Bull. Korean Math. Soc. **51** (2014), No. 6, pp. 1735–1748
http://dx.doi.org/10.4134/BKMS.2014.51.6.1735

MEROMORPHIC SOLUTIONS OF A COMPLEX DIFFERENCE EQUATION OF MALMQUIST TYPE

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Abstract. In this paper, we investigate the finite order transcendental meromorphic solutions of complex difference equation of Malmquist type

$$\prod_{i=1}^{n} f(z+c_i) = R(z,f),$$

where $c_1, \ldots, c_n \in \mathbb{C}\setminus\{0\}$, and $R(z,f)$ is an irreducible rational function in $f(z)$ with meromorphic coefficients. We obtain some results on deficiencies of the solutions. Using these results, we prove that the growth order of the finite order solution $f(z)$ is 1, if $f(z)$ has Borel exceptional values $a(\in\mathbb{C})$ and ∞. Moreover, we give the forms of $f(z)$.

1. Introduction and results

Let $f(z)$ be a meromorphic function in the complex plane $\mathbb{C}$. We assume that the reader is familiar with the basic notions of Nevanlinna's theory (see [7, 14]). We use $\sigma(f)$ to denote the order of growth of $f(z)$; and $\lambda(f)$ and $\lambda(1/f)$ to denote, respectively, the exponents of convergence of zero and pole sequences of $f(z)$. Moreover, we use $\delta(a,f)$ to denote the Nevanlinna deficiency of $f(z)$.

We denote by $S(r,f)$ any real function of growth $o(T(r,f))$ as $r\to\infty$ outside a possible exceptional set of finite logarithmic measure.

We now recall the celebrated Malmquist–Yosida theorem.

Theorem A ([12, p. 193]). *Let $R(z,y)$ be rational and irreducible in y with meromorphic coefficients. If the differential equation*

$$(y')^n = R(z,y) \tag{1.1}$$

Received November 12, 2013.
2010 *Mathematics Subject Classification.* 30D35, 39A10.
Key words and phrases. difference equation, meromorphic function, deficiency.
This work was supported by National Natural Science Foundation of China (11226090) and Natural Science Foundation of Guangdong Province in China (S2012040006865).

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admits a meromorphic solution y such that $T(r,\alpha)=S(r,y)$ for all coefficients $\alpha(z)$ of $R(z,y)$, then (1.1) *reduces into*

$$(y')^n=\sum_{i=0}^{2n}\alpha_i(z)y^i,$$

where at least one of the coefficients $\alpha_i(z)$ does not vanish.

Recently, a number of papers (including [2, 3, 4, 8, 9, 10, 17, 18]) are focused on complex difference equations. In these papers, the authors mainly studied the properties of finite order meromorphic solutions of difference equations, and obtained many meaningful results. In particular, Heittokangas et al. [8] discussed the following difference equation

$$\prod_{i=1}^{n}f(z+c_i)=R(z,f), \tag{1.2}$$

where $c_1,\dots,c_n\in\mathbb{C}\setminus\{0\}$, and $R(z,f)$ is an irreducible rational function in $f(z)$ with meromorphic coefficients. Writing $R(z,f)$ as the quotient of two relatively prime polynomials in $f(z)$, we see that the equation (1.2) takes the form

$$\prod_{i=1}^{n}f(z+c_i)=\frac{a_0(z)+a_1(z)f+\cdots+a_p(z)f^p}{b_0(z)+b_1(z)f+\cdots+b_q(z)f^q}, \tag{1.3}$$

where the coefficients $a_i(z)$, $b_j(z)$ are meromorphic functions such that

$$a_p(z)b_q(z)\not\equiv 0.$$

In what follows, we always assume that the polynomials $a_0(z)+a_1(z)f+\cdots+a_p(z)f^p$ and $b_0(z)+b_1(z)f+\cdots+b_q(z)f^q$ are relatively prime in $f(z)$.

The equation (1.3) can be viewed as difference analogue of the equation (1.1). Two results are obtained in [8] about the equation (1.3).

Theorem B ([8])**.** *Let $c_1,\dots,c_n\in\mathbb{C}\setminus\{0\}$. If the difference equation* (1.3) *with rational coefficients $a_i(z)$, $b_j(z)$ admits a transcendental meromorphic solution of finite order, then* $\max\{p,q\}\le n$.

Remark 1.1. It is easy to prove that if the coefficients $a_i(z)$, $b_j(z)$ in Theorem B are meromorphic and of growth $S(r,f)$, then the conclusion "$\max\{p,q\}\le n$" still holds.

Theorem C ([8])**.** *Let $c_1,\dots,c_n\in\mathbb{C}\setminus\{0\}$ and suppose that f is a non-rational meromorphic solution of the equation* (1.3) *with meromorphic coefficients $a_i(z)$, $b_j(z)$ of growth $S(r,f)$ such that $a_p(z)b_q(z)\not\equiv 0$. If*

$$\max(\lambda(f),\lambda(1/f))<\sigma(f),$$

then (1.3) *is of the form*

$$\prod_{i=1}^{n}f(z+c_i)=c(z)f(z)^k,$$

where $c(z)$ is meromorphic, $T(r,c) = S(r,f)$ and $k \in \mathbb{Z}$.

In this paper, we continue to study the properties of meromorphic solutions of the equation (1.3). First, we consider the deficiencies of the solutions and get the following result.

Theorem 1.1. *Let $c_1, \ldots, c_n \in \mathbb{C} \setminus \{0\}$. Suppose that f is a finite order transcendental meromorphic solution of the equation* (1.3) *with meromorphic coefficients $a_i(z)$, $b_j(z)$ of growth $S(r,f)$ such that $a_p(z)b_q(z) \not\equiv 0$.*

(i) *If $q \geq 1$, then $\delta(\infty, f) = 0$.*

(ii) *If at least one of $a_0(z), a_1(z), \ldots, a_{p-1}(z)$ is not identically zero, then $\delta(0,f) = 0$.*

(iii) *If $a(\in \mathbb{C})$ is not a solution of the equation* (1.3)*, then $\delta(a,f) = 0$.*

The following corollary extends Theorem C, and its proof is different from that in Theorem C. In fact, we can easily prove the following corollary by Theorem 1.1.

Corollary 1.1. *Let $c_1, \ldots, c_n \in \mathbb{C} \setminus \{0\}$. Suppose that f is a finite order transcendental meromorphic solution of the equation* (1.3) *with meromorphic coefficients $a_i(z)$, $b_j(z)$ of growth $S(r,f)$ such that $a_p(z)b_q(z) \not\equiv 0$. For $a \in \mathbb{C}$, if*

$$\max(\lambda(f-a), \lambda(1/f)) < \sigma(f),$$

then (1.3) *is of the form*

$$\prod_{i=1}^{n} f(z+c_i) = \frac{a_0(z)}{b_0(z)} + \frac{a_1(z)}{b_0(z)} f(z) + \cdots + \frac{a_n(z)}{b_0(z)} f(z)^n.$$

In particular, if $a = 0$, then (1.3) *is of the form*

$$\prod_{i=1}^{n} f(z+c_i) = \frac{a_n(z)}{b_0(z)} f(z)^n.$$

The following two examples show that the conditions "$q \geq 1$", "at least one of $a_0(z), a_1(z), \ldots, a_{p-1}(z)$ is not identically zero" and "$a(\in \mathbb{C})$ is not a solution of the equation (1.3)" in Theorem 1.1 cannot be dropped.

Example 1. The function $f(z) = e^z$ is a solution of the equation

$$f(z+2)f(z+1) = e^3 f^2,$$

where $q = 0$ and $a_0(z), a_1(z), \ldots, a_{p-1}(z)$ are identically zeros. We know that $\delta(\infty, f) = 1$, $\delta(0,f) = 1$.

Example 2. The function $f(z) = \tan(\frac{\pi}{2} z)$ is a solution of the equation

$$f(z+1)f(z-1) = 1/f^2,$$

where i and $-i$ are two solutions of this equation. We know that $\delta(i,f) = 1$, $\delta(-i,f) = 1$.

The following example satisfies all conditions in Theorem 1.1.

Example 3. The function

$$f(z) = \frac{1}{e^{2\pi iz} + z} + z$$

is a solution of the equation

$$f(z+1)f(z-1) = \frac{(4-z^2)f^2 + (2z^3 - 6z)f - z^4 + 3z^2 - 1}{-f^2 + 2zf - z^2 + 1}.$$

We know that $\delta(\infty, f) = 0$, $\delta(0, f) = 0$ and $\delta(a, f) = 0$ for all $a \in \mathbb{C} \setminus \{0\}$.

We now consider the forms of meromorphic solutions of the equation (1.3) and get the following result which deepens Theorem C when the solutions are of finite order. From this result, we see that the growth order of these solutions is 1.

Theorem 1.2. *Let $c_1, \ldots, c_n \in \mathbb{C}\backslash\{0\}$ such that $c_1+c_2+\cdots+c_n \neq 0$. Let $a_i(z)$, $b_j(z)$ be meromorphic functions of order less than 1 such that $a_p(z)b_q(z) \not\equiv 0$. Suppose that $f(z)$ is a finite order transcendental meromorphic solution of the equation* (1.3) *such that*

$$\max(\lambda(f-a), \lambda(1/f)) < \sigma(f),$$

where $a \in \mathbb{C}$. Then $f(z)$ is of the form

$$f(z) = H(z)e^{cz} + a,$$

where c is a nonzero constant, $H(z)$ is a meromorphic function with $\sigma(H) < 1$.

In particular, if $a_p(z)$ and $b_q(z)$ are nonzero constants, then $f(z)$ is of the form

$$f(z) = de^{cz} + a,$$

where d and c are nonzero constants.

The following two examples show that the condition "$c_1 + c_2 + \cdots + c_n \neq 0$" in Theorem 1.2 cannot be dropped.

Example 4. The function $f(z) = ze^{z^2}$ is a solution of the equation

$$f(z+1)f(z-1) = \frac{(e^2z^2 - e^2)f^2}{z^2}.$$

Obviously, $f(z) = ze^{z^2}$ cannot takes the form $f(z) = H(z)e^{cz}$, where c is a nonzero constant, $H(z)$ is a meromorphic function with $\sigma(H) < 1$.

Example 5. The function $f(z) = e^{z^2}$ is a solution of the equation

$$f(z+1)f(z-1) = e^2f^2.$$

Obviously, $f(z) = e^{z^2}$ cannot takes the form $f(z) = de^{cz}$, where d and c are nonzero constants.

At last, we discuss the coefficients of the equation (1.3). The following result tells us that solutions having Borel exceptional values $a(\in \mathbb{C})$ and ∞ appear in special situations only.

Theorem 1.3. *Let $c_1, \ldots, c_n \in \mathbb{C}\backslash\{0\}$ such that $c_1+c_2+\cdots+c_n \neq 0$. Let $a_i(z)$, $b_j(z)$ be meromorphic functions of order less than 1 such that $a_p(z)b_q(z) \not\equiv 0$. If the equation* (1.3) *admits a finite order transcendental meromorphic solution such that*

$$\max(\lambda(f-a), \lambda(1/f)) < \sigma(f),$$

then there exists an ε-set E such that

$$\frac{a_p(z)}{b_q(z)} \to e^{c(c_n+\cdots+c_1)} \quad \textit{as } z \to \infty \textit{ in } \mathbb{C} \setminus E.$$

Remark 1.2. An ε-set is a countable union of open discs not containing the origin and subtending angles at the origin whose sum is finite. If E is an ε-set, then the set of $r \geq 1$ for which the circle $S(0, r)$ meets E has finite logarithmic measure.

2. Proof of Theorem 1.1 and Corollary 1.1

We need the following lemmas.

Lemma 2.1 ([11])**.** *Let $T : [0, +\infty) \to [0, +\infty)$ be a non-decreasing continuous function, let $\delta \in (0, 1)$, and let $s \in (0, \infty)$. If T is of finite order, i.e.,*

$$\overline{\lim_{r\to\infty}} \frac{\log T(r)}{\log r} < \infty,$$

then

$$T(r+s) = T(r) + o(T(r)/r^{\delta}),$$

where r runs to infinity outside a set of finite logarithmic measure.

By Corollaries 2.2 and 3.4 in [6] and the above Lemma 2.1, we get the following two lemmas.

Lemma 2.2. *Let $f(z)$ be a non-constant meromorphic function of finite order, and let η_1, η_2 be two arbitrary complex numbers. Then we have*

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = S(r, f).$$

Lemma 2.3. *Let $f(z)$ be a non-constant finite order meromorphic solution of*

$$P(z, f) = 0,$$

where $P(z, f)$ is a difference polynomial in $f(z)$. If $P(z, a) \not\equiv 0$ for a meromorphic function $a(z)$ satisfying $T(r, a) = S(r, f)$, then

$$m\left(r, \frac{1}{f-a}\right) = S(r, f).$$

Applying Lemma 2.1 to [13, Theorem 2.3], we get the following lemma.

Lemma 2.4. *Let $f(z)$ be a transcendental finite order meromorphic solution of*

$$U(z,f)Q(z,f) = P(z,f),$$

where $U(z,f)$, $P(z,f)$, $Q(z,f)$ are difference polynomials in $f(z)$ with meromorphic coefficients of growth $S(r,f)$, $\deg_f U = n$ and $\deg_f P \le n$. Moreover, we assume that $U(z,f)$ contains just one term of maximal total degree. Then

$$m\big(r, Q(z,f)\big) = S(r,f).$$

Lemma 2.5 ([16])**.** *Let $f(z)$ be a meromorphic function of finite order such that $N(r,f) = S(r,f)$. Suppose that $H(z,f)$ is a difference polynomial in $f(z)$ with meromorphic coefficients of growth $S(r,f)$, and $H(z,f)$ contains just one term of maximal total degree. Then*

$$T(r,H) = (\deg_f H)T(r,f) + S(r,f).$$

Proof of Theorem 1.1. By Theorem B and Remark 1.1, we get

$$\max\{p,q\} \le n. \tag{2.1}$$

(i) We deduce from the equation (1.3) that

$$\begin{aligned}&(b_0(z) + b_1(z)f + \cdots + b_q(z)f^q)f(z+c_n)\cdots f(z+c_2)f(z+c_1)\\ &= a_0(z) + a_1(z)f + \cdots + a_p(z)f^p.\end{aligned}$$

Set

$$\begin{aligned}U(z,f) &= (b_0(z) + b_1(z)f + \cdots + b_q(z)f^q)f(z+c_n)\cdots f(z+c_2),\\ P(z,f) &= a_0(z) + a_1(z)f + \cdots + a_p(z)f^p.\end{aligned}$$

We have

$$U(z,f)f(z+c_1) = P(z,f). \tag{2.2}$$

Since $b_q(z) \not\equiv 0$ and $q \ge 1$, we get $\deg_f U \ge n$. By $a_p(z) \not\equiv 0$ and (2.1), we get $\deg_f P = p \le n$. It is obviously that $U(z,f)$ contains just one term of maximal total degree. So by (2.2) and Lemma 2.4, we get

$$m(r, f(z+c_1)) = S(r,f).$$

Together with Lemma 2.2, we obtain

$$m(r,f) \le m\left(r, \frac{f(z)}{f(z+c_1)}\right) + m(r, f(z+c_1)) = S(r,f),$$

and so

$$\delta(\infty, f) = 0.$$

(ii) Set $y(z) = 1/f(z)$. Then we conclude from the equation (1.3) that

$$\frac{1}{y(z+c_n)\cdots y(z+c_1)} = \frac{(a_p(z) + a_{p-1}(z)y + \cdots + a_0(z)y^p)y^q}{(b_q(z) + b_{q-1}(z)y + \cdots + b_0(z)y^q)y^p}.$$

Therefore,

$$(a_p(z) + a_{p-1}(z)y + \cdots + a_0(z)y^p)y^q y(z+c_n)\cdots y(z+c_2)y(z+c_1)$$

$$= (b_q(z) + b_{q-1}(z)y + \cdots + b_0(z)y^q)y^p.$$

Set

$$U(z, y) = (a_p(z) + a_{p-1}(z)y + \cdots + a_0(z)y^p)y^q y(z + c_n) \cdots y(z + c_2),$$
$$P(z, y) = (b_q(z) + b_{q-1}(z)y + \cdots + b_0(z)y^q)y^p.$$

We have

$$U(z, y)y(z + c_1) = P(z, y). \tag{2.3}$$

Since at least one of $a_0(z), a_1(z), \ldots, a_{p-1}(z)$ is not identically zero, we have $\deg_y U \geq n + q$. By (2.1), we get $\deg_y P \leq p + q \leq n + q$. It is obviously that $U(z, y)$ contains just one term of maximal total degree. Thus, we deduce from (2.3) and Lemma 2.4 that

$$m(r, y(z + c_1)) = S(r, y).$$

Together with Lemma 2.2, we obtain

$$m(r, y) = S(r, y).$$

Noting that $y(z) = 1/f(z)$, we get

$$m(r, 1/f) = S(r, f),$$

and so

$$\delta(0, f) = 0.$$

(iii) Set

$$Q(z, f) = f(z + c_n) \cdots f(z + c_1)(b_0(z) + b_1(z)f + \cdots + b_q(z)f^q)$$
$$- (a_0(z) + a_1(z)f + \cdots + a_p(z)f^p).$$

Since $a(\in \mathbb{C})$ is not a solution of the equation (1.3), we get

$$Q(z, a) \not\equiv 0.$$

Thus, we deduce from Lemma 2.3 that

$$m\left(r, \frac{1}{f - a}\right) = S(r, f),$$

and

$$\delta(a, f) = 0.$$

Theorem 1.1 is proved. □

Proof of Corollary 1.1. Since $f(z)$ satisfies $\max(\lambda(f - a), \lambda(1/f)) < \sigma(f) < \infty$, by Hadamard's factorization theory, we see that $f(z)$ takes the form

$$f(z) = H(z)e^{h(z)} + a,$$

where $h(z)$ is a polynomial and $H(z)$ is a meromorphic function such that $\sigma(H) < \sigma(f)$. So $\sigma(f) = \deg h \geq 1$, and $f(z)$ is of regular growth, i.e.,

$$\lim_{r \to \infty} \frac{\log T(r, f)}{\log r} = \sigma(f). \tag{2.4}$$

Fix constants α, β such that

$$\max(\lambda(f-a), \lambda(1/f)) < \alpha < \beta < \sigma(f). \tag{2.5}$$

By (2.4) and (2.5), when r is sufficiently large, we have

$$T(r,f) > r^{\beta}, \quad N\left(r, \frac{1}{f-a}\right) < r^{\alpha}, \quad N(r,f) < r^{\alpha}. \tag{2.6}$$

Therefore,

$$\delta(a,f) = 1 \quad \text{and} \quad \delta(\infty, f) = 1. \tag{2.7}$$

Thus, we deduce from (2.7) and Theorem 1.1(i) that $q = 0$, and so the equation (1.3) reduces into

$$\prod_{i=1}^{n} f(z+c_i) = \frac{a_0(z)}{b_0(z)} + \frac{a_1(z)}{b_0(z)} f(z) + \cdots + \frac{a_p(z)}{b_0(z)} f(z)^p, \tag{2.8}$$

where $a_p(z)b_0(z) \not\equiv 0$. By (2.6), we see that $N(r,f) = S(r,f)$. Therefore, we deduce from Lemma 2.5 that

$$T\left(r, \prod_{i=1}^{n} f(z+c_i)\right) = nT(r,f) + S(r,f)$$

and

$$T\left(r, \frac{a_0(z)}{b_0(z)} + \frac{a_1(z)}{b_0(z)} f(z) + \cdots + \frac{a_p(z)}{b_0(z)} f(z)^p\right) = pT(r,f) + S(r,f).$$

The above two equalities show that $p = n$, and (2.8) becomes

$$\prod_{i=1}^{n} f(z+c_i) = \frac{a_0(z)}{b_0(z)} + \frac{a_1(z)}{b_0(z)} f(z) + \cdots + \frac{a_n(z)}{b_0(z)} f(z)^n. \tag{2.9}$$

In particular, if $a = 0$, we get $\delta(0,f) = 1$ from (2.7). So by (2.9) and Theorem 1.1(ii), we get

$$\prod_{i=1}^{n} f(z+c_i) = \frac{a_n(z)}{b_0(z)} f(z)^n.$$

Corollary 1.1 is proved. $\square$

3. Proof of Theorem 1.2

We need the following lemmas for the proof of Theorem 1.2.

Lemma 3.1 ([4]). *Let η_1, η_2 be two complex numbers such that $\eta_1 \neq \eta_2$ and let $f(z)$ be a finite order meromorphic function. Let σ be the order of $f(z)$. Then for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O(r^{\sigma-1+\varepsilon}).$$

Lemma 3.2 ([1]). *Let $g(z)$ be a function transcendental and meromorphic in the plane of order less than 1. Let $h > 0$. Then there exists an ε-set E such that*

$$\frac{g'(z+c)}{g(z+c)} \to 0 \quad \text{and} \quad \frac{g(z+c)}{g(z)} \to 1 \quad \text{as } z \to \infty \text{ in } \mathbb{C} \setminus E,$$

uniformly in c for $|c| \le h$. Further, E may be chosen so that for large z not in E, the function $g(z)$ has no zeros or poles in $|\zeta - z| \le h$.

Lemma 3.3 ([1]). *Let $f(z)$ be a function transcendental and meromorphic in the plane of order less than 1. Let $h > 0$. Then there exists an ε-set E such that*

$$f(z+c) - f(z) = cf'(z)(1 + o(1)) \quad \text{as } z \to \infty \text{ in } \mathbb{C} \setminus E,$$

uniformly in c for $|c| \le h$.

Remark 3.1. It is easy to prove that if $f(z)$ is a rational function, the conclusions in Lemmas 3.2 and 3.3 still hold.

Lemma 3.4. *Let $c_1, \ldots, c_n \in \mathbb{C} \setminus \{0\}$. Suppose that $f(\not\equiv 0)$ is a meromorphic solution of the equation*

$$f(z+c_n)f(z+c_{n-1}) \cdots f(z+c_1) = f(z)^n. \tag{3.1}$$

If $\sigma(f) < 1$, then f is a constant.

Proof of Lemma 3.4. By Lemma 3.3 and Remark 3.1, there exists an ε-set E_1 such that, for all $i = 1, 2, \ldots, n$,

$$f(z+c_i) = c_i f'(z)(1 + o(1)) + f(z) \tag{3.2}$$

as $z \to \infty$ in $\mathbb{C} \setminus E_1$. Substituting (3.2) into (3.1), we get

$$\Big(c_n f'(z)(1+o(1)) + f(z)\Big)\Big(c_{n-1} f'(z)(1+o(1)) + f(z)\Big) \\ \cdots \Big(c_1 f'(z)(1+o(1)) + f(z)\Big) = f(z)^n$$

as $z \to \infty$ in $\mathbb{C} \setminus E_1$. Therefore, when $z \to \infty$ in $\mathbb{C} \setminus E_1$, we have

$$\begin{aligned} &K_0(f'(z))^n(1+o(1)) + K_1(f'(z))^{n-1} f(z)(1+o(1)) + \\ &\cdots + K_{n-1} f'(z) f(z)^{n-1}(1+o(1)) = 0, \end{aligned} \tag{3.3}$$

where $K_0 = c_n c_{n-1} \cdots c_1 \neq 0$, and $K_1, K_2, \ldots, K_{n-1}$ are constants.

Now we prove that $f'(z) \equiv 0$. Thus, we divide our discussion into two cases.

Case (1). $K_{n-1} = K_{n-2} = \cdots = K_1 = 0$. Then by (3.3), when $z \to \infty$ in $\mathbb{C} \setminus E_1$, we get

$$K_0(f'(z))^n(1 + o(1)) = 0.$$

Since $K_0 \neq 0$, we obtain $f'(z) \equiv 0$, when z is sufficiently large and $z \in \mathbb{C} \setminus E_1$. So $f'(z) \equiv 0$ in $\mathbb{C}$.

Case (2). $K_{n-1}, K_{n-2}, \ldots, K_1$ are not all zeros. In this case, we can assume that $K_j \neq 0$ $(1 \leq j \leq n-1)$ and $K_{j+1}, K_{j+2}, \ldots, K_{n-1}$ are all zeros. Thus, the equation (3.3) reduces into

$$\begin{aligned} &K_0(f'(z))^n(1+o(1)) + K_1(f'(z))^{n-1}f(z)(1+o(1))+ \\ &\cdots + K_j(f'(z))^{n-j}f(z)^j(1+o(1)) = 0 \end{aligned} \tag{3.4}$$

as $z \to \infty$ in $\mathbb{C} \setminus E_1$.

If $f'(z) \not\equiv 0$, we conclude from (3.4) that

$$K_0\left(\frac{f'(z)}{f(z)}\right)^j(1+o(1)) + K_1\left(\frac{f'(z)}{f(z)}\right)^{j-1}(1+o(1)) + \cdots + K_j(1+o(1)) = 0 \tag{3.5}$$

as $z \to \infty$ in $\mathbb{C} \setminus E_1$. Since $\sigma(f) < 1$, we deduce from Lemma 3.2 and Remark 3.1 that there exists an ε-set E_2 such that

$$\frac{f'(z)}{f(z)} \to 0 \tag{3.6}$$

as $z \to \infty$ in $\mathbb{C} \setminus E_2$. Thus, when $z \to \infty$ in $\mathbb{C} \setminus (E_1 \cup E_2)$, we obtain from (3.5) and (3.6) that

$$K_j(1+o(1)) \to 0.$$

This contradicts $K_j \neq 0$. So we proved that $f'(z) \equiv 0$. Therefore, $f(z)$ is a constant. Lemma 3.4 is proved. □

Lemma 3.5 ([4]). *Let $f(z)$ be a meromorphic function with $\sigma(f) < \infty$, and let $\eta \neq 0$ be a fixed nonzero complex number. Then for each $\varepsilon > 0$, we have*

$$T\big(r, f(z+\eta)\big) = T(r,f) + O(r^{\sigma(f)-1+\varepsilon}) + O(\log r).$$

Lemma 3.6 ([5, pp. 69–70 or 15, p. 82]). *Suppose that $f_1(z), f_2(z), \ldots, f_n(z)$ are meromorphic functions and that $g_1(z), g_2(z), \ldots, g_n(z)$ are entire functions satisfying the following conditions.*

(1) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;

(2) $g_j(z) - g_k(z)$ *are not constants for* $1 \leq j < k \leq n$;

(3) *for* $1 \leq j \leq n, 1 \leq h < k \leq n$,

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\} \quad n.e. \quad as\ r \to \infty.$$

Then $f_j(z) \equiv 0$ $(j = 1, 2, \ldots, n)$.

Proof of Theorem 1.2. Since $f(z)$ is a transcendental meromorphic function such that $\max(\lambda(f-a), \lambda(1/f)) < \sigma(f) < \infty$, we see that $f(z)$ takes the form

$$f(z) = H(z)e^{h(z)} + a, \tag{3.7}$$

where $h(z)$ is a polynomial and $H(z)$ is a meromorphic function such that $\sigma(H) < \sigma(f)$. So $\sigma(f) = \deg h \geq 1$, and $f(z)$ is of regular growth. Noting that

$\sigma(a_i) < 1$ and $\sigma(b_j) < 1$, we see that $a_i(z)$ and $b_j(z)$ are of growth $S(r, f)$. Therefore, by Corollary 1.1, the equation (1.3) reduces into

$$f(z+c_n)\cdots f(z+c_1) = \frac{a_0(z)}{b_0(z)} + \frac{a_1(z)}{b_0(z)}f(z) + \cdots + \frac{a_n(z)}{b_0(z)}f(z)^n, \tag{3.8}$$

where

$$a_n(z) = a_p(z) \not\equiv 0, \quad b_0(z) = b_q(z) \not\equiv 0. \tag{3.9}$$

Now we prove that $\sigma(f) = 1$. Suppose that, on the contrary, $\sigma(f) = k > 1$. By $\sigma(f) = \deg h$, we get $\sigma(f) = k = \sigma(e^h) \geq 2$.

Substituting (3.7) into (3.8), we get

(3.10)
$$\begin{aligned}&\Big(H(z+c_n)e^{h(z+c_n)-h(z)}e^{h(z)} + a\Big)\cdots\Big(H(z+c_1)e^{h(z+c_1)-h(z)}e^{h(z)} + a\Big)\\ &= \frac{a_0(z)}{b_0(z)} + \frac{a_1(z)}{b_0(z)}(H(z)e^{h(z)} + a) + \cdots + \frac{a_n(z)}{b_0(z)}(H(z)e^{h(z)} + a)^n.\end{aligned}$$

Since $\sigma(a_i) < 1$, $\sigma(b_0) < 1$ and $\sigma(H) < \sigma(f) = \sigma(e^h) \geq 2$, we have, for $i = 0, 1, \ldots, n$,

$$T(r, a_i) = S(r, e^h), \quad T(r, b_0) = S(r, e^h), \quad T(r, H) = S(r, e^h). \tag{3.11}$$

Let

$$h(z) = l_k z^k + l_{k-1}z^{k-1} + \cdots + l_1 z + l_0, \tag{3.12}$$

where $l_k \neq 0$. We have

$$h(z+c_j) - h(z) = kl_k c_j z^{k-1} + o(z^{k-1}).$$

Thus, for $j = 1, 2, \ldots, n$,

$$T(r, e^{h(z+c_j)-h(z)}) = S(r, e^{h(z)}). \tag{3.13}$$

By (3.11), (3.13) and Lemma 3.5, we have, for $j = 1, 2, \ldots, n$,

$$T\left(r, H(z+c_j)e^{h(z+c_j)-h(z)}\right) = S(r, e^{h(z)}). \tag{3.14}$$

Therefore, we deduce from (3.10) that

$$A_n(z)e^{nh(z)} + A_{n-1}(z)e^{(n-1)h(z)} + \cdots + A_0(z) = 0, \tag{3.15}$$

where

$$A_n(z) = H(z+c_n)e^{h(z+c_n)-h(z)}\cdots H(z+c_1)e^{h(z+c_1)-h(z)} - \frac{a_n(z)}{b_0(z)}H(z)^n,$$

and $A_n(z), A_{n-1}(z), \ldots, A_0(z)$ are of growth $S(r, e^h)$ by (3.11) and (3.14). Thus, we deduce from (3.15) and Lemma 3.6 that $A_n(z) \equiv A_{n-1} \equiv A_0 \equiv 0$. By $A_n(z) \equiv 0$, we get

$$e^{h(z+c_n)+\cdots+h(z+c_1)-nh(z)} = \frac{a_n(z)}{b_0(z)}\frac{H(z)}{H(z+c_n)}\cdots\frac{H(z)}{H(z+c_1)}. \tag{3.16}$$

We conclude from (3.12) that

$$h(z+c_n)+\cdots+h(z+c_1)-nh(z)=kl_k(c_n+c_{n-1}+\cdots+c_1)z^{k-1}+o(z^{k-1}).$$

Since $l_k\neq 0$, $k\neq 0$ and $c_n+c_{n-1}+\cdots+c_1\neq 0$, we have

$$\sigma\left(e^{h(z+c_n)+\cdots+h(z+c_1)-nh(z)}\right)=k-1. \tag{3.17}$$

Since $\sigma(H)<\sigma(f)=k$ and $\max\{\sigma(a_n),\sigma(b_0)\}=\sigma_1<1$, there exists a constant ε_0 such that $\sigma(H)<k-2\varepsilon_0$ and $\sigma_1<1-\varepsilon_0$. Thus, we deduce from Lemma 3.1 and $k\geq 2$ that

$$\begin{aligned}
&m\left(r,\frac{a_n(z)}{b_0(z)}\frac{H(z)}{H(z+c_n)}\cdots\frac{H(z)}{H(z+c_1)}\right)\\
&\leq m\left(r,\frac{a_n(z)}{b_0(z)}\right)+m\left(r,\frac{H(z)}{H(z+c_n)}\right)+\cdots+m\left(r,\frac{H(z)}{H(z+c_1)}\right)\\
&=O(r^{1-\varepsilon_0})+O(r^{\sigma(H)-1+\varepsilon_0})\\
&\leq O(r^{1-\varepsilon_0})+O(r^{k-1-\varepsilon_0})\\
&=O(r^{k-1-\varepsilon_0}).
\end{aligned}$$

By (3.16), we see that $\frac{a_n(z)}{b_0(z)}\frac{H(z)}{H(z+c_n)}\cdots\frac{H(z)}{H(z+c_1)}$ is entire. Thus,

$$T\left(r,\frac{a_n(z)}{b_0(z)}\frac{H(z)}{H(z+c_n)}\cdots\frac{H(z)}{H(z+c_1)}\right)\leq O(r^{k-1-\varepsilon_0}),$$

and

$$\sigma\left(\frac{a_n(z)}{b_0(z)}\frac{H(z)}{H(z+c_n)}\cdots\frac{H(z)}{H(z+c_1)}\right)\leq k-1-\varepsilon_0. \tag{3.18}$$

Therefore, we deduce a contradiction from (3.16)–(3.18). This shows that $\sigma(f)=1$ and (3.7) can be written as

$$f(z)=H(z)e^{cz}+a, \tag{3.19}$$

where c is a nonzero constant, $H(z)$ is a meromorphic function with $\sigma(H)<1$.

In particular, if $a_p(z)$ and $b_q(z)$ are nonzero constants, we obtain from (3.9) that $a_n(z)$ and $b_0(z)$ are nonzero constants. Substituting (3.19) into (3.8) and using the similar method as in (3.10)–(3.16), we get

$$e^{c(c_n+\cdots+c_1)}=\frac{a_n(z)}{b_0(z)}\frac{H(z)}{H(z+c_n)}\cdots\frac{H(z)}{H(z+c_1)}. \tag{3.20}$$

Since $\sigma(H)<1$, by Lemma 3.2 and Remark 3.1, there exists an ε-set E such that, for $i=1,2,\ldots,n$,

$$\frac{H(z)}{H(z+c_i)}\to 1 \tag{3.21}$$

as $z \to \infty$ in $\mathbb{C} \setminus E$. Since $e^{c(c_n+\cdots+c_1)}$ and $\frac{a_n(z)}{b_0(z)}$ are constants, we conclude from (3.20) and (3.21) that

$$e^{c(c_n+\cdots+c_1)} = \frac{a_n(z)}{b_0(z)}.$$

Thus (3.20) becomes

$$H(z+c_n)\cdots H(z+c_1) = H(z)^n.$$

By Lemma 3.4, we see that $H(z)$ is a constant. Thus, we obtain from (3.19) that

$$f(z) = de^{cz} + a,$$

where d and c are nonzero constants. Theorem 1.2 is proved. □

4. Proof of Theorem 1.3

Suppose that $f(z)$ is a finite order transcendental meromorphic solution of the equation (1.3) such that $\max(\lambda(f-a), \lambda(1/f)) < \sigma(f)$. By Theorem 1.2, we get

$$f(z) = H(z)e^{cz} + a, \tag{4.1}$$

where c is a nonzero constant, $H(z)$ is a meromorphic function with $\sigma(H) < 1$. As in Theorem 1.2, we also get (3.8) and (3.9). Substituting (4.1) into (3.8) and using the similar method as in (3.10)–(3.16), we get

$$e^{c(c_n+\cdots+c_1)} = \frac{a_n(z)}{b_0(z)} \frac{H(z)}{H(z+c_n)} \cdots \frac{H(z)}{H(z+c_1)}. \tag{4.2}$$

Since $\sigma(H) < 1$, by Lemma 3.2 and Remark 3.1, there exists an ε-set E such that, for $i = 1, 2, \ldots, n$,

$$\frac{H(z)}{H(z+c_i)} \to 1 \quad \text{as } z \to \infty \text{ in } \mathbb{C} \setminus E. \tag{4.3}$$

By (4.2) and (4.3), we get

$$\frac{a_n(z)}{b_0(z)} \to e^{c(c_n+\cdots+c_1)} \quad \text{as } z \to \infty \text{ in } \mathbb{C} \setminus E.$$

So by (3.9), Theorem 1.3 is proved.

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