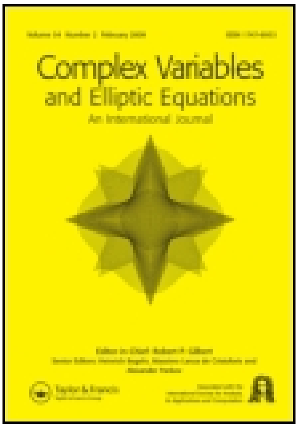

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The properties of the meromorphic solutions of some difference equations

Zhi-Bo Huang [a] [b] , Zong-Xuan Chen [a] & Qian Li [c]
[a] School of Mathematical Sciences , South China Normal University , Guangzhou 510631 , P.R. China
[b] Department of Physics and Mathematics , University of Eastern Finland , P.O. Box 111, 80101 Joensuu , Finland
[c] Department of Applied Mathematics , South China Agricultural University , Guangzhou 510640 , P.R. China
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The properties of the meromorphic solutions of some difference equations

Zhi-Bo Huang[ab]*, Zong-Xuan Chen[a] and Qian Li[c]

[a]*School of Mathematical Sciences, South China Normal University, Guangzhou 510631, P.R. China;* [b]*Department of Physics and Mathematics, University of Eastern Finland, P.O. Box 111, 80101 Joensuu, Finland;* [c]*Department of Applied Mathematics, South China Agricultural University, Guangzhou 510640, P.R. China*

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In this article, we treat complex difference equation of the form

$$A_n(z)f(z+n)+\cdots+A_1(z)f(z+1)+A_0(z)f(z)=A_{n+1}(z),$$

where $A_j(z)(j=0,1,\ldots,n,n+1)$ are meromorphic functions. We give answers to the growth estimates of the meromorphic solutions, and firstly consider the deficiency and fixed points of the meromorphic solutions of these equations. Some examples are listed to show that the result about the properties of fixed points is the best possible in a certain sense.

Keywords: growth; difference equations; deficiency; fixed points

AMS Subject Classifications: 30D35; 39B32

1. Introduction

In this article, we are concerned with the properties of the meromorphic solution of linear difference equations of the forms

$$A_n(z)f(z+n)+\cdots+A_1(z)f(z+1)+A_0(z)f(z)=0, \tag{1.1}$$

and

$$A_n(z)f(z+n)+\cdots+A_1(z)f(z+1)+A_0(z)f(z)=A_{n+1}(z), \tag{1.2}$$

where $A_j(z)(j=0,1,\ldots,n,\ n+1)$ are meromorphic functions.

We use the standard notations of Nevanlinna theory in this article [1–4].

We know that the lemma on the logarithmic derivative of a meromorphic function plays a key role in the study of meromorphic functions and complex differential equations. Thus, in order to use Nevanlinna theory to difference operator and difference equations [5–15], it is necessary to have a difference analogue of the lemma on the logarithmic derivative. Fortunately, there are two papers [9,10]

*Corresponding author. Email: hzbo20019@sina.com

containing very similar results about a difference analogue of the lemma on the logarithmic derivative.

THEOREM 1.1 [10, Theorem 2.1] *Let f(z) be a non-constant meromorphic function,* $c \in \mathbb{C}$, $0 < \delta < 1$ *and* $\varepsilon > 0$. *Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r+|c|,f)^{1+\varepsilon}}{r^{\delta}}\right)$$

for all r outside of a possible exceptional set E with finite logarithmic measure $\int_E \frac{\mathrm{d}r}{r} < +\infty$.

THEOREM 1.2 [9, Corollary 2.6] *Let* η_1, η_2 *be two complex numbers such that* $\eta_1 \neq \eta_2$ *and let f(z) be a finite-order meromorphic function. Let* $\sigma(f)$ *be the order of f(z), then for each* $\varepsilon > 0$, *we have*

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O(r^{\sigma(f)-1+\varepsilon}).$$

In order to relate our results, we also need the following preliminaries.

Let $g(z)$ be an entire function. The order $\sigma(g)$ and the type $\tau(g)$ of $g(z)$ are defined, respectively,

$$\sigma(g) = \limsup_{r\to+\infty} \frac{\log\log M(r,g)}{\log r} = \limsup_{r\to+\infty} \frac{\log T(r,g)}{\log r}, \quad \tau(g) = \limsup_{r\to+\infty} \frac{\log M(r,g)}{r^{\sigma}}.$$

2. The growth of the solutions of the difference equations

Y.M. Chiang and S.J. Feng considered the growth of meromorphic solutions of a general linear difference equations (1.1), and they obtained the following theorem.

THEOREM 2.1 [9, Theorem 9.2] *Let* $A_0(z), A_1(z), \ldots, A_n(z)$ *be entire functions such that there exists an integer l,* $0 \le l \le n$, *such that*

$$\max_{0\le j\neq l\le n} \{\sigma(A_j)\} < \sigma(A_l). \tag{2.1}$$

If f(z) is a meromorphic solution of equations (1.1), *then* $\sigma(f) \ge \sigma(A_l) + 1$.

In Theorem 2.1, the coefficients of (1.1) should satisfy the condition (2.1). If the condition (2.1) was replaced by $\sigma(A_l) = \max_{0\le j\neq l\le n}\{\sigma(A_j)\}$, what will be the results? Regarding this, I. Laine and C.C. Yang obtained the following theorem.

THEOREM 2.2 [15, Theorem 5.2] *Let* $A_0(z), A_1(z), \ldots, A_n(z)$ *be entire functions of finite order such that among those having the maximal order* $\sigma = \max_{0\le j\neq l\le n}\{\sigma(A_j)\}$, *exactly one has its type strictly greater than the others. Then for any meromorphic solution of* (1.1), *we have* $\sigma(f) \ge \sigma(A_l) + 1$.

Remark 2.1 In [15], Laine and Yang asked whether all meromorphic solutions $f(z)$ of equation (1.1) satisfy $\sigma(f) \ge 1 + \max_{0\le j\le n}\{\sigma(A_j)\}$, even if there is no dominating coefficient.

Here, we assert that the above conclusion does not hold identically if there is no dominating coefficient in (1.1). For example:

Example 2.1 Let

$$\Delta f(z) = f(z+1) - f(z), \quad \Delta^{n+1} f(z) = \Delta(\Delta^n f(z)).$$

By $n(\in\mathbb{N})$ times iteration of the above difference operator to $f(z)$, we have

$$\Delta^n f(z) = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} f(z+j), \tag{2.2}$$

and

$$f(z+n) = \sum_{j=0}^{n} \binom{n}{j} \Delta^j f(z).$$

Then, by the Theorem 1.1 in [14], the equation

$$(6z^2 + 19z + 15)\Delta^3 f(z) + (z+3)\Delta^2 f(z) - \Delta f(z) - f(z) = 0, \tag{2.3}$$

admits an entire solution $f(z)$ of order $\sigma(f) = \frac{1}{3}$.

By making use of the relation (2.2), we can rewrite Equation (2.3) to an equation of the form (1.1), i.e.,

$$\begin{aligned}(6z^2 + 19z + 15)f(z+3) - (18z^2 + 56z + 42)f(z+2) \\ + (18z^2 + 55z + 38)f(z+1) - (6z^2 + 18z + 12)f(z) = 0.\end{aligned} \tag{2.4}$$

Thus, Equation (2.4) also has an entire solution $f(z)$ with order $\sigma(f) = \frac{1}{3}$. Here, all the coefficients of (2.4) have order 0 and type $+\infty$. Obviously, the conclusion of Theorem 2.2 does not hold if there is no dominating coefficient.

But the following example shows that, if there is no dominating coefficient, $f(z)$ may satisfy $\sigma(f) \geq 1 + \max_{0\leq j\leq n}\{\sigma(A_j)\}$.

Example 2.2 $f(z) = e^z + z$ is a solution of the equation

$$[(e-1)z - 1]f(z+2) - [(e^2-1)z - 2]f(z+1) + [(e^2-2)z + (e^2 - 2e)]f(z) = 0. \tag{2.5}$$

Here, the all coefficients of (2.5) have the order 0 and the type $+\infty$, but $\sigma(f) = 1$ satisfies the conclusion of Theorem 2.2.

Now, we will discuss the properties of the solutions of Equation (1.2) and obtain the following theorems.

THEOREM 2.3 *Suppose that the coefficients* $A_j(z)(j = 0, 1, \ldots, n, n+1)$ *in* (1.2) *are meromorphic functions with finite order* $\leq \sigma$. *If for any given* $\varepsilon > 0$, *there exists some* $l \in \{0, 1, \ldots, n, n+1\}$ *and an unbounded domain* $D \subset \mathbb{C}$ *such that*

$$|A_l(z)| \geq \exp\{\alpha r^{\sigma-\varepsilon}\},$$

$$|A_j(z)| \leq \exp\{\beta r^{\sigma-\varepsilon}\}, \quad j \in \{0, 1, \ldots, n, n+1\}\backslash\{l\}$$

for all $z \in D$, *where* $\alpha > \beta > 0$ *are real numbers. Then each nontrivial meromorphic solution* $f(z)$ *of* (1.2) *satisfies* $\sigma(f) \geq \sigma$.

THEOREM 2.4 *Suppose that* $A_j(z) = B_j(z)e^{a_j z}(j = 0, 1, \dots, n), A_{n+1}(z) = B_{n+1}(z)$, *where* $B_j(z)(j = 0, 1, \dots, n, n+1)$ *are meromorphic functions with order* $\sigma(B_j) < 1$ $(j = 0, 1, \dots, n, n+1)$, $a_j = \alpha_j e^{i\theta}$, $\alpha_j \geq 0$, $\theta \in [0, 2\pi)(j = 0, 1, \dots, n)$. *If there exists* $l \in \{0, 1, \dots, n\}$ *such that* $\alpha_l > \alpha = \max\{\alpha_j\colon\ j \neq l,\ 0 \leq j \leq n\}$, *then each nontrivial meromorphic solution* $f(z)$ *of equation* (1.2) *satisfies* $\sigma(f) \geq 1$.

COROLLARY 2.5 *Suppose that* $A_j(z) = P_j(z)e^{a_j z}(j = 0, 1, \dots, n)$, where $P_j(z)(j = 0, 1, \dots, n)$ *are polynomials,* $a_j = \alpha_j e^{i\theta}$, $\alpha_j \geq 0$, $\theta \in [0, 2\pi)(j = 0, 1, \dots, n)$. *If there exists* $l \in \{0,\ 1, \dots, n\}$ *such that* $\alpha_l > \alpha = \max\{\alpha_j\colon\ j \neq l,\ 0 \leq j \leq n\}$ *and* $A_{n+1}(z)$ *is an entire function with the order* $\sigma(A_{n+1}(z)) < 1$, *then each nontrivial entire solution* $f(z)$ *of the equation* (1.2) *satisfies* $\sigma(f) \geq 1$.

In order to prove Theorems 2.3 and 2.4, we need the following lemmas.

LEMMA 2.1 [5, Lemma 1] *Given* $\varepsilon > 0$ *and meromorphic function* $f(z)$, *the Nevanlinna characteristic function* $T(r, f)$ *satisfies*

$$T(r, f(z + c)) \leq (1 + \varepsilon)T(r + |c|, f), \quad T(r + |c|, f) = (1 + \varepsilon)T(r, f)$$

for all $r > \frac{1}{\varepsilon}$, where c is a complex number.

LEMMA 2.2 [16, Lemma 2.1] *Let* $g(z)$ *be a meromorphic function of order* $\sigma(g) = \beta < +\infty$. *Then for any given* $\varepsilon > 0$, *there exists a set* $E \subset [0, 2\pi)$ *that has finite linear measure* mE, *such that for all* z *satisfying arg* $z = \phi \in [0, 2\pi)\backslash E$ *and* $|z| = r \geq R > 1$, *we have*

$$\exp\{-r^{\beta+\varepsilon}\} \leq |g(re^{i\phi})| \leq \exp\{r^{\beta+\varepsilon}\}.$$

LEMMA 2.3 *Suppose that* $H(z) = h(z)e^{az}$, *where* $h(z)$ *is a nonzero meromorphic function with order* $\sigma(h) = \alpha < 1$, $a = \mathrm{d}e^{i\theta}$, $\theta \in [0, 2\pi)$, $d \geq 0$ *a constant. Set* $E_0 = \{\phi \in [0, 2\pi)\colon\ \cos(\theta + \phi) = 0\}$. *Then for any given* $\varepsilon(0 < \varepsilon < 1 - \alpha)$, *there exists a set* E *that has linear measure zero, if* $z = re^{i\phi}, \phi \in [0, 2\pi)\backslash(E \bigcup E_0)$, *we have* r *sufficiently large,*

(i) *if* $\cos(\theta + \phi) > 0$, *then*

$$\exp\{(1 - \varepsilon)\mathrm{d}r\cos(\theta + \phi)\} \leq |H(re^{i\phi})| \leq \exp\{(1 + \varepsilon)\mathrm{d}r\cos(\theta + \phi)\},$$

(ii) *if* $\cos(\theta + \phi) < 0$, *then*

$$\exp\{(1 + \varepsilon)\mathrm{d}r\cos(\theta + \phi)\} \leq |H(re^{i\phi})| \leq \exp\{(1 - \varepsilon)\mathrm{d}r\cos(\theta + \phi)\}.$$

Proof of Lemma 2.3 We can use the similar method used in [17] to prove it. Here, we omit it.

LEMMA 2.4 [9, Corollary 8.3] *Let* η_1, η_2 *be two arbitrary complex numbers, and let* $f(z)$ *be a meromorphic function of finite order* $\sigma(f)$. *Let* $\varepsilon > 0$ *be given, then there exists a subset* $E \subset \mathbb{R}$ *with finite logarithmic measure such that for all* $r \notin E \cup [0, 1]$, we have

$$\exp\left\{-r^{\sigma(f)-1+\varepsilon}\right\} \leq \left|\frac{f(z + \eta_1)}{f(z + \eta_2)}\right| \leq \exp(r^{\sigma(f)-1+\varepsilon}).$$

Proof of Theorem 2.3 Suppose that the conclusion does not hold, i.e., $\sigma(f)=\rho<\sigma$. By Lemma 2.1, we have $\sigma(f(z+j))=\rho<\sigma$ for all $j=0, 1, \ldots, n$.

By Lemma 2.2, for any given $\varepsilon(0<2\varepsilon<\sigma-\rho)$, there exists a set $E_1\subset[0, 2\pi)$ that has finite linear measure, such that for all z satisfying $\arg z=\phi\in[0, 2\pi)\backslash E_1$ and $|z|=r\geq R>1$, we have

$$\exp\{-r^{\rho+\varepsilon}\}\leq|f(re^{i\phi}+j)|\leq\exp\{r^{\rho+\varepsilon}\},\quad j=0,1,\ldots,n. \tag{2.6}$$

By assumption, we can choose a sequence of points $\{z_k=r_ke^{i\theta_k}\}\subset D\subseteq\mathbb{C}$, where $\theta_k\in[0,2\pi)\backslash E_1$ and $|z_k|=r_k\geq R>1$, $r_k\to+\infty$ as $k\to+\infty$,such that

$$|A_l(r_ke^{i\theta_k})|\geq\exp\{\alpha {r_k}^{\sigma-\varepsilon}\}, \tag{2.7}$$

$$|A_j(r_ke^{i\theta_k})|\leq\exp\{\beta {r_k}^{\sigma-\varepsilon}\},\quad j\in\{0,1,\ldots,n,n+1\}\backslash\{l\}, \tag{2.8}$$

where $\alpha>\beta>0$ are real numbers.

If $l\neq n+1$, It follows from (1.2) and (2.6–2.8) that

$$\begin{aligned}\exp\{\alpha {r_k}^{\sigma-\varepsilon}-{r_k}^{\rho+\varepsilon}\}&\leq|A_l(r_ke^{i\theta_k})|\cdot|f(r_ke^{i\theta_k}+l)|\\&\leq\sum_{\substack{j=0\\j\neq l}}^{n}|A_j(r_ke^{i\theta_k})|\cdot|f(r_ke^{i\theta_k}+j)|+|A_{n+1}(r_ke^{i\theta_k})|\\&\leq(n+1)\exp\{\beta {r_k}^{\sigma-\varepsilon}+{r_k}^{\rho+\varepsilon}\},\end{aligned}$$

which implies that

$$\exp\{(\alpha-\beta){r_k}^{\sigma-\varepsilon}-2{r_k}^{\rho+\varepsilon}\}\leq n+1$$

holds for all sufficiently large r_k. This is a contradiction since $0<2\varepsilon<\sigma-\rho$ and $\alpha>\beta>0$.

If $l=n+1$, we can use the same method to deduce a similar contradiction.

The proof of Theorem 2.3 is completed.

Proof of Theorem 2.4 Suppose that the conclusion does not hold, i.e., $\sigma(f)=\rho<1$.

Set

$$z=re^{i\phi},\quad \delta(\theta,\phi)=\cos(\theta+\phi),\quad E_0=\{\phi\colon\cos(\theta+\phi)=0\},\quad \beta=\max_{0\leq j\leq n+1}\{\sigma(B_j)\}.$$

Then E_0 is a finite set and $\beta<1$. Thus, for any $\phi\in[0,2\pi)\backslash E_0$, we have $\delta(\theta, \phi)>0$ or $\delta(\theta,\phi)<0$.

Here, we only prove the case $\delta(\theta, \phi)>0$ and deduce a contradiction.

By Lemma 2.3, for any given

$$\varepsilon\left(0<2\varepsilon<\min\left\{\frac{\alpha_l-\alpha}{\alpha_l+\alpha},1-\rho,1-\beta\right\}\right),$$

there exists a set E_2 that has linear measure zero, if $z=re^{i\phi}$, $\phi\in[0,2\pi)\backslash(E_0\bigcup E_2)$, we have r sufficiently large,

$$\left|A_j(re^{i\phi})\right|\leq\exp\{(1+\varepsilon)\alpha r\delta(\theta,\phi)\},\quad j\in\{0,1,\ldots,n,n+1\}\setminus\{l\}, \tag{2.9}$$

$$|A_l(re^{i\phi})|\geq\exp\{(1-\varepsilon)\alpha_l r\delta(\theta,\phi)\}. \tag{2.10}$$

Since we have assumed that $\sigma(f)=\rho<1$, we also have $\sigma(f(z+l))=\sigma\left(\frac{1}{f(z+l)}\right)=\rho<1$. By Lemma 2.2, for any given ε above, there exists a set $E_3\in[0,2\pi)$ that has finite linear measure mE_3, such that for all z satisfying arg $z=\phi\in[0,2\pi)\backslash E_3$ and $|z|=r\geq R>1$, we have

$$\exp\{-r^{\rho+\varepsilon}\}\leq\left|\frac{1}{f(re^{i\phi}+l)}\right|\leq\exp\{r^{\rho+\varepsilon}\}, \tag{2.11}$$

$$\exp\{-r^{\beta+\varepsilon}\}\leq|A_{n+1}(re^{i\phi})|\leq\exp\{r^{\beta+\varepsilon}\}. \tag{2.12}$$

It follows from (1.2), (2.9)–(2.12) and Lemma 2.4 that, for any given

$$\varepsilon\left(0<2\varepsilon<\min\left\{\frac{\alpha_l-\alpha}{\alpha_l+\alpha},1-\rho,1-\beta\right\}\right)$$

and $z=re^{i\phi},\phi\in[0,2\pi)\backslash(E_0\bigcup E_2\bigcup E_3)$, we have r sufficiently large,

$$\begin{aligned}\exp\{(1-\varepsilon)\alpha_l r\delta(\theta,\phi)\}&\leq|A_l(re^{i\phi})|\\&\leq|A_{n+1}(re^{i\phi})|\cdot\frac{1}{|f(re^{i\phi}+l)|}+\sum_{\substack{j=0\\j\neq l}}^{n}|A_j(re^{i\phi})|\cdot\left|\frac{f(re^{i\phi}+j)}{f(re^{i\phi}+l)}\right|\\&\leq\exp\{r^{\beta+\varepsilon}\}\cdot\exp\{r^{\rho+\varepsilon}\}+n\exp\{(1+\varepsilon)\alpha r\delta(\theta,\phi)\}\cdot\exp\{r^{\rho-1+\varepsilon}\}\\&\leq(n+1)\exp\{(1+\varepsilon)\alpha r\delta(\theta,\phi)+r^{\beta+\varepsilon}+r^{\rho+\varepsilon}\},\end{aligned}$$

which implies that

$$\exp\left\{\frac{1}{2}(\alpha_l-\alpha)r\delta(\theta,\phi)-r^{\beta+\varepsilon}-r^{\rho+\varepsilon}\right\}\leq n+1,$$

for all sufficiently large r. This is a contradiction since

$$0<2\varepsilon<\min\left\{\frac{\alpha_l-\alpha}{\alpha_l+\alpha},1-\rho,1-\beta\right\},$$

$\alpha_l>\alpha$ and $\delta(\theta,\phi)>0$.

The proof of Theorem 2.4 is completed.

Now, we consider the more general coefficients of (1.2).

Define

$$\sigma=\max_{0\leq j\leq n+1}\{\sigma(A_j)\},\quad I=\{j\in\{0,1,\ldots,n+1\}:\sigma(A_j)=\sigma\}. \tag{2.13}$$

According to these notations, we obtain the following theorem.

Theorem 2.6 *Suppose that $\sigma>0$ and that $A_j(z)=B_j(z)e^{a_jz^\sigma}$ for all $j\in I$, where $a_j\in\mathbb{C}\backslash\{0\}(j\in I)$ and $B_j(z)(j\in I)$ are meromorphic functions with finite order $\sigma(B_j)<\sigma$. If the constants $a_j(j\in I)$ are distinct, then each nontrivial meromorphic solution $f(z)$ of Equation* (1.2) *satisfies $\sigma(f)\geq\sigma$.*

In order to prove Theorem 2.6, we need the following lemma.

Lemma 2.5 [4, p. 79–80] *Let $f_j(z)(j=1,2,\ldots,n)(n\geq2)$ be meromorphic functions, $g_j(z)(j=1,2,\ldots,n)$ be entire functions, and satisfy*

(1) $\sum_{j=1}^{n}f_j(z)e^{g_j(z)}=0$;

(2) *when* $1 \le j < k \le n$, $g_j(z) - g_k(z)$ *is not a constant*;
(3) *when* $1 \le j \le n$, $1 \le h < k \le n$,

$$T(r, f_j) = o\left\{T(r, e^{g_h - g_k})\right\}(r \to +\infty, r \notin E),$$

where $E \subset (1, +\infty)$ *is of finite linear measure or finite logarithmic measure. Then* $f_j(z) \equiv 0 (j = 1, 2, \ldots, n)$.

Proof of Theorem 2.6 We suppose that $\sigma(f) < +\infty$. Assume that the assertion does not hold, i.e., every nontrivial solution $f(z)$ of Equation (1.2) satisfies $\sigma(f) = \rho < \sigma$. By Lemma 2.1, we have $\sigma(f(z+j)) = \sigma(f) = \rho < \sigma$ for all $j = 0, 1, 2, \ldots, n$.

Now, we can rewrite Equation (1.2) the form

$$\sum_{j \in I} G_j(z) e^{a_j z^\sigma} + B(z) = 0. \tag{2.14}$$

In (2.14), if $n+1 \in I$,

$$G_j(z) = B_j(z) f(z+j) (j \in I \backslash \{n+1\}), G_{n+1}(z) = -B_{n+1}, B(z) = \sum_{j \notin I} A_j(z) f(z+j),$$

if $n+1 \notin I$,

$$G_j(z) = B_j(z) f(z+j) (j \in I), \quad B(z) = \sum_{j \notin I} A_j(z) f(z+j) - A_{n+1}(z).$$

By (2.13) and the assumption of Theorem 2.6 $\sigma(G_j) < \sigma$ and $B(z)$ is a meromorphic function with finite order $\sigma(B) < \sigma$.

It follows from Lemma 2.5 and (2.14) that $G_j(z) \equiv 0$, $j \in I$. This is impossible. The proof of Theorem 2.3 is completed.

Now, we consider that the coefficients of Equation (1.2) are transcendental meromorphic functions and obtain the following theorem.

Theorem 2.7 *Suppose that* $\sigma = +\infty$ *and that* $A_j(z) = B_j(z) e^{g_j(z)}$ *for all* $j \in I$, *where* $g_j(z)$ *are transcendental entire functions and* $B_j(z) (j \in I)$ *are meromorphic functions with finite order. Moreover, suppose that* $g_i(z) - g_j(z)$ *is transcendental entire function for all* $i, j \in I$, $i \neq j$. *Then each nontrivial solution of Equation* (1.2) satisfies $\sigma(f) = +\infty$.

Proof of Theorem 2.7 The proof is similar to the proof of Theorem 2.6. Here we omit it.

Remark From Theorems 2.6 and 2.7, if the coefficients $A_j(z) (j = 0, 1, \ldots, n, n+1)$ are entire functions, We can find that those coefficients have the maximum order σ, exactly one of its type is strictly greater than the others. So we obtain the following theorem.

Theorem 2.8 *Suppose that the coefficients* $A_j(z) (j = 0, 1, \ldots, n, n+1)$ *in* (1.2) *are entire functions with finite order such that among those coefficients having the maximum order* $\sigma = \max_{0 \le j \le n+1}\{\sigma(A_j(z))\}$, *exactly one has its type strictly greater than the others. Then each nontrivial entire solution* $f(z)$ *of* (1.2) *satisfies* $\sigma(f) \ge \sigma$. *Moreover, if* $f(z)$ *is an entire solution of* (1.2) *with finite order* $\sigma(f) = \sigma$ *and if* $l \in I$ *and* $\tau(A_l) > \tau = \max\{\tau(A_j) \colon j \in I \backslash \{l\}\}$, *then* $\tau(f) \ge \tau(A_l) - \max\{\tau(A_j) \colon j \in I \backslash \{l\}\}$.

In order to prove Theorem 2.8, we need the following lemmas.

LEMMA 2.6 [18, Lemma 4] *Let $f(z)$ be an entire function of order $\sigma(f) = \sigma < +\infty$. Then for any given $\varepsilon > 0$, there is a set $E \subset [1, +\infty)$ that has finite linear measure mE and finite logarithmic measure lmE, such that for all z satisfying $|z| = r \notin [0, 1] \cup E$,*

$$\exp\{-r^{\sigma+\varepsilon}\} \le |f(z)| \le \exp\{r^{\sigma+\varepsilon}\}.$$

LEMMA 2.7 *Let $f(z)$ be an entire function with the order $\sigma(f) = \sigma(0 < \sigma < +\infty)$ and the type $\tau(f) = \tau(0 < \tau \le +\infty)$. Then for any given positive number $\beta < \tau$, there exists a set $E \subset [1, +\infty)$ that has infinite linear measure mE and infinite logarithmic measure lmE, such that for all $r \in E$,*

$$\log M(r, f) > \beta r^{\sigma}.$$

Remark 2.2 In [19], Tu and Yi obtained the same result when $0 < \tau < +\infty$ and $E \subset [1, +\infty)$ that has infinite logarithmic measure. The main idea of the proof of Lemma 2.7 comes from [19], but the details are somewhat different. For the convenience, we give a complete proof.

Proof of Lemma 2.7 We prove the conclusion by considering the following two cases.

Case 1 If $0 < \tau < +\infty$.

By the definition of type function, there exists an increasing sequences $\{r_n\}(r_n \to +\infty, n \to +\infty)$ satisfying $\left(1 + \frac{1}{n}\right)r_n < r_{n+1}$ and

$$\lim_{n\to+\infty} \frac{\log M(r_n, f)}{r_n^{\sigma}} = \tau.$$

Then for any given positive number $\beta < \tau$ and for any given $\varepsilon(0 < \varepsilon < \tau - \beta)$, there exists $N_0 \in \mathbb{N}$ such that for all $n \ge N_0$, we have

$$\log M(r_n, f) > (\tau - \varepsilon) r_n^{\sigma}. \tag{2.15}$$

Since $0 < \varepsilon < \tau - \beta$, we have $0 < \frac{\beta}{\tau - \varepsilon} < 1$. Thus, there exists $N_1 \in \mathbb{N}$ such that for all $n \ge N_1$, we have

$$\left(\frac{n}{n+1}\right)^{\sigma} > \frac{\beta}{\tau - \varepsilon}. \tag{2.16}$$

It follows from (2.15) and (2.16) that for all $n \ge N = \max\{N_0, N_1\}$ and for all $r \in \left[r_n, \left(1 + \frac{1}{n}\right)r_n\right]$, we have

$$\log M(r, f) \ge \log M(r_n, f) > (\tau - \varepsilon) r_n^{\sigma} \ge (\tau - \varepsilon)\left(\frac{n}{n+1} r\right)^{\sigma} > \beta r^{\sigma}. \tag{2.17}$$

Set

$$E = \bigcup_{n=N}^{+\infty} \left[r_n, \left(1 + \frac{1}{n}\right) r_n\right],$$

then

$$mE = \sum_{n=N}^{+\infty} \frac{r_n}{n} = +\infty \tag{2.18}$$

since

$$\lim_{n\to+\infty}\left[\frac{r_{n+1}}{n+1}/\frac{r_n}{n}\right]=\lim_{n\to+\infty}\frac{r_{n+1}}{r_n}\cdot\frac{n}{n+1}>\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)\cdot\frac{n}{n+1}=1,$$

and

$$lmE=\sum_{n=N}^{+\infty}\int_{r_n}^{\left(1+\frac{1}{n}\right)r_n}\frac{1}{t}dt=\sum_{n=N}^{+\infty}\log\left(1+\frac{1}{n}\right)=+\infty. \tag{2.19}$$

Case 2 If $\tau=+\infty$.

By the definition of type function, there exists an increasing sequence $\{r_n\}(r_n\to+\infty,\ n\to+\infty)$ satisfying $\left(1+\frac{1}{n}\right)r_n<r_{n+1}$ such that

$$\log M(r_n,f)>Ar_n^{\sigma} \tag{2.20}$$

for any given positive number $A<+\infty$.

For any given positive number $\beta<A$, there exists $N\in\mathbb{N}$ such that for all $n\geq N$, we have

$$\left(\frac{n}{n+1}\right)^{\sigma}>\frac{\beta}{A}. \tag{2.21}$$

By using the same method similar to case 1, we also can prove that (2.17)–(2.19) hold.

Together with Case 1 and Case 2, the proof of Lemma 2.7 is completed.

Proof of Theorem 2.8 Suppose that the conclusion does not hold, i.e., $\sigma(f)=\rho<\sigma$, then $\sigma(f+j)=\sigma\left(\frac{1}{f(z+j)}\right)=\rho<\sigma$ for all $j=0,1,\ldots,n$.

Since the coefficients of (1.2) have the maximum order $\sigma=\max_{0\leq j\leq n+1}\{\sigma(A_j(z))\}$, exactly one has its type strictly greater than the others, without loss of generality, we can set $l\in I$ and satisfies $\tau(A_l)>\tau=\max\{\tau(A_j): j\in I\backslash\{l\}\}$.

Define

$$\beta=\max\{\sigma(A_j): j\in\{0,1,\ldots,n,n+1\}\backslash I\},$$

then $\beta<\sigma$ by (2.13).

By Lemma 2.6, for any given $\varepsilon(0<2\varepsilon<\min\{\sigma-\rho,\ \sigma-\beta\})$, there exists a set $E_4\subset[1,+\infty)$ that has finite linear measure and finite logarithmic measure , such that for all z satisfying $|z|=r\notin[0,1]\cup E_4$, we have

$$\left|\frac{1}{f(z+l)}\right|\leq\exp\{r^{\rho+\varepsilon}\},\quad |f(z+j)|\leq\exp\{r^{\rho+\varepsilon}\},\quad j=0,1,\ldots,n, \tag{2.22}$$

$$|A_j(z)|\leq\exp\{r^{\beta+\varepsilon}\}\leq\exp\{r^{\sigma-\varepsilon}\},\quad j\in\{0,1,\ldots,n,n+1\}\backslash I. \tag{2.23}$$

Let α_1,α_2 be positive real numbers such that $\tau<\alpha_1<\alpha_2<\tau(A_l)$. By Lemma 2.7 and the definition of type of an entire function, there exists a set $E_5\subset[1,+\infty)$ that has infinite linear measure and infinite logarithmic measure , such that for all z satisfying $|z|=r\in E_5$, we have

$$M(r,A_l(z))>\exp\{\alpha_2 r^{\sigma}\}, \tag{2.24}$$

$$M(r,A_j(z))<\exp\{\alpha_1 r^{\sigma}\}, j\in I\backslash\{l\}. \tag{2.25}$$

Thus, for all z satisfying $|z| = r \in E_5 \backslash ([0, 1] \cup E_4)$, (2.22)–(2.25) hold. So there exists a subsequence $\{r_n: |z| = r_n\} \in E_5 \backslash ([0, 1] \cup E_4)$ such that $|A_l(z)| = M(r, A_l(z))$ and

$$\exp\{(\alpha_2 - \alpha_1) r_n^{\sigma} - 2 r_n^{\rho+\varepsilon}\} \leq n + 1,$$

whether $l = n + 1$ or $l \neq n + 1$.

Now we will show that $\tau(f) \geq \tau(A_l) - \tau$. Suppose that the conclusion does not hold, i.e., $\tau(f) < \tau(A_l) - \tau$.

Since we suppose that $f(z)$ is an entire solution of (1.2) with finite order $\sigma(f) = \sigma$, we have $\sigma(f(z+j)) = \sigma\left(\frac{1}{f(z+j)}\right) = \sigma < +\infty$ for all $j = 0, 1, \ldots, n$.

By Lemma 2.6, for any given $\varepsilon (0 < 4\varepsilon < \min\{\sigma - \beta, \tau(A_l) - \tau\})$, there exists a set $E_6 \subset [1, +\infty)$ that has finite linear measure and finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_6$, we have

$$\left| \frac{1}{f(z+l)} \right| \leq \exp\{r^{\sigma+\varepsilon}\}, \tag{2.26}$$

$$|A_j(z)| \leq \exp\{r^{\beta+\varepsilon}\} \leq \exp\{r^{\sigma-\varepsilon}\}, \quad j \in \{0, 1, \ldots, n, n+1\} \backslash I. \tag{2.27}$$

By Lemma 2.7 and the definition of type of an entire function, there exists a set $E_7 \subset [1, +\infty)$ that has infinite linear measure and infinite logarithmic measure, such that for all z satisfying $|z| = r \in E_7$, we have

$$M(r, A_l(z)) > \exp\{(\tau(A_l) - \varepsilon) r^{\sigma}\}, \tag{2.28}$$

$$M(r, f(z+j)) \leq \exp\{(\tau(f) + \varepsilon) r^{\sigma}\}, \quad j = 0, 1, \ldots, n, \tag{2.29}$$

$$M(r, A_j(z)) < \exp\{(\tau + \varepsilon) r^{\sigma}\}, \quad j \in I \backslash \{l\}. \tag{2.30}$$

Thus, for all z satisfying $|z| = r \in E_7 \backslash ([0, 1] \cup E_6)$, (2.26)–(2.30) hold. So there exists a subsequence $\{r_n : |z| = r_n\} \in E_7 \backslash ([0, 1] \cup E_6)$ such that $|A_l(z)| = M(r, A_l(z))$ and

$$\begin{aligned} &\exp\{(\tau(A_l) - \tau - 2\varepsilon) r^{\sigma} - (\tau(f) + \varepsilon) r^{\sigma} - r^{\sigma+\varepsilon}\} \\ &< \exp\{(\tau(A_l) - \tau - 2\varepsilon) r^{\sigma} - (\tau(f) + \varepsilon) r^{\sigma}\} \leq n + 1, \end{aligned}$$

whether $l = n + 1$ or $l \neq n + 1$.

This is a contradiction. The proof of Theorem 2.8 is completed.

3. The deficiency and fixed points of the solutions of the difference equations

We first briefly recall some of the basic definition of Nevanlinna theory. We refer to [9,13] for a comprehensive description of the value distribution theory. Denote the Nevanlinna deficiency of a by

$$\delta(a, f) = \liminf_{r \to +\infty} \frac{m(r, a)}{T(r, f)} = 1 - \limsup_{r \to +\infty} \frac{N\left(r, \frac{1}{f-a}\right)}{T(r, f)}$$

for a non-constant meromorphic function $f(z)$ and for all $a \in \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$.

Let $f(z)$ be meromorphic function. Set $g(z) = f(z) - z$, then z is a fixed point of $f(z)$ if and only if $g(z) \equiv 0$.

In [6,7], Bergweiler et al. considered the zeros and fixed points of differences of meromorphic functions. Here, we consider the deficiency and the fixed points of solutions of difference equations (1.1) and (1.2).

In [3], Lanie considered the differential equation and obtained the following Theorem.

THEOREM 3.1 [3, Theorem 4.3] *Let $f(z)$ be an admissible meromorphic solution of equation*

$$a_n(z)f^{(n)}(z) + a_{n-1}(z)f^{(n-1)}(z) + \cdots + a_0(z)f(z) = 0, \quad a_0(z)a_n(z) \not\equiv 0, \tag{3.1}$$

where $T(r, a_j) = S(r,f)$ for all $j = 0, 1, \ldots, n$. Then

$$\delta(\alpha, f) = 0$$

for all $\alpha \neq 0$, ∞. Especially this is true for transcendental solutions of (3.1) *with polynomial coefficients.*

If we consider the difference equation (1.1), by using the similar method as Theorem 3.1, we also obtain the similar result as follows.

THEOREM 3.2 *Let $f(z)$ be a finite-order meromorphic solution of Equation* (1.1), *where the entire coefficients $A_j(z)(j = 0, 1, \ldots, n)$ are small functions relative to $f(z)$.*

(1) *If $a(z)$ is a small meromorphic function relative to $f(z)$ and satisfies*

$$\sum_{j=0}^{n} A_j(z)a(z+j) \not\equiv 0, \tag{3.2}$$

we obtain

$$\delta(a(z), f) = 0.$$

(2) *If $a(z) = z$ is a small function relative to $f(z)$ and satisfies*

$$\sum_{j=0}^{n} A_j(z)(z+j) \not\equiv 0, \tag{3.3}$$

we obtain that $f(z)$ has infinitely many fixed points and satisfies $\lambda(f(z) - z) = \sigma(f)$.

COROLLARY 3.3 *Suppose that the coefficients $A_j(z)(j = 0, 1, \ldots, n)$ in* (1.1) *are entire functions with finite order such that among those coefficients having the maximum order $\sigma = \max_{0 \leq j \leq n}\{\sigma(A_j)\}$, exactly one has its type strictly greater than the others. Let $f(z)$ be a finite order meromorphic solution of Equation* (1.1).

(1) *If $a(z)$ is a meromorphic function with finite order $< \sigma + 1$ and satisfies*

$$\sum_{j=0}^{n} A_j(z)a(z+j) \not\equiv 0,$$

we obtain

$$\delta(a(z), f) = 0.$$

(2) *If $a(z) = z$ and satisfies*

$$\sum_{j=0}^{n} A_j(z)(z+j) \not\equiv 0,$$

we obtain that $f(z)$ has infinitely many fixed points and satisfies $\lambda(f(z) - z) = \sigma(f)$.

Now, we will consider the fixed point of the solutions of the more general equations (1.2), and obtain the following theorems.

THEOREM 3.4 *Suppose that* $\sigma > 0$ *and that* $A_j(z) = B_j(z)e^{a_j z^\sigma}$ *for all* $j \in I$, *where* $a_j \in \mathbb{C}\backslash\{0\}(j \in I)$ *and* $B_j(z)(j \in I)$ *are meromorphic functions with finite order* $\sigma(B_j) < \sigma$. *If the constants* $a_j(j \in I)$ *are distinct, then every nontrivial finite-order meromorphic solution* $f(z)$ *of Equation* (1.2) *has infinitely many fixed points and satisfies* $\lambda(f(z) - z) = \sigma(f) \geq \sigma$.

Proof By using Theorem 2.6, we obtain that every nontrivial solution $f(z)$ of Equation (1.2) satisfies $\sigma(f) \geq \sigma$. Now we suppose that the assertion does not hold, i.e., $\lambda(f(z) - z) < \sigma(f) < \infty$. This shows that there exists a positive integer $k(\geq\sigma)$ such that $\sigma(f(z) - z) = \sigma(f) = k$. Thus we can rewrite $f(z) - z$ the form

$$f(z) - z = P(z)e^{\beta z^k}, \tag{3.4}$$

where β is a nonzero constant and $P(z)$ is a meromorphic function with

$$\sigma(P) \leq \max\{\lambda(f(z) - z), k - 1\} < k.$$

From (3.4), we have

$$f(z + j) = z + j + P(z + j)Q_j(z)e^{\beta z^k}, \tag{3.5}$$

where

$$Q_j(z) = \exp\{\beta C_k^1 z^{k-1} j + \beta C_k^2 z^{k-2} j^2 + \cdots + \beta j^k\}, \quad \sigma(Q_j) = k - 1,\ j = 1, \ldots, n.$$

By (1.2), (3.4) and (3.5), we obtain

$$\sum_{j=0}^{n} P(z + j)Q_j(z)A_j(z)e^{\beta z^k} + \sum_{j=0}^{n}(z + j)A_j(z) = A_{n+1}(z). \tag{3.6}$$

Together with (2.13) and $A_j(z) = B_j(z)e^{a_j z^\sigma}$ for all $j \in I$, if $n + 1 \in I$, (3.6) can be rewritten in the form

$$\begin{aligned}\sum_{j \in I\backslash\{n+1\}} P(z + j)Q_j(z)B_j(z)e^{a_j z^\sigma + \beta z^k} + \sum_{j \in I\backslash\{n+1\}} (z + j)B_j(z)e^{a_j z^\sigma} - B_{n+1}(z)e^{a_{n+1} z^\sigma} \\ + \left(\sum_{j \notin I} P(z + j)Q_j(z)A_j(z)\right)e^{\beta z^k} + \sum_{j \notin I}(z + j)A_j(z) = 0,\end{aligned} \tag{3.7}$$

if $n + 1 \notin I$, (3.6) can be rewritten in the form

$$\begin{aligned}\sum_{j \in I} P(z + j)Q_j(z)B_j(z)e^{a_j z^\sigma + \beta z^k} + \sum_{j \in I}(z + j)B_j(z)e^{a_j z^\sigma} \\ + \left(\sum_{j \notin I} P(z + j)Q_j(z)A_j(z)\right)e^{\beta z^k} + \left(\sum_{j \notin I}(z + j)A_j(z) - A_{n+1}(z)\right) = 0.\end{aligned} \tag{3.8}$$

By the above assumptions, we find that (3.7) and (3.8) satisfy the conditions of Lemma 2.5 respectively. Hence, we obtain

$$P(z+j)Q_j(z)B_j(z) \equiv 0, j \in I \backslash \{n+1\}.$$

This is a contradiction. The proof of Theorem 3.4 is completed.

THEOREM 3.5 *Suppose that the coefficients* $A_j(z)(j=0,1,\ldots,n,n+1)$ *in* (1.2) *are entire functions with finite order such that among those coefficients having the maximum order* $\sigma = \max_{0 \le j \le n+1}\{\sigma(A_j(z))\}$, *exactly one has its type strictly greater than the others. Then every nontrivial finite-order meromorphic solution* $f(z)$ *of Equation* (1.2) *has infinitely many fixed points and satisfies* $\lambda(f(z)-z)=\sigma(f) \ge \sigma$.

The remaining part of this section is devoted to show that the result of Theorem 3.2(2) is the best possible in certain senses.

Example 3.1 $f(z)=e^z+1$ is a solution of the equation

$$(z^2+1)f(z+2)-(e+1)(z^2+1)f(z+1)+e(z^2+1)f(z)=0. \tag{3.9}$$

Obviously, the coefficients of Equation (3.9) are small functions relative to $f(z)$, and

$$(z^2+1)(z+2)-(e+1)(z^2+1)(z+1)+e(z^2+1)z=(1-e)(z^2+1) \not\equiv 0,$$

also satisfies (3.3). Here, $f(z)=e^z+1$ has infinitely many fixed points and satisfies $\lambda(f(z)-z)=\sigma(f)=1$.

The following example shows that the condition (3.3) cannot be omitted.

Example 3.2 $f(z)=e^z-z$ and $f(z)=e^z+z$ both are the solutions of the equation

$$[(e-1)z-1]f(z+2)-[(e^2-1)z-2]f(z+1)+[(e^2-e)z+(e^2-2e)]f(z)=0. \tag{3.10}$$

Obviously, the coefficients of Equation (3.10) are small functions relative to $f(z)$. But

$$[(e-1)z-1](z+2)-[(e^2-1)z-2](z+1)+[(e^2-e)z+(e^2-2e)]z \equiv 0.$$

Thus, it does not satisfy the condition (3.3). In this case, $f(z)=e^z-z$ has infinitely many fixed points and satisfies $\lambda(f(z)-z)=\sigma(f)=1$, but $f(z)=e^z+z$ has no fixed point.

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