Iterative Post algebras
I. A. Malcev (Novosibirsk)

Let $A$ be a nonempty set, $P_A^{(n)}$ be a set of all $n$-ary functions defined on $A$ and taking values in this same set. Let $P_A = \bigcup_n P_A^{(n)}$. Let iterative Post algebra be an algebra $P_A = \langle P_A; \zeta, \tau, \Delta, \nabla, * \rangle$, with operations defined by the following equations:

\[
\begin{align*}
(\zeta f)(x_1, \ldots, x_n) &= f(x_2, x_3, \ldots, x_n, x_1), \\
(\tau f)(x_1, \ldots, x_n) &= f(x_2, x_1, x_3, \ldots, x_n), \\
(\Delta f)(x_1, \ldots, x_n) &= f(x_1, x_1, x_2, \ldots, x_{n-1}), \\
(\nabla f)(x_1, \ldots, x_n) &= f(x_2, x_3, \ldots, x_{n+1}), \\
(f \ast g)(x_1, \ldots, x_{n+m-1}) &= f(g(x_1, \ldots, x_m), x_{m+1}, \ldots, x_{m+n-1}).
\end{align*}
\]

If the function $f$ is unary, then $\zeta f = \tau f = \Delta f = f$. Let preiterative Post algebra be $P_A^* = \langle P_A; \zeta, \tau, \Delta, * \rangle$. The subalgebras of $P_A$ and $P_A^*$ are called iterative and preiterative algebras respectively. Clones are preiterative algebras containing all selector functions (projections) $e^n_i(x_1, \ldots, x_n) = x_i$.

Clones are important part in the theory of universal algebras, because if $A = \langle A; (f^A_i)_{i \in I} \rangle$ is an universal algebra of a given type then the set of all its term functions is the clone generated by the fundamental operations $(f^A_i)_{i \in I}$ of $A$. Clones may also be regarded as algebras with operations $\zeta, \tau, \Delta, *, e^2_1$. But it is more natural to use iterative algebras rather than clones in applications, such as theory of automata, because every subalgebra of iterative algebra that contains some function $f$ contains all functions that differ from $f$ by unessential variables.

Let $A$ be an algebra and let $T(A)$ be the clone of all its term functions. The clone identities of $T(A)$ correspond to the hyperidentities of $A$. Recall that an identity $t \approx t'$ is said to be a hyperidentity in $A$ iff $t = t'$ holds identically in $A$ for any substitution of term functions of $A$ of corresponding arities for the operation symbols appearing in $t$ and $t'$.

In this case we write $A \models_{\text{hyp}} t \approx t'$ or $T(A) \models_{\text{hyp}} t \approx t'$. If $C$ and $C'$ are clones of functions then we will say that $C$ can be separated from $C'$ by hyperidentities if there is a hyperidentity $t \approx t'$ in $C$ which fails in $C'$: $C \models_{\text{hyp}} t \approx t'$ but $C' \not\models_{\text{hyp}} t \approx t'$. For isomorphic clones $C$ and $C'$ we have $\text{Id } C = \text{Id } C'$; i.e., using the correspondence between the clone identities and the hyperidentities we see that the sets of hyperidentities of $C$ and $C'$ coincide.

We give a general criterion for separation of clones by means of hyperidentities and some examples of such hyperidentities.

Let $\prod_i P_{A_i} = P_{A_1} \times \ldots \times P_{A_m}$ be the set of all possible sequences of functions $(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$

for which $f_i \in P_{A_i}$. The operations $\zeta, \tau, \Delta, \nabla, *$ could be naturally defined on the set $\prod_i P_{A_i}$. For example, $(\zeta f)(x_1, \ldots, x_n) = ((\zeta f_1)(x_1, \ldots, x_n), \ldots, (\zeta f_m)(x_1, \ldots, x_n))$. The algebra $\prod_i P_{A_i} = \langle \prod_j P_{A_j}; \zeta, \tau, \cdot, \nabla, * \rangle$ is called a coordinated product of algebras $P_{A_j}$. In the theory of universal algebras to this conception correspond non-indexed product of universal algebras.

We deduce some conditions for a subalgebra of the algebra $P_{k_1} \times \ldots \times P_{k_m}$ to contain a subalgebra which is a coordinated product of iterative algebras. As a corollary we get a theorem
that each coordinated product of iterative Post algebras has only one Slupecki subalgebra if $k_i \geq 3$ for each $i \in 1, \ldots, m$. 