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Research Article

On q-Difference Riccati Equations and Second-Order Linear q-Difference Equations

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We consider q-difference Riccati equations and second-order linear q-difference equations in the complex plane. We present some basic properties, such as the transformations between these two equations, the representations and the value distribution of meromorphic solutions of q-difference Riccati equations, and the q-Casorati determinant of meromorphic solutions of second-order linear q-difference equations. In particular, we find that the meromorphic solutions of these two equations are concerned with the q-Gamma function when $q \in \mathbb{C}$ such that $0 < |q| < 1$. Some examples are also listed to illustrate our results.

1. Introduction and Main Results

In this paper, a meromorphic function means meromorphic in the whole complex plane $\mathbb{C}$, unless stated otherwise. We also assume that the reader is familiar with the standard symbols and fundamental results such as $m(r, f), N(r, f)$, and $T(r, f)$, of Nevanlinna theory, see, for example, [1, 2], for a given meromorphic function $f(z)$. A meromorphic function $a(z)$ is said to be a small function relative to $f(z)$ if $T(r, a) = S(r, f)$, where $S(r, f)$ is used to denote any quantity satisfying $S(r, f) = o(\{T(r, f)\}$ as $r \to \infty$, possibly outside of a set of finite logarithmic measure, furthermore, possibly outside of a set E of logarithmic density $\operatorname{logdens}(E) = \lim_{r\to\infty} \int_{[1,r]\cap E} (dt/t)/\log r = 0$. For a small function $a(z)$ relative to $f(z)$, we define

$$\delta(a, f) = \varliminf_{r\to\infty} \frac{m(r, 1/(f-a))}{T(r, f)} = 1 - \varlimsup_{r\to\infty} \frac{N(r, 1/(f-a))}{T(r, f)}. \tag{1}$$

Recently, Ishizaki [3] considered difference Riccati equation

$$\Delta f(z) + \frac{f(z)^2 + A(z)}{f(z) - 1} = 0, \tag{2}$$

and second-order linear difference equation

$$\Delta^2 y(z) + A(z) y(z) = 0, \tag{3}$$

where $A(z)$ is meromorphic function, and gave surveys of basic properties of (2) and (3), which are analogues in the differential cases.

Now, we are concerned with q-difference Riccati equation

$$g(qz) = -\frac{a_1(z) g(z) + a_0(z)}{g(z)}, \tag{4}$$

and second-order linear q-difference equation

$$f(q^2 z) + a_1(z) f(qz) + a_0(z) f(z) = 0, \tag{5}$$

where $q \in \mathbb{C} \setminus \{0\}$, $|q| \neq 1$, $a_1(z)$ and $a_0(z) \not\equiv 0$ are rational functions and will obtain some parallel results for q-difference case. For a meromorphic function $h(z)$, the q-difference operator Δ_q is defined by $\Delta_q h(z) = h(qz) - h(z)$.

This paper is organized as follows. In Section 2, we describe the transformation between q-difference Riccati equation (4) and second-order linear q-difference equation (5). In Section 3, we present some properties of q-difference Riccati equation (4), such as q-difference analogue on the property of a cross ratio for four distinct meromorphic solutions of a differential Riccati equation, the meromorphic solutions concerning with q-Gamma function. In Section 4, we study the value distribution of transcendental meromorphic solutions of q-difference Riccati equation (4) and the form of meromorphic solutions of second-order linear q-difference equation (5). In Section 5, we discuss the properties on the q-Casorati determinant of meromorphic solutions of second-order linear q-difference equation (5).

2. Transformations between q-Difference Riccati Equations and Linear q-Difference Equations of Second-Order

It is well known that a differential Riccati equation

$$w'(z) + w(z)^2 + A(z) = 0 \tag{6}$$

and second-order linear differential equation

$$u''(z) + A(z)u(z) = 0 \tag{7}$$

are closely related by the transformation

$$w(z) = -\frac{u'(z)}{u(z)}, \tag{8}$$

where $A(z)$ is a meromorphic function, see, for example, [4, pages 103–106].

Ishizaki [3] considered a difference analogue of (6) and (7) and obtained that difference Riccati equation (2) and second-order linear difference equation (3) are closely linked by the transformation

$$f(z) = -\frac{\Delta y(z)}{y(z)}, \tag{9}$$

where $A(z)$ is a meromorphic function.

Here, we are concerned with a transformation between (4) and (5), see [5]. For a nontrivial meromorphic solution $f(z)$ of (5), we take

$$g(z) = \frac{f(qz)}{f(z)}. \tag{10}$$

Then $g(z)$ satisfies q-difference Riccati equation (4). In fact, we deduce from (5) that

$$\frac{f(q^2z)}{f(qz)} + a_1(z) + a_0(z)\frac{f(z)}{f(qz)} = 0, \tag{11}$$

which implies the desired form of (4).

Conversely, if (4) admits a nontrivial meromorphic solution $g(z)$, then meromorphic function $f(z)$ of first-order q-difference equation (10) satisfies (5). In fact, we conclude from (4) and (10) that

$$\begin{aligned} f(q^2z) &= g(qz)f(qz) = \left(-\frac{a_1(z)g(z)+a_0(z)}{g(z)}\right)f(qz) \\ &= -a_1(z)f(qz) - a_0(z)f(z), \end{aligned} \tag{12}$$

which implies (5).

Example 1. Suppose that $q \in \mathbb{C} \setminus \{0\}$ and $|q| \neq 1$. Let $a_0(z) = (q^2z^2 - (q^2 - 2q - 1)z + 1)/(1 - z^2)$ and $a_1(z) = 2/(z - 1)$. Then $g(z) = (qz + 1)/(z + 1)$ and $f(z) = z + 1$ satisfy q-difference Riccati equation (4) and second-order linear q-difference equation (5), respectively, which both satisfy the transformation (10).

3. Representations of Solutions of q-Difference Riccati Equations

The representations on meromorphic solutions of Riccati equations are interesting. Bank et al. [6, pages 371–373] obtained that differential Riccati equation (6) possesses a one parameter family of meromorphic solutions $(f_c)_{c\in\mathbb{C}}$ if (6) has three distinct meromorphic solutions $\alpha_1(z)$, $\alpha_2(z)$, and $\alpha_3(z)$. Ishizaki extended this property to difference Riccati equation (2) and obtained a difference analogue of this property, see [3, Proposition 2.1]. Now, we present this property for q-difference case below, which can also be seen as a q-difference analogue of the fact that a cross ratio for four distinct meromorphic solutions of a differential Riccati equation is a constant, see, for example, [4, pages 108-109]. Furthermore, we find that meromorphic solutions of q-difference Riccati equations (4) are concerned with q- Gamma function if $q \in \mathbb{C}$ such that $0 < |q| < 1$.

Theorem 2. *Suppose that* (4) *possesses three distinct meromorphic solutions $g_1(z)$, $g_2(z)$, and $g_3(z)$. Then any meromorphic solution $g(z)$ of* (4) *can be represented by*

$$\begin{aligned} g(z) = {} & (g_1(z)g_2(z) - g_2(z)g_3(z) - g_1(z)g_2(z)\phi(z) \\ & + g_1(z)g_3(z)\phi(z)) \\ & \times (g_1(z) - g_3(z) - g_2(z)\phi(z) + g_3(z)\phi(z))^{-1}, \end{aligned} \tag{13}$$

where $\phi(z)$ is a meromorphic function satisfying $\phi(qz) = \phi(z)$. Conversely, if for any meromorphic function $\phi(z)$ satisfying $\phi(qz) = \phi(z)$, we define a function $g(z)$ by (13)*, then $g(z)$ is a meromorphic solution of* (4).

Proof of Theorem 2. Let $h_j(z)$, $j = 1, 2, 3, 4$ be distinct meromorphic functions. We denote a cross ratio of $h_j(z)$, $j = 1, 2, 3, 4$ by

$$R(h_1, h_2, h_3, h_4; z) = \frac{h_1(z) - h_3(z)}{h_1(z) - h_4(z)} : \frac{h_2(z) - h_3(z)}{h_2(z) - h_4(z)}. \tag{14}$$

Suppose that $g(z)$ is meromorphic solution of (4) and is also distinct from $g_1(z)$, $g_2(z)$, and $g_3(z)$. We first show that $g(z)$ is a meromorphic solution of q-difference Riccati equation (4) if and only if $R(qz) = R(z)$, where $R(z) = R(g_1, g_2, g_3, g; z)$. In fact, we conclude from (4) that

$$\begin{aligned} R(qz) &= \frac{g_1(qz) - g_3(qz)}{g_1(qz) - g(qz)} : \frac{g_2(qz) - g_3(qz)}{g_2(qz) - g(qz)} \\ &= \frac{a_0(z)(g_1(z) - g_3(z))/g_1(z)g_3(z)}{a_0(z)(g_1(z) - g(z))/g_1(z)g(z)} \\ &\quad : \frac{a_0(z)(g_2(z) - g_3(z))/g_2(z)g_3(z)}{a_0(z)(g_2(z) - g(z))/g_2(z)g(z)} \\ &= \frac{g_1(z) - g_3(z)}{g_1(z) - g(z)} : \frac{g_2(z) - g_3(z)}{g_2(z) - g(z)} = R(z). \end{aligned} \tag{15}$$

Conversely, if $R(qz) = R(z)$, then

$$\begin{aligned} &\frac{a_0(z)(g_1(z) - g_3(z))/g_1(z)g_3(z)}{-(a_1(z)g(z) + a_0(z)/g(z)) - g(qz)} \\ &\quad : \frac{a_0(z)(g_2(z) - g_3(z))/g_2(z)g_3(z)}{-(a_2(z)g(z) + a_0(z)/g(z)) - g(qz)} \\ &= \frac{g_1(z) - g_3(z)}{g_1(z) - g(z)} : \frac{g_2(z) - g_3(z)}{g_2(z) - g(z)}. \end{aligned} \tag{16}$$

We conclude from (16) that $g(qz) = -(a_1(z)g(z) + a_0(z))/g(z)$, which shows that $g(z)$ satisfies (4).

Thus, for any meromorphic function $\phi(z)$ satisfying $\phi(qz) = \phi(z)$, we define $g(z)$ by

$$R(g_1, g_2, g_3, g; z) = \phi(z). \tag{17}$$

Then $g(z)$ is represented by (13), and also satisfies q-difference Riccati equation (4). The proof of Theorem 2 is completed. □

Now, we recall some results of transcendental meromorphic solutions concerned with q-difference Riccati equation (4). Bergweiler et al. [7, 8] pointed out that all transcendental meromorphic solutions of (5) satisfy $T(r, f) = O((\log r)^2)$ if $q \in \mathbb{C}$ and $0 < |q| < 1$. Since (10) is a transformation between (4) and (5), we obtain that all transcendental meromorphic solutions of (4) are of order zero if $q \in \mathbb{C}$ and $0 < |q| < 1$. On the other hand, if $g(z)$ is a transcendental meromorphic solution of

$$g(qz) = R(z, g(z)), \tag{18}$$

where $q \in \mathbb{C}, |q| > 1$ and the coefficients of $R(z, g(z))$ are small functions relative to $g(z)$, Gundersen et al. [9] showed that the order of growth of (18) is equal to $\log \deg_g(R)/\log|q|$, where $\deg_g(R)$ is the degree of irreducible rational function $R(z, g(z))$ in $g(z)$. Thus, from the above two cases, we obtain that all transcendental meromorphic solutions of (4) are of order zero for all $q \in \mathbb{C} \setminus \{0\}$ and $|q| \neq 1$.

We also illustrate some of the results on q-difference equations, which are explicitly solvable in terms of known zero-order meromorphic functions (see [5]). Let $q \in \mathbb{C}$ be such that $0 < |q| < 1$. Then q-Gamma function $\Gamma_q(x)$ is defined by

$$\Gamma_q(x) := \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \tag{19}$$

where $(a;q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k)$. It is a meromorphic function with poles at $x = -n \pm 2\pi ik/\log q$, where k and n are nonnegative integers, see [10]. By defining

$$\gamma_q(z) := (1-q)^{x-1}\Gamma_q(x), \quad z = q^x, \tag{20}$$

and $\gamma_q(0) := (q;q)_\infty$, we see that $\gamma_q(z)$ is a meromorphic function of zero-order with no zeros, having its poles at $\{q^k\}_{k=0}^{\infty}$.

Therefore, the first-order linear q-difference equation

$$h(qz) = (1 - z)h(z) \tag{21}$$

is solved by the function $\gamma_q(z)$. Moreover, for general first-order linear q-difference equation,

$$h(qz) = a(z)h(z), \tag{22}$$

where $a(z)$ is a rational function. If $a(z) \equiv a$ is a constant, (22) is solvable in terms of rational functions if and only if $\log_q a$ is an integer. If $a(z)$ is nonconstant, let $\alpha_i, i = 1, 2, \ldots, n$ and $\beta_j, j = 1, 2, \ldots, m$ be the zeros and poles of $a(z)$, respectively, repeated according to their multiplicities. Then $a(z)$ can be written in the form

$$a(z) = \frac{c(1 - z/\alpha_1)(1 - z/\alpha_2)\cdots(1 - z/\alpha_n)}{(1 - z/\beta_1)(1 - z/\beta_2)\cdots(1 - z/\beta_m)}, \tag{23}$$

where $c \neq 0$ is a complex number depending on $a(z)$. So, (22) is solved by

$$h(z) = z^{\log_q c}\frac{\gamma_q(z/\alpha_1)\gamma_q(z/\alpha_2)\cdots\gamma_q(z/\alpha_n)}{\gamma_q(z/\beta_1)\gamma_q(z/\beta_2)\cdots\gamma_q(z/\beta_m)}, \tag{24}$$

which is meromorphic if and only if $\log_q c$ is an integer.

Now, let $c_1(z)$ and $c_2(z)$ be two distinct rational solutions of the differential Riccati equation (6). If there exists a rational solution $c_3(z)$ distinct from $c_j(z), j = 1, 2$, then all meromorphic solutions of (6) are rational solutions. If there exists a transcendental meromorphic solution $w(z)$, then there is no rational solution other than $c_j(z), j = 1, 2$, see, for example, [6, pages 393-394]. For difference Riccati equation (2), Ishizaki obtained a difference analogue, see [3, Proposition 2.2]. In the following, we give a q-difference case for q-difference Riccati equation (4).

Theorem 3. *Let $q \in \mathbb{C}$ be such that $0 < |q| < 1$. Suppose that q-difference Riccati equation* (4) *possesses two distinct rational solutions $g_1(z)$ and $g_2(z)$. Then there exists a meromorphic solution $g_3(z)$ distinct from $g_1(z)$ and $g_2(z)$ so that any meromorphic solution $g(z)$ of* (4) *is represented in the form* (13).

Proof of Theorem 3. Since $g_1(z)$ and $g_2(z)$ are two distinct rational solutions of (4), we define a translation

$$g(z) = \frac{g_1(z)h(z) + g_2(z)}{h(z)+1}. \qquad (25)$$

Then $\sigma(h) = \sigma(g) = 0$. Substituting (25) into (4), we conclude that

$$h(qz) = \frac{g_1(z)}{g_2(z)}h(z), \qquad (26)$$

which is type of (22). So, $h(z)$ is a meromorphic solution of (26) as in the form (24). Therefore, we conclude from (25) that $g_3(z)$ is a meromorphic solution of (4), which is distinct from $g_1(z)$ and $g_2(z)$. So, we now deduce from Theorem 2 that any meromorphic solution of (4) is represented in the form (13). The proof of Theorem 3 is completed. □

Example 4. Let $q = -(1/2)$, $a_1(z) = (5z+4)/2(z+2)$ and $a_0(z) = (z-4)/(z+2)$ in (4) and (5). Then functions

$$g_1(z) = -2, \qquad g_2(z) = -\frac{z-2}{2(z+1)} \qquad (27)$$

satisfy q-difference Riccati equation (4), and (26) turns into

$$h\left(-\frac{1}{2}z\right) = \frac{4(z+1)}{2(z-2)}h(z) = \frac{-2[1-z/(-1)]}{1-z/2}h(z). \qquad (28)$$

We note that

$$\gamma_q(z) = (1-q)^{x-1}\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty} = \frac{(q;q)_\infty}{(z;q)_\infty}. \qquad (29)$$

Thus, we conclude from (24) and (29) that

$$\begin{aligned} h(z) &= z^{\{\log_{-1/2}-2\}}\cdot\frac{\gamma_{-1/2}(-z)}{\gamma_{-1/2}(z/2)} \\ &= z^{-1}\frac{(z/2;-1/2)_\infty}{(-z;-1/2)_\infty} \\ &= z^{-1}\frac{\prod_{k=0}^{\infty}\left(1-(z/2)(-1/2)^k\right)}{\prod_{k=0}^{\infty}\left(1+z(-1/2)^k\right)} \\ &= z^{-1}\frac{\prod_{k=0}^{\infty}\left(1+z(-1/2)^{k+1}\right)}{\prod_{k=0}^{\infty}\left(1+z(-1/2)^k\right)} \\ &= z^{-1}\left(\left(\left[1+\left(-\frac{1}{2}\right)z\right]\left[1+\left(-\frac{1}{2}\right)^2 z\right]\right.\right. \\ &\qquad \left.\cdots\left[1+\left(-\frac{1}{2}\right)^k z\right]\cdots\right) \\ &\qquad\times\left((1+z)\left[1+\left(-\frac{1}{2}\right)z\right]\left[1+\left(-\frac{1}{2}\right)^2 z\right]\right. \\ &\qquad \left.\left.\cdots\left[1+\left(-\frac{1}{2}\right)^k z\right]\cdots\right)^{-1}\right) = \frac{1}{z(z+1)}, \end{aligned}$$

$$g_3(z) = \frac{g_1(z)h(z)+g_2(z)}{h(z)+1} = -\frac{(z-2)^2}{2(z^2+z+1)} \qquad (30)$$

is a meromorphic solution of (4), which is distinct from $g_1(z)$ and $g_2(z)$. Moreover, we also conclude from (10), (27), and (5) that

$$f_1\left(-\frac{1}{2}z\right) = -2f_1(z), \qquad f_2\left(-\frac{1}{2}z\right) = \frac{1-z/2}{1-z/(-1)}f_2(z), \qquad (31)$$

which are corresponding to $g_1(z)$ and $g_2(z)$, respectively, and are also the types of (22). Thus, we deduce from (24) that

$$f_1(z) = \frac{1}{z}, \qquad f_2(z) = \frac{\gamma_{-1/2}(z/2)}{\gamma_{-1/2}(-z)} = z+1 \qquad (32)$$

satisfy second-order linear q-difference equation (5).

4. Value Distribution of Solutions of q-Difference Riccati Equations and Form of Solutions of Second-Order Linear q-Difference Equations

We first consider the value distribution of transcendental meromorphic solution of q-difference Riccati equation (4).

Theorem 5. *Let $a_1(z)$ and $a_0(z)$ be nonconstant rational functions. If $g(z)$ is a zero-order transcendental meromorphic solution of q-difference Riccati equation*

$$g(qz) = -\frac{a_1(z)g(z)+a_0(z)}{g(z)} \qquad (33)$$

with $q \in \mathbb{C}\setminus\{0\}$ and $|q| \neq 1$, then

(i) *if*

$$|q| > 1, \qquad \overline{N}(r,g) + \overline{N}\left(r,\frac{1}{g}\right) = S(r,g), \qquad (34)$$

then $g(z)$ has at most one Borel exceptional value;

(ii) *if $|q| \neq 1$, then Nevanlinna deficiencies $\delta(0,g) = \delta(\infty,g) = 0$;*

(iii) *if $|q| \neq 1$ and $qz^2 + za_1(z) + a_0(z) \not\equiv 0$, then $g(z)$ has infinitely many fixed points.*

In particular, we obtain the following theorem.

Theorem 6. *If $a_1(z) = a_1$ and $a_0(z) = a_0(\neq 0)$ are constants, and if $q \in \mathbb{C}\setminus\{0\}$ and $|q| \neq 1$, then q-difference Riccati equation (4) has only rational solutions. Furthermore, if $a_1(z) \equiv 0$ and $a_0(z) = a_0$ is nonzero constant, then (4) has only a nonzero constant solution $g(z) = d$, which satisfies $d^2 + a_0 = 0$.*

We need some preliminaries to prove Theorems 5 and 6.

The theorem of Tumura and Clunie is an important result in Nevanlinna theory, see [11, 12]. Weissenborn extended it and obtained the following lemma.

Lemma 7 (see [13, Theorem]). *Let $h(z)$ be a meromorphic function and let ϕ be given by*

$$\begin{aligned} &\phi(z) = c_n(z) h(z)^n + c_{n-1}(z) h(z)^{n-1} + \cdots + c_0(z), \\ &T(r, c_j) = S(r, h), \quad j = 0, 1, \ldots, n-1, n. \end{aligned} \tag{35}$$

Then either

$$\phi \equiv \left(h + \frac{c_{n-1}(z)}{nc_n(z)}\right)^n, \tag{36}$$

or

$$T(r, h) \le \overline{N}\left(r, \frac{1}{\phi}\right) + \overline{N}(r, h) + S(r, h). \tag{37}$$

Lemma 8. *Suppose that $h(z)$ is a nonconstant meromorphic function satisfying*

$$\overline{N}(r, h) + \overline{N}\left(r, \frac{1}{h}\right) = S(r, h). \tag{38}$$

Let

$$\phi(z) = c_n(z) h(z)^n + c_{n-1}(z) h(z)^{n-1} + \cdots + c_0(z) \tag{39}$$

be a polynomial in $h(z)$ with $n \in \mathbb{N}$, and coefficients satisfying

$$T(r, c_j) = S(r, h), \quad j = 0, 1, \ldots, n-1, n, \ c_n(z) c_0(z) \not\equiv 0. \tag{40}$$

Then

$$N\left(r, \frac{1}{\phi}\right) = nT(r, h) + S(r, h) \tag{41}$$

or

$$\overline{N}\left(r, \frac{1}{\phi}\right) \ge T(r, h) + S(r, h). \tag{42}$$

Thus, $\phi(z) \not\equiv 0$.

Proof of Lemma 8. By differentiating both sides of (39), we conclude that

$$\phi'(z) = \sum_{j=1}^{n} \left(c_j'(z) + jc_j(z) \frac{h'(z)}{h(z)}\right) h(z)^j. \tag{43}$$

Thus, we deduce from (39) and (43) that

$$\begin{aligned} &\left(c_n'(z) + nc_n(z) \frac{h'(z)}{h(z)}\right) \phi(z) - c_n \phi'(z) \\ &= \sum_{j=1}^{n-1} \left[c_j(z) \left(c_n'(z) + nc_n(z) \frac{h'(z)}{h(z)}\right) \right. \\ &\qquad \left. - c_n(z) \left(c_j'(z) + jc_j(z) \frac{h'(z)}{h(z)}\right) \right] h(z)^j \\ &\quad + c_0(z) \left(c_n'(z) + nc_n(z) \frac{h'(z)}{h(z)}\right). \end{aligned} \tag{44}$$

Therefore, $(c_n'(z) + nc_n(z)(h'(z)/h(z)))\phi(z) - c_n\phi'(z)$ is a polynomial in $h(z)$ with degree no greater than $n-1$ and the term of degree zero is $c_0(z)(c_n'(z) + nc_n(z)(h'(z)/h(z))) \not\equiv 0$. Then

$$c_n'(z) + nc_n(z) \frac{h'(z)}{h(z)} \not\equiv 0. \tag{45}$$

Otherwise, if $c_n'(z) + nc_n(z)(h'(z)/h(z)) \equiv 0$, then $c_n(z)h(z)^n$ is a nonzero constant, a contradiction. We also note that $c_0(z)(c_n'(z) + nc_n(z)(h'(z)/h(z)))$ is a small function relative to $h(z)$ by (38) and the lemma of logarithmic derivative. Set

$$\mu_1(z) = c_n'(z) + nc_n(z) \frac{h'(z)}{h(z)}, \qquad \nu_1(z) = c_n. \tag{46}$$

Then $\mu_1(z)$ and $\nu_1(z)$ are small functions relative to $h(z)$ and

$$\phi_1(z) = \mu_1(z)\phi(z) - \nu_1(z)\phi'(z) \tag{47}$$

is a polynomial in $h(z)$ with degree no greater than $n-1$ and the term of degree zero is small function relative to $h(z)$.

If the degree of $\phi_1(z)$ is greater than zero, then by repeating the above process, we can get two small functions $\mu_2(z)$ and $\nu_2(z)$ such that

$$\phi_2(z) = \mu_2(z)\phi_1(z) - \nu_1(z)\phi_1'(z) \tag{48}$$

is a polynomial in $h(z)$ with a degree less than the degree of $\phi_1(z)$ and the term of degree zero is a small function relative to $h(z)$.

We note that such process will be terminated at most n times. Thus, We can proceed this process to obtain small functions $\mu_j(z)$ and $\nu_j(z)$, where $j = 1, 2, \ldots, s, s+1$ and $s \le n$, such that

$$\phi_j(z) = \mu_j(z)\phi_{j-1}(z) - \nu_j(z)\phi_{j-1}'(z) \tag{49}$$

are polynomial in $h(z)$ with $\deg\phi_j(z) > \deg\phi_{j-1}(z)$ ($j = 1, 2, \ldots, s$), where $\phi_0(z) \equiv \phi(z)$ and

$$\phi_{s+1}(z) = \mu_{s+1}(z)\phi_s(z) - \nu_{s+1}(z)\phi_s'(z) \tag{50}$$

is a small function relative to $h(z)$. Thus, we deduce that the small function $\phi_{s+1}(z)$ can be expressed as a linear differential polynomial in $\phi(z)$ with coefficients being small functions relative to $h(z)$. So,

$$m\left(r, \frac{1}{\phi}\right) = S(r, h). \tag{51}$$

On the other hand, we deduce from Lemma 7 that either

$$\phi \equiv \left(h + \frac{c_{n-1}(z)}{nc_n(z)}\right)^n \tag{52}$$

or

$$T(r, h) \le \overline{N}\left(r, \frac{1}{\phi}\right) + \overline{N}(r, h) + S(r, h). \tag{53}$$

Thus, we deduce from Valiron-Mohon'ko Lemma, (51), and (52) that (41) holds and obtain from (38) and (53) that (42) holds. Therefore, $\phi(z) \not\equiv 0$. The proof of Lemma 8 is completed. □

Lemma 9 (see [9, Theorem 5.2]). *Let $h(z)$ be a transcendental meromorphic solution of*

$$h(qz) = R(z, h(z)) = \frac{\sum_{i=0}^{p} a_i(z) h(z)^i}{\sum_{j=0}^{q} b_j b_j(z) h(z)^j} \tag{54}$$

with meromorphic coefficients $a_i(z), b_j(z)$ relative to $h(z)$ and $q \in \mathbb{C}$ such that $|q| > 1$. If $\overline{N}(r, h) + \overline{N}(r, 1/h) = S(r, h)$, then (54) is either of the form

$$f(qz) = a_p(z) f(z)^p \quad \text{or} \quad f(qz) = \frac{a_0(z)}{f(z)^q}. \tag{55}$$

Lemma 10 (see [5, Theorem 2.2]). *Let $f(z)$ be a nonconstant zero-order meromorphic solution of*

$$P(z, f) = 0, \tag{56}$$

where $P(z, f)$ is a q-difference polynomials in $f(z)$. If $P(z, \alpha) \not\equiv 0$ for a small function $\alpha(z)$ relative to $f(z)$, then

$$m\left(r, \frac{1}{f - \alpha}\right) = o(T(r, f)) \tag{57}$$

on a set of logarithmic density 1.

Lemma 11 (see [2, Theorem 2.2.5 and Corollary 2.2.7]). *Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in $f(z)$,*

$$R(z, f(z)) = \frac{\sum_{i=0}^{p} a_i(z) f(z)^i}{\sum_{j=0}^{q} b_j(z) f(z)^j}, \tag{58}$$

with meromorphic coefficients $a_i(z), b_j(z)$, the characteristic function of $R(z, f(z))$ satisfies

$$T(R(z, f(z))) = dT(r, f) + O(\Psi(r)), \tag{59}$$

where $d = \max\{p, q\}$ and $\Psi(r) = \max_{i,j}\{T(r, a_i), T(r, b_j)\}$.

In the particular case when

$$\begin{aligned} T(r, a_i) &= S(r, f), \quad i = 0, 1, \ldots, p, \\ T(r, b_j) &= S(r, f), \quad j = 0, 1, \ldots, q, \end{aligned} \tag{60}$$

we have

$$T(r, R(z, f(z))) = dT(r, f) + S(r, f). \tag{61}$$

We also use the observation [7, page 2] that, for any meromorphic function $f(z)$ and any constant $q \in \mathbb{C} \setminus \{0\}$,

$$T(r, f(qz)) = T(|q| r, f) + O(1). \tag{62}$$

Proof of Theorem 5. Suppose that $g(z)$ is a zero-order transcendental meromorphic solution of q-difference Riccati equation (4).

(i) Suppose that $g(z)$ has two finite Borel exceptional values a and $b(\neq 0, a)$. For the case where one of a and b is infinite, we can use a similar method to prove. Set

$$h(z) = \frac{g(z) - a}{g(z) - b}. \tag{63}$$

Since $\overline{N}(r, g) + \overline{N}(r, 1/g) = S(r, g)$, we deduce from (63) that

$$\overline{N}(r, h) + \overline{N}\left(r, \frac{1}{h}\right) = S(r, h). \tag{64}$$

We also conclude from (63) that

$$g(z) = \frac{a - bh(z)}{1 - h(z)}. \tag{65}$$

Now, substituting (65) into (4), we conclude that

$$h(qz) = \frac{(ba_1(z) + a_0(z) + ab) h(z) - \left(aa_1(z) + a_0(z) + a^2\right)}{\left(ba_1(z) + a_0(z) + b^2\right) h(z) - (aa_1(z) + a_0(z) + ab)}. \tag{66}$$

By the assumptions of Theorem 5, we get

$$(ba_1(z) + a_0(z) + ab) \not\equiv 0, \qquad \left(ba_1(z) + a_0(z) + b^2\right) \not\equiv 0. \tag{67}$$

Thus, we deduce from Lemma 9, (64), and (66) that

$$h(qz) = c(z) h(z)^k, \quad k \in \mathbb{Z} \setminus \{0\}, \tag{68}$$

where $T(r, c(z)) = S(r, h)$.

If $k \geq 1$, we conclude from (66) and (68) that

$$\begin{aligned} & c(z)\left(ba_1(z) + a_0(z) + b^2\right) h(z)^{k+1} \\ & \quad - c(z)(aa_1(z) + a_0(z) + ab) h(z)^k \\ & \quad - (ba_1(z) + a_0(z) + ab) h(z) + \left(aa_1(z) + a_0(z) + a^2\right) \\ & = 0. \end{aligned} \tag{69}$$

Thus, we deduce from Lemma 8 and (64) that (69) is a contradiction. If $k \leq -1$, we use the same method as above to get another contradiction. Therefore, $g(z)$ at most one Borel exceptional value.

(ii) We first prove $\delta(0, g) = 0$. We obtain from (4) that

$$P_1(z, g) = g(z) g(qz) + a_1(z) g(z) + a_0(z) = 0. \tag{70}$$

Since $P_1(z, 0) = a_0(z) \not\equiv 0$, we deduce from Lemma 10 and (70) that

$$m\left(r, \frac{1}{g}\right) = S(r, g) \tag{71}$$

on a set E of logarithmic density 1. Therefore,

$$0 \leq \delta(0, g) = \varliminf_{r \to \infty} \frac{m(r, 1/g)}{T(r, g)} \leq \varliminf_{r \to \infty, r \in E} \frac{m(r, 1/g)}{T(r, g)} = 0. \tag{72}$$

Thus, $\delta(0, g) = 0$.

We second prove $\delta(\infty, g) = 0$. Set $y(z) = 1/g(z)$. Then

$$T(r, y) = T(r, g) + O(1), \qquad S(r, y) = S(r, g). \tag{73}$$

Now, substituting $g(z) = 1/y(z)$ into (4), we conclude that

$$P_2(z, y) = y(qz)(a_0(z)y(z) + a_1(z)) + 1 = 0. \quad (74)$$

Since $P_2(z, 0) = 1 \not\equiv 0$, we obtain from Lemma 10 and (74) that

$$m\left(r, \frac{1}{y}\right) = S(r, y) \quad (75)$$

on a set E of logarithmic density 1. Therefore,

$$N\left(r, \frac{1}{y}\right) = T(r, y) + S(r, y) \quad (76)$$

on a set E of logarithmic density 1. Thus, we conclude from $y(z) = 1/g(z)$ and (76) that

$$\begin{aligned} N(r, g) = N\left(r, \frac{1}{y}\right) &= T(r, y) + o(T(r, y)) \\ &= T(r, g) + S(r, g) \end{aligned} \quad (77)$$

on a set E of logarithmic density 1, and so,

$$\begin{aligned} 0 \le \delta(\infty, g) &= 1 - \overline{\lim_{r\to\infty}} \frac{N(r, g)}{T(r, g)} \\ &\le 1 - \overline{\lim_{r\to\infty, r\in E}} \frac{N(r, g)}{T(r, g)} = 0. \end{aligned} \quad (78)$$

Thus, $\delta(\infty, g) = 0$.

(iii) Set $y(z) = g(z) - z$. Then

$$T(r, y) = T(r, g) + S(r, g), \qquad S(r, y) = S(r, g). \quad (79)$$

Substituting $g(z) = y(z) + z$ into (4), we conclude that

$$\begin{aligned} P_3(z, y) = {} & y(z)y(qz) + zy(qz) + qzy(z) \\ & + qz^2 + za_1(z) + a_0(z) = 0. \end{aligned} \quad (80)$$

Since $P_3(z, 0) = qz^2 + za_1(z) + a_0(z) \not\equiv 0$, we deduce from Lemma 10 and (80) that

$$m\left(r, \frac{1}{y}\right) = S(r, y) \quad (81)$$

on a set E of logarithmic density 1. Therefore

$$\begin{aligned} N\left(r, \frac{1}{g - z}\right) = N\left(r, \frac{1}{y}\right) &= T(r, y) + o(T(r, y)) \\ &= T(r, g) + S(r, g) \end{aligned} \quad (82)$$

on a set E of logarithmic density 1. This shows that $g(z)$ has infinitely many fixed points if $qz^2 + za_1(z) + a_0(z) \not\equiv 0$. □

Proof of Theorem 6. Suppose first that $0 < |q| < 1$ and (4) with nonzero constant coefficients $a_1(z)$ and $a_0(z)$ admits a meromorphic solution $g(z)$. We assert that $g(z)$ is rational. In fact, we conclude from Lemma 11, (4), and (62) that

$$T(r, f) \le T(|q|r, f) + A, \quad r \ge R_0, \quad (83)$$

where $A > \Psi(r) = \max\{T(r, a_0), T(r, a_1)\} \ge 0, R_0(>0)$ is fixed number.

Thus, for any $r \ge R_0$, there exists an $n \in \mathbb{N}$ such that

$$\frac{R_0}{|q|^{n-1}} \le r < \frac{R_0}{|q|^n}. \quad (84)$$

By an inductive argument, we deduce from (84) that

$$\begin{aligned} T(r, f) &\le T(|q|^n r, f) + An \\ &\le T(R_0, f) + A\left(\frac{\log r}{\log(1/|q|)} - \frac{\log R_0}{\log(1/|q|)} + 1\right) \\ &= O(\log r). \end{aligned} \quad (85)$$

Suppose now that $|q| > 1$ and (4) with nonzero constant coefficients $a_1(z)$ and $a_1(z)$ admits a meromorphic solution $g(z)$. Replacing z by z/q in (4), we proceed in a similar method as above to get (85) again. Therefore, $g(z)$ is rational solution of (4).

Now, we affirm that $g(z)$ must be nonzero constant if $a_1(z) \equiv 0$ and $a_0(z) = a_0(\ne 0)$ is a constant. Otherwise, if $g(z)$ is nonconstant rational and has a pole $z_0 \ne 0$, we conclude from (4) that $g(z)$ has infinitely many poles of the forms $q^{2(n-1)}z_0$ and infinitely many zeros of the forms $q^{2(n-1)+1}z_0$ for all $n \in \mathbb{N}$. Conversely, If $g(z)$ is nonconstant rational and has a zero $z_0 \ne 0$, we conclude from (4) that $g(z)$ has infinitely many zeros of the forms $q^{2(n-1)}z_0$ and infinitely many poles of the forms $q^{2(n-1)+1}z_0$ for all $n \in \mathbb{N}$. These are both impossible since $g(z)$ is rational. Thus, the only possible pole (resp. zero) of $g(z)$ is at 0. So $g(z)$ may have the form $g(z) = dz^k (k \in \mathbb{Z})$, where d is a nonzero constant. If $k \ne 0$, we get a contradiction from (4). Therefore, $k = 0$ and (4) has only a nonzero constant solution $g(z) = d$, which satisfies $d^2 + a_0 = 0$. The proof of Theorem 6 is completed. □

We now consider the form of meromorphic solutions of (5), which is according to Theorem 6. In fact, more details about meromorphic solutions of (5) have been studied in [7, 14]. Here, we only consider the case that all coefficients are constants.

Theorem 12. *If $a_1(z) \equiv 0$ and $a_0(z) = a_0$ is constant, and if $q \in \mathbb{C} \setminus \{0\}$ and $|q| \ne 1$, then every meromorphic solution $f(z)$ of second-order linear q-difference equation (5) has the form $f(z) = \beta z^k$, where $\beta \in \mathbb{C} \setminus \{0\}$ and $k \in \mathbb{Z}$ satisfying $q^{2k} + a_0 = 0$.*

We first list a lemma needed below.

Lemma 13 (see [14, Theorem 2.1]). *Suppose that $q \in \mathbb{C} \setminus \{0\}$ and $|q| \ne 1$. Let $a_0, a_1, \dots, a_n$ be complex constants and let $Q(z)$ be of the reduced form $Q(z) = p_1(z)/z^l$, where $p_1(z)$ is a polynomial of degree d and $l \in \mathbb{N} \cup \{0\}$. Then all meromorphic solutions $f(z)$ of*

$$\sum_{j=0}^{n} a_j(z) f(q^j z) = Q(z) \quad (86)$$

are of the reduced form $f(z) = p_2(z)/z^p$, *where* $p_2(z)$ *is a polynomial and* $p \geq l$.

Proof of Theorem 12. We deduce from Lemma 13 that all meromorphic solutions $f(z)$ of (5) are of the form $f(z) = p_2(z)/z^p$, where $p_2(z)$ and p are defined as Lemma 13. Thus, we conclude from Theorem 6 and (10) that

$$d = g(z) = \frac{f(qz)}{f(z)} = \frac{1}{q^p} \cdot \frac{p_2(qz)}{p_2(z)}, \tag{87}$$

where d is defined as Theorem 6. From (87), we obtain that there exists $\beta \in \mathbb{C}\backslash\{0\}$ and $m \in \mathbb{N}\cup\{0\}$ such that $p_2(z) = \beta z^m$, and so $f(z) = p_2(z)/z^p = \beta z^m/z^p =: \beta z^k$, where $k = m-p \in \mathbb{Z}$. Now, substituting $f(z) = \beta z^k$ into (5), we conclude that k satisfies $q^{2k} + a_0 = 0$. The proof of Theorem 12 is completed. □

Example 14. Let $q \in \mathbb{C} \setminus \{0\}, |q| \neq 1$, $a_1(z) \equiv 0$ and $a_0(z) = -1/q^2$. Then second-order q-difference equation (5) is solved by $f(z) = 1/z$. Obviously, $f(z) = 1/z$ and $k = -1$ satisfy the conclusions described by Theorem 12.

5. Linear q-Difference Equations of Second-Order

Let $y_1(z)$ and $y_2(z)$ be meromorphic solutions of (5). We define the q-Casorati determinant of meromorphic functions $y_1(z)$ and $y_2(z)$ by

$$\widehat{C_q}(z) = \widehat{C_q}(y_1, y_2; z) = \begin{vmatrix} y_1(z) & y_2(z) \\ y_1(qz) & y_2(qz) \end{vmatrix}. \tag{88}$$

Then the q-Casorati determinant $\widehat{C_q}(z)$ vanishes identically on $\mathbb{C}$ if and only if the functions $y_1(z)$ and $y_2(z)$ are linearly dependent over the field of functions $\phi(qz) = \phi(z)$. On the other hand, $g_1(z)$ and $g_2(z)$ are linear independent if and only if $\widehat{C_q}(g_1, g_2; z) \not\equiv 0$. From this definition, we have some properties on the q-Casorati determinant $\widehat{C_q}(z)$ as follows.

Theorem 15. *If* $y_1(z)$ *and* $y_2(z)$ *are nontrivial meromorphic solutions of* (5), *then q-Casorati determinant* $\widehat{C_q}(y_1, y_2; z)$ *satisfies a first-order q-difference equation*

$$\Delta_q \widehat{C_q}(z) = (a_0 - 1)\widehat{C_q}(z). \tag{89}$$

Conversely, we assume that $y_1(z)(\not\equiv 0)$ *and* $y_2(z)$ *satisfy* (89). *If* $y_1(z)$ *is a meromorphic solution of* (5), *then* $y_2(z)$ *is also a meromorphic solution of* (5).

Proof of Theorem 15. Suppose first that $y_1(z)$ and $y_2(z)$ are nontrivial meromorphic solutions of (5), we conclude that

$$\begin{aligned} &\widehat{C_q}(qz) \\ &= \widehat{C_q}(y_1, y_2; qz) = \begin{vmatrix} y_1(qz) & y_2(qz) \\ y_1(q^2z) & y_2(q^2z) \end{vmatrix} \\ &= \begin{vmatrix} y_1(qz) & y_2(qz) \\ -a_1 y_1(qz) - a_0(z) y_1(z) & -a_1 y_2(qz) - a_0(z) y_2(z) \end{vmatrix} \\ &= \begin{vmatrix} y_1(qz) & y_2(qz) \\ -a_0(z) y_1(z) & -a_0(z) y_2(z) \end{vmatrix} \\ &= a_0(z) \begin{vmatrix} y_1(z) & y_2(z) \\ y_1(qz) & y_2(qz) \end{vmatrix} = a_0(z)\widehat{C_q}(z). \end{aligned} \tag{90}$$

Therefore,

$$\Delta_q \widehat{C_q}(z) = \widehat{C_q}(qz) - \widehat{C_q}(z) = (a_0(z) - 1)\widehat{C_q}(z). \tag{91}$$

Second, if $y_1(z)(\not\equiv 0)$ and $y_2(z)$ satisfy (89), then we have

$$\begin{vmatrix} y_1(qz) & y_2(qz) \\ y_1(q^2z) & y_2(q^2z) \end{vmatrix} = a_0(z) \begin{vmatrix} y_1(z) & y_2(z) \\ y_1(qz) & y_2(qz) \end{vmatrix}. \tag{92}$$

We note that, for any meromorphic function $c(z) \not\equiv 0$,

$$\begin{aligned} &\begin{vmatrix} y_1(qz) & y_2(qz) \\ y_1(q^2z) & y_2(q^2z) \end{vmatrix} \\ &\quad = \begin{vmatrix} y_1(qz) & y_2(qz) \\ y_1(q^2z) + c(z) y_1(qz) & y_2(q^2z) + c(z) y_2(qz) \end{vmatrix}. \end{aligned} \tag{93}$$

In particular, we take $c(z) = a_1(z)$. Thus,

$$\begin{aligned} &\begin{vmatrix} y_1(qz) & y_2(qz) \\ y_1(q^2z) + a_1(z) y_1(qz) & y_2(q^2z) + a_1(z) y_2(qz) \end{vmatrix} \\ &\quad = a_0(z) \begin{vmatrix} y_1(z) & y_2(z) \\ y_1(qz) & y_2(qz) \end{vmatrix}. \end{aligned} \tag{94}$$

So, we have

$$\begin{aligned} &\begin{vmatrix} y_1(qz) & y_2(qz) \\ y_1(q^2z) + a_1(z) y_1(qz) & y_2(q^2z) + a_1(z) y_2(qz) \end{vmatrix} \\ &\quad + \begin{vmatrix} y_1(qz) & y_2(qz) \\ a_0(z) y_1(z) & a_0(z) y_2(z) \end{vmatrix} = 0. \end{aligned} \tag{95}$$

From this, we conclude that

$$y_1(qz)\left[y_2\left(q^2z\right)+a_1(z)y_2(qz)+a_0(z)y_2(z)\right] = y_2(qz)\left[y_1\left(q^2z\right)+a_1(z)y_1(qz)+a_0(z)y_1(z)\right]. \tag{96}$$

Since $y_1(z)(\not\equiv 0)$ is a meromorphic solution of (5), we have

$$y_1\left(q^2z\right)+a_1(z)y_1(qz)+a_0(z)y_1(z)=0, \tag{97}$$

and so,

$$y_2\left(q^2z\right)+a_1(z)y_2(qz)+a_0(z)y_2(z)=0. \tag{98}$$

This shows that $y_2(z)$ is a meromorphic solution of (5). The proof of Theorem 15 is completed. □

Theorem 16. *(i) Let $y_1(z)$ and $y_2(z)$ be linear independent meromorphic solutions of* (5)*, and let $\widehat{C_q}(z)$ be the q-Casoratian determinant of $y_1(z)$ and $y_2(z)$. Then $y_2(z)$ is represented as $y_2(z)=g(z)y_1(z)$, where $g(z)$ satisfies*

$$\Delta_q g(z)=\frac{\widehat{C_q}(z)}{y_1(z)y_1(qz)}. \tag{99}$$

(ii) Let $y_1(z)$ be a nontrivial meromorphic solution of (5)*, and let $\widehat{C_q}(z)$ be a meromorphic solution of* (89)*. If $g(z)$ satisfies* (99)*, then $y_2(z) = g(z)y_1(z)$ is a meromorphic solution of* (5).

Proof of Theorem 16. (i) From the definition of $\widehat{C_q}(z)$, we obtain

$$\widehat{C_q}(z)=y_1(z)y_2(qz)-y_2(z)y_1(qz). \tag{100}$$

This shows that $y_2(z)$ satisfies first-order q-difference equation of type

$$y_2(qz)=y_2(z)\cdot\frac{y_1(qz)}{y_1(z)}+\frac{\widehat{C_q}(z)}{y_1(z)}. \tag{101}$$

By substituting $y_2(z) = g(z)y_1(z)$ into (101), we conclude that

$$g(qz)y_1(qz)=g(z)y_1(z)\cdot\frac{y_1(qz)}{y_1(z)}+\frac{\widehat{C_q}(z)}{y_1(z)}, \tag{102}$$

and so we obtain the desired form (99).

(ii) Obviously, we conclude from (99) and (89) that

$$\begin{aligned} g(qz)&=g(z)+\frac{\widehat{C_q}(z)}{y_1(z)y_1(qz)},\\ g\left(q^2z\right)&=g(qz)+\frac{\widehat{C_q}(qz)}{y_1(qz)y_1(q^2z)}\\ &=g(qz)+\frac{a_0(z)\widehat{C_q}(z)}{y_1(qz)y_1(q^2z)}. \end{aligned} \tag{103}$$

Since $y_2(z)=g(z)y_1(z), \widehat{C_q}(z)=y_1(z)y_2(qz)-y_2(z)y_1(qz)$, and

$$y_1\left(q^2z\right)+a_1(z)y_1(qz)+a_0(z)y_1(z)=0, \tag{104}$$

we conclude from (103), and (104) that

$$\begin{aligned} &y_2\left(q^2z\right)\\ &=g\left(q^2z\right)y_1\left(q^2z\right)\\ &=\left(g(z)+\frac{\widehat{C_q}(z)}{y_1(z)y_1(qz)}+\frac{a_0(z)\widehat{C_q}(z)}{y_1(qz)y_1(q^2z)}\right)y_1\left(q^2z\right)\\ &=g(z)y_1\left(q^2z\right)+\frac{y_1\left(q^2z\right)+a_0(z)y_1(z)}{y_1(z)y_1(qz)}\cdot\widehat{C_q}(z)\\ &=g(z)y_1\left(q^2z\right)+\frac{-a_1(z)y_1(qz)}{y_1(z)y_1(qz)}\cdot\widehat{C_q}(z)\\ &=\frac{y_2(z)}{y_1(z)}\cdot\left(-a_1(z)y_1(qz)-a_0(z)y_1(z)\right)-\frac{a_1(z)}{y_1(z)}\\ &\quad\cdot\left(y_1(z)y_2(qz)-y_2(z)y_1(qz)\right)\\ &=-a_1(z)y_2(qz)-a_0(z)y_2(z). \end{aligned} \tag{105}$$

This yields that $y_2(z)=g(z)y_1(z)$ is a meromorphic solution of (5). The proof of Theorem 16 is completed. □

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