omega(\alpha, \beta)$ with $|z| = r \notin E_3$, where $\text{mes}E_3 < \infty$ and K, M are positive constants. Substituting (7.1), (7.3) and (7.4) into (1.1), we obtain that, for $\theta \in (\alpha, \beta)$ and sufficiently large $r \notin H_2 \cup E_3$,

$$\exp(r^{\rho(B)-\varepsilon}) \le |B(z)| \le Kr^M\left(1+|c|+\exp\{(1-\varepsilon)\delta(Q,\theta)r^d\}\right),$$

which is impossible since $\delta(Q,\theta) < 0$.

(2) Since $A(z)$ has a finite deficient value a, we obtain from Lemma 2.11 that there exists an increasing and unbounded sequence $\{r_k\}$ such that

$$\text{mes}D(r_k) \ge \sigma - t/4,$$

where $D(r) = \{\theta \in [-\pi, \pi) : \log|A(re^{i\theta}) - a| < 1\}$ for all $r_k \notin \{|z| : z \in H_3\}$ with a R-set H_3.

Obviously,

$$|A(r_ke^{i\theta})| \le e + |a| \tag{7.5}$$

for $\theta \in D(r_k)$.

Thus, we have from (1.1), (2.8), (7.1) and (7.5) that

$$\exp(r_{k'}^{\rho(B)-\varepsilon}) \le \left|\frac{f''(r_{k'}e^{i\theta})}{f(r_{k'}e^{i\theta})}\right| + |A(r_{k'}e^{i\theta})|\left|\frac{f'(r_{k'}e^{i\theta})}{f(r_{k'}e^{i\theta})}\right| \le C(T(2r_{k'},f))^4(1+e+|a|)$$

for all sufficient large $r_{k'}(\in \{r_k\}) \notin E_2 \cup [0, 1]$ and for $\theta \in D(r_{k'}) \cap S(r_{k'})$. Therefore, we obtain $\mu(f) = \infty$.

Next, we assume that

$$\text{mes}E(f) < \sigma := \min\left\{2\pi, \frac{4}{\mu(A)}\arcsin\sqrt{\frac{\delta(a,A)}{2}}\right\},$$

then $t = \sigma - \text{mes}E(f) > 0$. Similarly as in the proof of Theorem 1.5, we denote $F_2(r_k) = D(r_k) \cap S(r_k)$ and so

$$\text{mes}F_2(r_k) = \text{mes}\left\{\theta \in [-\pi, \pi) \middle| \begin{array}{c} |A(r_k e^{i\theta})| \leq e + |a|, \\ |B(r_k e^{i\theta})| \geq \exp(r_k^{\rho(B)-\varepsilon}) \end{array}\right\} \geq \sigma - \frac{t}{4}. \tag{7.6}$$

Clearly,

$$\begin{aligned} \text{mes}\left(\left(\bigcup_{i=1}^{m} I_i\right) \bigcap F_2(r_k)\right) &= \text{mes}\left(\Phi \bigcap F_2(r_k)\right) - \text{mes}\left(\left(\Phi \setminus \bigcup_{i=1}^{m} I_i\right) \bigcap F_2(r_k)\right) \\ &= \text{mes}F_2(r_k) - \text{mes}E(f) - \text{mes}\left(\Phi \setminus \bigcup_{i=1}^{m} I_i\right) \\ &\geq \sigma - \frac{t}{4} - \text{mes}E(f) - \frac{t}{4} = \frac{t}{2}. \end{aligned}$$

According to (3.8), we have

$$\text{mes}\left(F_2(r_k) \cap \bigcup_{i=1}^{m} (I_i^*)\right) \geq \frac{t}{4}.$$

Furthermore, there exists an open interval I_i^* such that for infinitely many k,

$$\text{mes}(F_2(r_k) \cap I_i^*) > \frac{t}{4m} > 0.$$

Hence, we obtain from (1.1), (3.6) and (7.6) that

$$\exp(r_k^{\rho(B)-\varepsilon}) \leq |B(r_k e^{i\theta})| \leq O(r_k^M)(1 + |a| + e)$$

for $\theta \in F_2(r_k) \cap I_i^*$, a contradiction. Thus, we have $\text{mes}E(f) \geq \sigma$. □

8. Proof of Theorem 1.14

Proof. Since the number of accumulation rays of the zero sequence of $A(z)$ is strictly less than $n+2$, there exists a $j_0 \in \{0, \cdots, n+1\}$ such that $A(z)$ decays to zero exponentially in S_{j_0} and (2.1) holds.

(1) $n + 2 < 2m$.

We affirm that there exists an odd number k_0 $(k_0 = 1, 3, \cdots, 2m-1)$ such that $\delta(Q, \theta) < 0$ and $S_{k_0}(Q, \theta) \cap S_{j_0}$ is a non-empty open interval. Otherwise, there exists an even number k' such that S_{j_0} contained in $S_{k'}(Q, \theta)$. Since

$$\text{mes}S_{j_0} = \frac{2\pi}{n+2} \quad \text{and} \quad \text{mes}S_{k'}(Q, \theta) = \frac{\pi}{m},$$

we have $\frac{2\pi}{n+2} < \frac{\pi}{m}$, contradicting to $n + 2 < 2m$.

(2) $n + 2 = 2m$ and $\arg a_n - 2\arg b_m \neq (2s+1)\pi, s \in \mathbb{Z}$.

We also affirm that there exists an odd number k_0 $(k_0 = 1, 3, \cdots, 2m-1)$ such that $\delta(Q, \theta) < 0$ and $S_{k_0}(Q, \theta) \cap S_{j_0}$ is a non-empty open interval. Otherwise, there must exist an even number k_1 such that $S_{j_0} = S_{k_1}(Q, \theta)$. Since $n + 2 = 2m$, then

$$\theta_{j_0} = \frac{2j_0\pi - \arg a_n}{n+2} \quad \text{and} \quad \theta'_{k_1} = -\frac{\arg b_m}{m} + \frac{(2k_1 - 1)\pi}{2m}.$$

This implies that $\theta_{j_0} = \theta'_{k_1}$, and so $\arg a_n - 2\arg b_m = [2(j_0 - k_1) + 1]\pi$, a contradiction.

From above two cases, there exists $\theta_1, \theta_2 \in S_{k_0}(Q, \theta) \cap S_{j_0}$ satisfying $\theta_1 < \theta_2$ such that $\delta(Q, \theta) < 0$ for $\theta \in (\theta_1, \theta_2)$ and $A(re^{i\theta})$ and $B(re^{i\theta})$ decay to zero exponentially. By the Phragmén-Lindelöf principle, we know $|A(z)|$ and $|B(z)|$ are bounded for all $z \in \overline{\Omega}(\theta_1, \theta_2)$. Therefore,

$$\max\{|A(z)|, |B(z)|\} < M \quad \text{for} \quad \text{all} \quad z \in \overline{\Omega}(\theta_1, \theta_2),$$

where $M > 0$ is a constant.

For $n \leq 0$, we obtain from (1.1) that $g(z) = f^{(n)}(z)$ must satisfy equation

$$g^{(m)} + A(z)g^{(m-1)} + B(z)g^{(m-2)} = 0, \tag{8.1}$$

where $m = -n + 2$. Set $h(r) = g(re^{i\theta})$, and so $h^{(k)}(r) = e^{ki\theta}g^{(k)}(re^{i\theta})$ for $k \in \mathbb{N}$. Then

$$h^{(m)} + A(re^{i\theta})e^{i\theta}h^{(m-1)} + B(re^{i\theta})e^{2i\theta}h^{(m-2)} = 0. \tag{8.2}$$

Define $V(r) = \exp(2Mr)$. Then $V(r)$ satisfies the equation

$$V^{(m)} - MV^{(m-1)} - 2M^2V^{(m-2)} = 0. \tag{8.3}$$

Set

$$M_0 = \max\{1, |g(0)|, |2M|^{-j}|g^{(j)}(0)|, j = 1, 2, \cdots, m\}.$$

Then

$$|g(0)| \leq M_0V(0), |g^{(j)}(0)| \leq M_0V^{(j)}(0)(1, 2, \cdots, m).$$

We obtain from (8.1), (8.2) and [5, Satz 1] that

$$|g(re^{i\theta})| = |h(r)| \leq M_0V(r) = M_0 \exp(2Mr)$$

for all $\theta \in [\theta_1, \theta_2]$. Thus,

$$\log^+ |f^{(n)}(re^{i\theta})| \leq Kr, z \in \overline{\Omega}(\theta_1, \theta_2),$$

where $K > 0$ is a constant. Since $f(z)$ is entire, so for $n \leq 0$, we have

$$S_{\theta_1,\theta_2}(r, f^{(n)}) = O(r). \tag{8.4}$$

For $n > 0$, we obtain from (8.4) and Lemma 2.4 that, for $\varepsilon > 0$,

$$S_{\theta_1+\varepsilon,\theta_2-\varepsilon}(r, f^{(n)}) = O(r) \tag{8.5}$$

for $r \notin E_4$, E_4 is a set of finite linear measure.

We now assume that $g = f^{(n)}$ has a Baker wandering domain, and so $\mathcal{J}(g)$ only has bounded component. It follows from Lemma 2.11 that there exists d $(0 < d < 1)$ such that

$$|g(z)| \geq M(r, g)^d, \quad r \in H_0,$$

where H_0 is a set of infinite logarithmic measure. Thus,

$$
\begin{aligned}
S_{\alpha,\beta}(r,g) &\geq B_{\alpha,\beta}(r,g)\frac{2\omega}{\pi r^{\omega}}\int\limits_{\alpha}^{\beta}\log^{+}|g(re^{i\theta})|\sin\omega(\theta-\alpha)d\theta \\
&\geq \frac{2\omega}{\pi r^{\omega}}\int\limits_{\alpha}^{\beta} d\log^{+}M(r,g)\frac{2}{\pi}\omega(\theta-\alpha)d\theta \\
&= \frac{2d}{r^{\omega}}\log M(r,g), \quad r \in H_0 \setminus E_4,
\end{aligned}
$$

where $\alpha = \theta_1 + \varepsilon, \beta = \theta_2 - \varepsilon$, and $\omega = \pi/(\theta_1 - \theta_2 - 2\varepsilon)$ for $n > 0$, while $\alpha = \theta_1, \beta = \theta_2$, and $\omega = \pi/(\theta_1 - \theta_2)$ for $n \leq 0$. Combining this with (8.4) and (8.5), we obtain

$$
\log M(r,g) \leq \frac{r^{\omega}}{2d}S_{\alpha,\beta}(r,g) = \frac{r^{\omega}}{2d}S_{\alpha,\beta}(r,f^{(n)}) = O(r^{1+\omega}), \quad r \in H_0 \setminus E_4,
$$

which implies $\mu(g) < \infty$.

Similarly, there exists an even number k' such that $S_{k'}(Q,\theta) \cap S_{j_0}$ is not empty open interval. We obtain from (1.1), (2.1) and (2.8) that

$$
\begin{aligned}
\exp\{(1-\varepsilon)\delta(Q,\theta)r^m\} &< \left|\frac{f''(z)}{f(z)}\right| + |A(z)|\left|\frac{f'(z)}{f(z)}\right| \\
&\leq C(T(2r,f))^4(1+\exp(-r^{\frac{n+2}{2}-\varepsilon}) \\
&\leq C(T(2r,f))^4(1+o(1))
\end{aligned}
$$

for all $z \in S_{k'}(Q,\theta) \cap S_{j_0}$ and for sufficient large $|z| = r \notin E_2 \cup [0,1]$, where E_2 is a set of finite measure. Thus we obtain $\mu(f) = \infty$, contradicting to $\mu(f) = \mu(f^{(n)}) = \mu(g) < \infty$ for all $n \in \mathbb{Z}$. Therefore, $g = f^{(n)}$ has no Baker wandering domain. □

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