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ON DIFFERENCE EQUATIONS RELATING TO GAMMA FUNCTION*

Chen Zongxuan (陈宗煊) *Huang Zhibo* (黄志波) *Zhang Ranran* (张然然)
School of Mathematical Sciences, South China Normal University, Guangzhou 510631, China
E-mail: chzx@vip.sina.com; hzbo20019@sina.com; zhrr19820315@163.com

Abstract We consider the existence, the growth, poles, zeros, fixed points and the Borel exceptional value of solutions for the following difference equations relating to Gamma function $y(z+1)-y(z)=R(z)$ and $y(z+1)=P(z)y(z)$.

Key words difference equation; meromorphic function; Borel exceptional value

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1 Introduction and Results

It is well known that Gamma function is defined

$$\Gamma(z)=\frac{\mathrm{e}^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}\mathrm{e}^{\frac{z}{n}},$$

where $\gamma=\lim\limits_{n\to\infty}\left[1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n\right]$ cannot satisfy any algebraic differential equation whose coefficients are rational functions. But it satisfies the difference equation

$$\Gamma(z+1)=z\Gamma(z).$$

And the Gaussian psi function is defined by $\Psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$, which satisfies the difference equation

$$\Psi(z+1)-\Psi(z)=\frac{1}{z}.$$

Both of $\Gamma(z)$ and $\Psi(z)$ are meromorphic functions in C with simple poles at $z=0,-1,\cdots$ and the order of growth $\sigma(\Gamma)=\sigma(\Psi)=1$.

In this paper, we consider difference equations more general than above

$$y(z+1)-y(z)=R(z) \tag{1.1}$$

and

$$y(z+1)=P(z)y(z), \tag{1.2}$$

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where $R(z)$ is a rational function, $P(z)$ is a polynomial.

Bank and Kaufman [1] gave the following result concerning the existence and the growth restriction of solutions of difference equations.

Theorem A For any rational function $R(z)$, equation (1.1) always has a meromorphic solution $y(z)$ such that $T(r,y)=O(r)$ as $r\to\infty$.

In this paper, we assume the reader is familiar with the basic notions of Nevanlinna's value distribution theory (see e.g. [2–5]). In addition, we use the notations $\sigma(f)$ to denote the order of growth of the meromorphic function $f(z)$, $\lambda(f)$ and $\lambda(\frac{1}{f})$ to denote, respectively, the exponents of convergence of zeros and poles of $f(z)$. We also use the notation $\tau(f)$ to denote the exponent of convergence of fixed points of f defined by

$$\tau(f)=\limsup_{r\to\infty}\frac{\log N\big(r,\frac{1}{f-z}\big)}{\log r}.$$

Recently, a number of papers (including [6–20]) focused on complex difference equations and difference analogues of Nevanlinna's theory. As the difference analogues of Nevanlinna's theory are investigated, many results on the complex difference equations were got rapidly. Many papers (including [6, 11, 12, 15–20]) mainly deal with the growth of meromorphic solutions of difference equations.

The goal of our research is to consider the existence, the growth, poles, zeros, fixed points and the Borel exceptional value of solutions of (1.1) and (1.2). We will prove the following four theorems.

Theorem 1 For any rational function $R(z)\not\equiv 0$, consider equation (1.1), the following statements holds.

(1) Equation (1.1) must have a meromorhpic solution $y(z)$ satisfies $\lambda(y)=\sigma(y)=1$, and all its transcendental meromorphic solutions of finite order satisfy $\lambda(y)=\sigma(y)\geq 1$.

(2) Every transcendental meromorphic solution $y(z)$ of finite order of it has at most one Borel exceptional value.

(3) If its solution $y(z)$ has infinitely many poles, then $\lambda\big(\frac{1}{y}\big)\geq 1$.

(4) If $R(z)\not\equiv 1$ and $y(z)$ is its transcendental meromorphic solution of finite order, then the exponent of convergence of fixed points of $y(z)$ satisfies $\tau(y)=\sigma(y)$.

Theorem 2 Let $R(z)=\frac{P(z)}{Q(z)}\not\equiv 0$ be a rational function where $P(z),\ Q(z)$ are irreducible polynomials, $\deg P(z)=p,\ \deg Q(z)=q,\ p-q=s$. Consider equation (1.1), the following statements hold.

(1) If $s=-1$, then (1.1) has no rational solution.

(2) If $s\geq 0$ and $y(z)=\frac{m(z)}{n(z)}$ is a rational solution of (1.1) where $m(z),\ n(z)$ are irreducible polynomials with $\deg m(z)=m,\ \deg n(z)=n$, then $m-n=s+1$.

(3) If $s\leq -2$ and $y(z)$ is defined as in (2), then $m-n=s+1$ or $m-n=0$.

(4) If $q=0$ (i.e. $Q(z)$ is a nonzero constant), then (1.1) must has a polynomial solution $y(z)=a_{s+1}z^{s+1}+\cdots+a_1z+a_0$ $(a_{s+1}\neq 0)$, coefficients $a_{s+1},\cdots,a_1$ of which can be decided by coefficients of $R(z)$, and the coefficient a_0 may take any constant.

By Theorem 1, we can obtain the following corollary.

Corollary 1 For the Gauss psi function $\Psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$ (see [21, Chapter 7]), we have

$$\lambda(\Psi)=\lambda\left(\frac{1}{\Psi}\right)=\tau(\Psi)=\sigma(\Psi)=1.$$

Theorem 3 Let $P(z)$ be a polynomial with $\deg P(z)=p\geq 1$. Consider the difference equation (1.2), then

(1) Equation (1.2) has no nonzero rational solution;

(2) Every transcendental meromorphic solution of (1.2) satisfies $\sigma(y)\geq 1$, and has at most one Borel exceptional value.

The following Examples 1–6 show the existences and the forms of rational solutions of Theorem 2. Examples 2–4 show that there assuredly exist two cases $m-n=s+1$ and $m-n=0$ under condition (3) of Theorem 2. Example 6 shows properties on transcendental solutions in Theorem 1.

Example 7 shows that the condition $R(z)\not\equiv 1$ in Theorem 1(4) cannot be omitted.

Example 1 The difference equation

$$y(z+1)-y(z)=\frac{z^2+3z+1}{(z+1)(z+2)}$$

has a solution $y(z)=\frac{z^2}{z+1}$, where $s=2-2=0$ and $m-n=2-1=1=s+1$.

Example 2 The difference equation

$$y(z+1)-y(z)=\frac{-1}{z(z+1)}$$

has a solution $y(z)=\frac{1}{z}$, where $s=-2$ and $m-n=-1=s+1$.

Example 3 The difference equation

$$y(z+1)-y(z)=\frac{3}{(z+3)(z+2)}$$

has a solution $y(z)=\frac{z-1}{z+2}$, where $s=-2$ and $m-n=0$.

Example 4 The difference equation

$$y(z+1)-y(z)=\frac{-4(z-1)}{z(z+1)(z-2)(z-3)}$$

has a solution $y(z)=\frac{(z-1)(z-2)}{z(z-3)}$, where $s=-3$ and $m-n=0$.

Example 5 The difference equation

$$y(z+1)-y(z)=\frac{1}{z}$$

has a transcendental solution $y(z)=\Psi(z)$ and has no rational solution. Equations

$$y(z+1)-y(z)=\frac{z}{(z-1)(z-2)}\quad\text{and}\quad y(z+1)-y(z)=\frac{z(z-1)}{((z-2)(z-3)(z-4)}$$

have no rational solutions.

Example 6 The difference equation $y(z+1)-y(z)=2z+1$ has the following three solutions:

(1) A polynomial solution $y_1(z) = z^2 + a_0$ (a_0 may be any constant), where $s = 1 - 0 = 1$ and $m - n = 2 - 0 = 2 = s + 1$;

(2) A finite order solution $y_2(z) = \mathrm{e}^{2\pi \mathrm{i} z} + z^2$ which satisfies $\lambda(y_2) = \tau(y_2) = \sigma(y_2) = 1$;

(3) A infinite order solution $y_3(z) = \mathrm{e}^{\mathrm{e}^{2\pi \mathrm{i} z}} + z^2$.

Example 7 The difference equation $y(z+1) - y(z) = 1$ has a solution $y(z) = \mathrm{e}^{2\pi \mathrm{i} z} + z$, which has no fixed point.

2 Proof of Theorem 1

We need the following lemmas for the proof of Theorem 1.

Lemma 2.1 [7] Let f be a transcendental meromorphic function satisfying

$$\limsup_{r\to\infty} \frac{T(r,f)}{r} = 0. \tag{2.1}$$

Then $f(z+1) - f(z)$ is transcendental.

Lemma 2.2 For any rational function $R(z) \not\equiv 0$, the difference equation (1.1) must have a transcendental meromorphic solution $y(z)$ satisfying $\sigma(y) = 1$.

Proof By Theorem A, we see that the difference equation (1.1) always has a meromorphic solution $y_1(z)$ such that $T(r, y_1) = O(r)$ as $r \to \infty$. If y_1 is transcendental, then we set $y(z) = y_1(z)$. Now suppose that y_1 is a rational function. Since any periodic function with period 1 satisfies the corresponding homogeneous difference equation

$$y(z+1) - y(z) = 0$$

of (1.1), we take a meromorphic periodic function $y_0(z)$ with period 1 and $\sigma(y_0) = 1$. Set $y(z) = y_1(z) + y_0(z)$. Then for two cases above, $y(z)$ satisfies (1.1) and $T(r, y) = O(r)$, namely, y is transcendental and $\sigma(y) \le 1$.

If $\sigma(y) < 1$, then y satisfies (2.1). By Lemma 2.1, we see $y(z+1) - y(z)$ is transcendental, which contradicts our assumption that $R(z)$ is a rational function. So $\sigma(y) = 1$.

Lemma 2.3 [11, 13] Let f be a meromorphic function of finite order and $c \in C$. Then

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r, f),$$

where $S(r, f)$ denotes $S(r, f) = o\{T(r, f)\}$.

Lemma 2.4 [13, 20] Let $w(z)$ be a nonconstant finite order meromorphic solution of

$$P(z, w) = 0,$$

where $P(z, w)$ is a difference polynomial in $w(z)$. If $P(z, a) \not\equiv 0$ for a meromorphic function $a(z)$ satisfying $T(r, a) = S(r, w)$, then

$$m\left(r, \frac{1}{w - a}\right) = S(r, w).$$

Lemma 2.5 (see [22]) Let $g(z)$ be an entire function of order $\sigma(g) = \alpha < \infty$. Then for any given $\varepsilon > 0$, there is a set $E \in (1, \infty)$ that has finite linear measure mE and finite logarithmic measure lmE, such that for all z satisfying $|z| = r \notin [0, 1] \bigcup E$,

$$\exp\{-r^{\alpha+\varepsilon}\} \le |g(z)| \le \exp\{r^{\alpha+\varepsilon}\}. \tag{2.2}$$

Remark 2.6 Let g be a meromorphic function of order $\sigma(g) = \alpha < \infty$. Then by Lemma 2.5, we easily obtain that for any given $\varepsilon > 0$, there is a set $E \in (1,\infty)$ that has finite linear measure mE and finite logarithmic measure lmE, such that for all z satisfying $|z| = r \notin [0,1] \bigcup E$, (2.2) holds.

Proof of Theorem 1 (1) By Lemma 2.2, we see that (1.1) must have a transcendental meromorphic solution $y(z)$ which is of order of growth $\sigma(y) = 1$. By Lemma 2.3 and (1.1), we obtain that

$$m\left(r, \frac{R(z)}{y(z)}\right) = m\left(r, \frac{y(z+1)}{y(z)} - 1\right) = m\left(r, \frac{y(z+1)}{y(z)}\right) = S(r,y).$$

So,

$$N\left(r, \frac{R(z)}{y(z)}\right) = N\left(r, \frac{1}{y(z)}\right) + O(\log r) = T(r,y) + S(r,y).$$

Hence $\lambda(y) = \sigma(y) = 1$.

If $y(z)$ is a transcendental solution with $\sigma(y) < 1$, then $y(z+1) - y(z)$ is transcendental by Lemma 2.1. This contradicts our supposition that $R(z)$ is a rational function. Hence $\sigma(y) \geq 1$. Using the same method as above, we have $\lambda(y) = \sigma(y)$.

(2) Now, we prove that a finite order meromorphic solution $y(z)$ has at most one Borel exceptional value.

Suppose that $y(z)$ has two Borel exceptional value a and $b(\neq 0, a)$. We will result in a contradiction.

First, we suppose that $a,\ b$ are finite value. We set

$$f(z) = \frac{y(z) - a}{y(z) - b}. \tag{2.3}$$

Then $\sigma(f) = \sigma(y)$ and

$$\lambda(f) = \lambda(y-a) < \sigma(y), \quad \lambda\left(\frac{1}{f}\right) = \lambda(y-b) < \sigma(y).$$

Hence, $f(z)$ may be rewritten as

$$f(z) = \pi(z)\mathrm{e}^{\delta z^n}, \tag{2.4}$$

where $\delta(\neq 0)$ is a constant, n is a positive integer and $\sigma(f) = n$, $\pi(z)(\not\equiv 0)$ is a meromorphic function with

$$\sigma(\pi) < \sigma(f) = n. \tag{2.5}$$

By (2.3), we have

$$y(z) = \frac{a - bf(z)}{1 - f(z)}. \tag{2.6}$$

Substituting (2.6) into (1.1), we obtain that

$$R(z)f(z+1)f(z) + (b - a - R(z))f(z+1) + (a - b - R(z))f(z) + R(z) = 0. \tag{2.7}$$

By (2.4), we have

$$f(z+1) = \pi(z+1)\mathrm{e}^{\delta(z+1)^n} = \pi(z+1)\mathrm{e}^{\delta z^n + \delta n z^{n-1} + \cdots + \delta} = \pi_1(z)\mathrm{e}^{\delta z^n}, \tag{2.8}$$

where $\pi_1(z) = \pi(z+1)\mathrm{e}^{\delta n z^{n-1}+\cdots+\delta}$ ($\not\equiv 0$) and

$$\sigma(\pi_1) \le \max\{\sigma(\pi),\ n-1\}. \tag{2.9}$$

Substituting (2.4) and (2.8) into (2.7), we obtain that

$$R(z)\pi(z)\pi_1(z)\mathrm{e}^{2\delta z^n} + [(b-a-R(z))\pi_1(z) + (a-b-R(z))\pi(z)]\mathrm{e}^{\delta z^n} + R(z) = 0. \tag{2.10}$$

By (2.5) and (2.9), we see that

$$\max\{\sigma(\pi),\ \sigma(\pi_1)\} = d < \sigma(f) = n.$$

By Lemma 2.5 and Remark 2.6, we see that, for any given ε $(0 < 2\varepsilon < n-d)$, there is a set $E \subseteq (1,\infty)$ with finite linear measure, such that for all z satisfying $|z| = r \notin [0,1]\bigcup E$,

$$\exp\{-r^{d+\varepsilon}\} \le |R(z)\pi_1(z)\pi(z)| \le \exp\{r^{d+\varepsilon}\} \tag{2.11}$$

and

$$|(b-a-R(z))\pi_1(z) + (a-b-R(z))\pi(z)| \le \exp\{r^{d+\varepsilon}\}. \tag{2.12}$$

For sufficiently large $|z| = r$, we have $|R(z)| \le r^k$ where k (> 0) is some constant.

Now we can choose points $z_j = r_j\mathrm{e}^{\mathrm{i}\theta_j},\ j = 1,2,\cdots$, satisfying $r_j \to \infty,\ r_j \notin [0,1]\bigcup E$, and

$$\exp\{2\delta z_j^n\} = \exp\{2|\delta| r_j^n\}. \tag{2.13}$$

Thus, by (2.11)–(2.13) and $d+\varepsilon < n$, we obtain that

$$\begin{aligned}&\left|R(z_j)\pi(z_j)\pi_1(z_j)\mathrm{e}^{2\delta z_j^n} + [(b-a-R(z_j))\pi_1(z_j) + (a-b-R(z_j))\pi(z_j)]\mathrm{e}^{\delta z_j^n} + R(z_j)\right| \\ &\ge \exp\left\{-r_j^{d+\varepsilon}\right\}\exp\left\{2|\delta| r_j^n\right\} - \exp\left\{r_j^{d+\varepsilon}\right\}\exp\left\{|\delta| r_j^n\right\} - r_j^k \\ &\ge \exp\left\{2|\delta| r_j^n(1-o(1))\right\}(1-o(1)).\end{aligned} \tag{2.14}$$

Thus, (2.14) contradicts (2.10).

Second, we suppose that a is a finite value and $b = \infty$. We set $f(z) = y(z) - a$. Then $\sigma(f) = \sigma(y)$ and

$$\lambda(f) = \lambda(y-a) < \sigma(y), \quad \lambda\left(\frac{1}{f}\right) = \lambda\left(\frac{1}{y}\right) < \sigma(y).$$

Hence, $f(z)$ may be rewritten as

$$f(z) = \pi(z)\mathrm{e}^{\delta z^n},$$

where $n,\ \delta$ and $\pi(z)$ are defined as above. And $f(z+1)$ may be rewritten as

$$f(z+1) = \pi_1(z)\mathrm{e}^{\delta z^n},$$

where $\pi_1(z)$ is defined as above. Since $f(z+1) - f(z) = y(z+1) - y(z)$, we have

$$f(z+1) - f(z) = (\pi_1(z) - \pi(z))\mathrm{e}^{\delta z^n} = R(z). \tag{2.15}$$

By $R(z) \not\equiv 0$, we see that $\pi_1(z) - \pi(z) \not\equiv 0$. Thus by (2.5), (2.9) and the fact that $R(z)$ is a rational function, we see that (2.15) is a contradiction.

(3) Now, we suppose that $y(z)$ has infinitely many poles. Then we will prove $\lambda(\frac{1}{y}) \geq 1$. Suppose that a set $A = \{x_j + \mathrm{i}y_j | j = 1, \cdots, s\}$ consists of all poles of $R(z)$. Set $M = \max\{|x_j| + |y_j| + 1 : 1 \leq j \leq s\}$. Then there is no pole of $R(z)$ in regions $D_1 = \{z : \mathrm{Re}\ z > M\}$, $D_2 = \{z : \mathrm{Re}\ z < -M\}$, $D_3 = \{z : \mathrm{Im}\ z > M\}$ and $D_4 = \{z : \mathrm{Im}\ z < -M\}$.

Since $y(z)$ has infinitely many poles, we see that there is at least one D_j such that y has infinitely many poles in D_j. If y has infinitely many poles in D_1, then there exists a point $z_1 \in D_1$ such that $y(z_1) = \infty$. Thus for any $n \in N$, $z_1 + n \in D_1$ and $R(z_1 + n) \neq \infty$, and by (1.1), we see $y(z_1 + n) = \infty$. Hence $\lambda(\frac{1}{y}) \geq 1$.

If y has infinitely many poles in D_3 or D_4, we can use the same method to prove $\lambda(\frac{1}{y}) \geq 1$.

If y has infinitely many poles in D_2, then there exists a point $z_2 \in D_2$ such that $y(z_2) = \infty$. We may rewrite (1.1) as

$$y(z) - y(z-1) = R(z-1).$$

In D_2, $R(z-1)$ has no pole, using the same method as above, we obtain that, for any $n \in N$, $y(z_2 - n) = \infty$. So, $\lambda(\frac{1}{y}) \geq 1$.

(4) Suppose $R(z) \not\equiv 1$. We prove that the exponent of convergence of fixed points of y satisfies $\tau(y) = \sigma(y)$. Set

$$f(z) = y(z) - z.$$

Then $f(z)$ is a transcendental meromorphic function and

$$\sigma(f) = \sigma(y), \quad \tau(y) = \lambda(f), \quad S(r, f) = S(r, y).$$

Substituting $y(z) = f(z) + z$ into (1.1), we obtain that

$$P(z, f) := f(z+1) - f(z) + 1 - R(z) = 0.$$

Since $P(z, 0) = 1 - R(z) \not\equiv 0$, by Lemma 2.4, we see that

$$N\left(r, \frac{1}{f}\right) = T(r, f) + S(r, f).$$

Hence $\tau(y) = \lambda(f) = \sigma(f) = \sigma(y)$.

3 Proof of Theorem 2

We need the following lemmas for the proof of Theorem 2.

Lemma 3.1 Let $m(z)$ and $n(z)$ be irreducible polynomials with $\deg m(z) = m$ and $\deg n(z) = n$, $m + n \geq 1$. Then

(1) If $m \neq n$, then $\deg[m(z+1)n(z) - m(z)n(z+1)] = m + n - 1$;

(2) If $m = n$, then $\deg[m(z+1)n(z) - m(z)n(z+1)] \leq m + n - 2$.

Proof Suppose that

$$m(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_0, \quad n(z) = b_n z^n + b_{n-1} z^{n-1} + \cdots + b_0,$$

where $a_m, \cdots, a_0$, $b_n, \cdots, b_0$ are constants, $a_m b_n \neq 0$. Since

$$m(z+1)n(z) - m(z)n(z+1) = Az^{m+n-1} + Bz^{m+n-2} + \cdots,$$

where $A = a_m b_n(m-n)$ and

$$B = \left\{\frac{1}{2}a_m b_n(m(m-1)-n(n-1)) + a_m b_{n-1}(m-n+1) + a_{m-1}b_n(m-n-1)\right\},$$

we see that

(1) If $m \neq n$, then by $a_m b_n(m-n) \neq 0$, we have

$$\deg[m(z+1)n(z) - m(z)n(z+1)] = m+n-1;$$

(2) If $m = n$, then by $a_m b_n(m-n) = 0$, we have

$$m(z+1)n(z) - m(z)n(z+1) = (a_m b_{n-1} - a_{m-1}b_n)z^{m+n-2} + \cdots,$$

so, $\deg[m(z+1)n(z) - m(z)n(z+1)] \leq m+n-2$.

Proof of Theorem 2 (1) Suppose that $s = -1$ and (1.1) has a rational solution $y(z) = \frac{m(z)}{n(z)}$ where $m(z)$, $n(z)$ are irreducible polynomials, $\deg m(z) = m$, $\deg n(z) = n$. Clearly, $y(z)$ cannot be a constant by $R(z)$ $(\not\equiv 0)$. So, $m+n \geq 1$. If $m > n$, then

$$y(z+1) - y(z) = \frac{m(z+1)n(z) - m(z)n(z+1)}{n(z+1)n(z)} = \frac{P(z)}{Q(z)}. \tag{3.1}$$

By Lemma 3.1, we have

$$m+n-1+q = 2n+p,$$

i.e.,

$$m-n = p-q+1 = s+1 = 0.$$

This contradicts our supposition that $m > n$.

If $m = n$, then by Lemma 3.1, we have

$$\deg[m(z+1)n(z) - m(z)n(z+1)] \leq m+n-2.$$

If $\deg[m(z+1)n(z) - m(z)n(z+1)] = m+n-2$, then by (3.1), we have

$$m+n-2+q = p+2n,$$

i.e.,

$$0 = m-n = p-q+2 = -1+2 = 1.$$

This is a contradiction. If $\deg[m(z+1)n(z) - m(z)n(z+1)] = m+n-k$ $(k \geq 3)$, then we obtain that

$$0 = m-n = s+k = k-1 \ (k \geq 3).$$

This is also a contradiction.

If $m < n$, then by Lemma 3.1 and (3.1), we obtain that

$$0 > m-n = p-q+1 = s+1 = -1+1 = 0.$$

This is also a contradiction. Hence (1.1) has no rational solution.

(2) Suppose that $s \geq 0$ and $y(z) = \frac{m(z)}{n(z)}$ is a rational solution of (1.1) where $m(z)$ and $n(z)$ are defined as in the proof of Lemma 3.1.

First, suppose that $s>0$. Then $\frac{P(z)}{Q(z)}\to\infty$ (as $z\to\infty$). If $m=n$, then, as $z\to\infty$, we have $y(z)\to\frac{a_m}{b_n}$, $y(z+1)\to\frac{a_m}{b_n}$. So, as $z\to\infty$,

$$y(z+1)-y(z)\to 0.$$

This contradicts (1.1). If $m<n$, then, using the same method as above, we have $y(z+1)-y(z)\to 0$. This is also a contradiction. Hence we have $m>n$. By Lemma 3.1 and (3.1), we obtain that $m-n=s+1$.

Second, suppose that $s=0$. Using the same method as above, we get $m-n=s+1$.

(3) Suppose that $s\le -2$ and $y(z)=\frac{m(z)}{n(z)}$ is a rational solution of (1.1).

If $m>n$, then by Lemma 3.1 and (3.1), we obtain that

$$m-n=s+1\le -2+1=-1.$$

This is a contradiction. Hence we have $m\le n$.

If $m<n$, then by Lemma 3.1, we have $m-n=s+1$. Hence $y(z)$ satisfies $m-n=s+1$ or $m=n$.

(4) Suppose that $q=0$, i.e., $R(z)\equiv P(z)$ is a polynomial of $\deg P=s$. Set

$$P(z)=d_sz^s+d_{s-1}z^{s-1}+\cdots+d_0\quad (d_s\ne 0). \tag{3.2}$$

Suppose that

$$y(z)=a_mz^m+a_{m-1}z^{m-1}+\cdots+a_0\quad (a_m\ne 0)$$

is a solution of (1.1). Then

$$\begin{aligned}y(z+1)-y(z)&=a_mmz^{m-1}+[C_m^2a_m-C_{m-1}^1a_{m-1}]z^{m-2}\\&\quad+\cdots+[C_m^ja_m+C_{m-1}^{j-1}a_{m-1}+\cdots+C_{m-(j-1)}^1a_{m-(j-1)}]z^{m-j}\\&\quad+\cdots+[a_m+a_{m-1}+\cdots+a_1],\end{aligned} \tag{3.3}$$

where C_h^d $(h=2,\cdots,m;\ d=1,\cdots,h)$ are usual notations for the binomial coefficients. Substituting (3.2) and (3.3) into (1.1), we obtain that

$$\begin{cases}m=s+1,\\ a_m=a_{s+1}=\dfrac{1}{s+1}d_{s+1-1},\\ a_{m-1}=a_{s+1-1}=\dfrac{1}{s+1-1}\left(d_{s+1-2}-C_{s+1}^2a_{s+1}\right),\\ a_{m-(j-1)}=a_{s+1-(j-1)}=\dfrac{1}{s+1-(j-1)}\Big(d_{s+1-j}-C_{s+1}^ja_{s+1}\\ \qquad\qquad -C_{s+1-1}^{j-1}a_{s+1-1}-\cdots-C_{s+1-(j-2)}^2a_{s+1-(j-2)}\Big),\\ \cdots\\ a_1=a_{s+1-s}=\dfrac{1}{s+1-s}\left(d_0-a_{s+1}-a_s-\cdots-a_2\right).\end{cases} \tag{3.4}$$

Thus, coefficients $a_{s+1},\cdots,a_1$ can be decided by coefficients of $R(z)$, and the coefficient a_0 may take any constant.

Theorem 2 is thus proved.

4 Proof of Corollary 1

By [21, Chapter 7], we see that $\Psi(z)$ satisfies the difference equation

$$\Psi(z+1)-\Psi(z)=\frac{1}{z}.$$

Since we have known that $\lambda\left(\frac{1}{\Psi}\right)=\sigma(\Psi)=1$, by Theorem 1(1), we see that $\lambda(\Psi)=\sigma(\Psi)=1$.

By Theorem 1(4) and $R(z)=\frac{1}{z}$ $(\not\equiv 1)$, we see that $\Psi(z)$ has infinitely many fixed points and the exponent of convergence of its fixed points satisfies $\tau(\Psi)=\sigma(\Psi)=1$.

Thus, Corollary 1 is proved.

5 Proof of Theorem 3

We need the following remark and lemmas for the proof of Theorem 3.

Remark 5.1 Following Hayman [23, p.75–76], we define an ε-set to be a countable union of open discs not containing the origin, and subtending angles at the origin whose sum is finite. If E is an ε-set, then the set of $r\geq 1$ for which the circle $S(0,r)$ meets E has finite logarithmic measure, and for almost all real θ the intersection of E with the ray $\arg z=\theta$ is bounded.

Lemma 5.2 [7] Let g be a function transcendental and meromorphic in the complex plane of order less than 1. Let $h>0$. Then there exists an ε-set E such that

$$\frac{g'(z+c)}{g(z+c)}\to 0,\quad \frac{g(z+c)}{g(z)}\to 1\quad (\text{as } z\to\infty \text{ in } C\backslash E)$$

uniformly in c for $|c|\leq h$. Furthermore, E may be chosen so that for large z not in E the function g has no zeros or poles in $|\zeta-z|\leq h$.

Lemma 5.3 [11] Let $c_1,\ c_2$ be two complex numbers such that $c_1\neq c_2$ and let $f(z)$ be a finite order meromorphic function. Let σ be the order of $f(z)$, then for each $\varepsilon>0$, we have

$$m\left(r,\frac{f(z+c_1)}{f(z+c_2)}\right)=O(r^{\sigma-1+\varepsilon}).$$

Proof of Theorem 3 Suppose that $y(z)$ $(\not\equiv 0)$ is a meromorphic solution of (1.2).

Clearly, $y(z)$ is not a constant by (1.2). Suppose that $y(z)=\frac{m(z)}{n(z)}$ is a rational function, where $m(z)$ and $n(z)$ are irreducible polynomials with $\deg m(z)=m,\ \deg n(z)=n$ and $m+n\geq 1$. By (1.2), we have

$$m(z+1)n(z)-P(z)m(z)n(z+1)=0. \tag{5.1}$$

But the left side of (5.1) is a polynomial of degree $p+m+n$. So, (5.1) is a contradiction. Hence $y(z)$ has no nonzero rational solution.

Now suppose that $y(z)$ is transcendental and $\sigma(y)<1$. Then by Lemma 5.2, there exists an ε-set E such that

$$y(z+1)=y(z)(1+o(1))\quad \text{as } z\to\infty \text{ in } C\backslash E. \tag{5.2}$$

By (1.2) and (5.2), we have that

$$y(z)(1+o(1)-P(z))=0\quad \text{as } z\to\infty \text{ in } C\backslash E. \tag{5.3}$$

Since for sufficiently large $|z|=r$ and z in $C\backslash E$,

$$|1+o(1)-P(z)|>1,$$

we obtain $y(z)\equiv 0$ by (5.3). This is a contradiction. So, $\sigma(y)\geq 1$.

Now, we prove that $y(z)$ has at most one Borel exceptional value. First, suppose that a ($\neq\infty$) and ∞ are two Borel exceptional values of $y(z)$. Set $f(z)=y(z)-a$. Then $f(z)$ has Borel exceptional values 0 and ∞, and can be rewritten as

$$f(z)=H(z)\mathrm{e}^{h(z)}, \tag{5.4}$$

where $H(z)$ is a meromorphic function, $h(z)$ is a nonconstant polynomial of $\deg h(z)=h$, and $\sigma(H)<\deg h(z)=h$. By (1.2) and (5.4), we obtain that

$$\frac{H(z+1)}{H(z)}\frac{1}{P(z)}\mathrm{e}^{h(z+1)-h(z)}=1. \tag{5.5}$$

If $h=1$, set $h(z)=\alpha z+\beta$ ($\alpha\neq 0,\ \beta$ are constants), then $h(z+1)-h(z)=\alpha$. Thus, by Lemma 5.2 and $\sigma(H)<h=1$, there exists an ε-set E such that

$$\frac{H(z+1)}{H(z)}=1+o(1)\quad \text{as } z\to\infty \text{ in } C\backslash E. \tag{5.6}$$

Thus, by $h(z+1)-h(z)=\alpha$, $\deg P(z)\geq 1$ and (5.6), we see that

$$\frac{H(z+1)}{H(z)}\frac{1}{P(z)}\mathrm{e}^{h(z+1)-h(z)}\to 0\quad \text{as } z\to\infty \text{ in } C\backslash E.$$

This contradicts (5.5).

If $h\geq 2$, then $\deg[h(z+1)-h(z)]=h-1\geq 1$ and $\sigma(H)=\sigma<h$. By (1.2) and (5.4), we have that

$$\frac{H(z+1)}{H(z)}=P(z)\mathrm{e}^{h(z)-h(z+1)}. \tag{5.7}$$

By Lemma 5.3, we see that, for any given ε ($0<2\varepsilon<h-\sigma$),

$$m\left(r,\frac{H(z+1)}{H(z)}\right)=O\left(r^{\sigma-1+\varepsilon}\right). \tag{5.8}$$

But

$$m\left(r,P(z)\mathrm{e}^{h(z)-h(z+1)}\right)=Cr^{h-1}, \tag{5.9}$$

where $C\neq 0$ is a constant.

Since $h-\sigma>2\varepsilon$, by (5.8) and (5.9), we see that (5.7) is a contradiction.

Second, suppose that a ($\neq\infty$) and b ($\neq 0,a,\infty$) are two Borel exceptional values of $y(z)$. Set

$$f(z)=\frac{y(z)-a}{y(z)-b}. \tag{5.10}$$

Then $f(z)$ has Borel exceptional values 0 and ∞, and can be rewritten as (5.4). By (5.10), we have

$$y(z)=\frac{a-bf(z)}{1-f(z)}. \tag{5.11}$$

Substituting (5.11) into (1.2), we obtain that

$$(1-P(z))bf(z+1)f(z)+(aP(z)-b)f(z+1)+(bP(z)-a)f(z)+a-aP(z)=0.$$

Using the same method as in proof of Theorem 1(2), we can deduce a contradiction.

Hence, $y(z)$ has at most one Borel exceptional value.

Theorem 3 is thus proved.

6 On the c-Separated Paired Value of $f(z)$ (see [14])

In [14], Halburd and Korhonen introduced the concept of the c-separated paired value for meromorphic functions $f(z)$.

The counting function $n_c(r,a),\ a\in \mathbf{C}$, which is the number of equal terms in the beginning of Taylor series expansions of $f(z)$ and $f(z+c)$ in a neighborhood of z_0. We call such points c-separated a-pairs of $f(z)$ in the disc $\{z:|z|\le r\}$.

The integrated counting function is defined as follows:

$$N_c(r,a):=\int_0^r \frac{n_c(t,a)-n_c(0,a)}{t}\mathrm{d}t+n_c(0,a)\log r.$$

Similarly,

$$N_c(r,\infty):=\int_0^r \frac{n_c(t,\infty)-n_c(0,\infty)}{t}\mathrm{d}t+n_c(0,\infty)\log r,$$

where $n_c(r,\infty)$ is the number of c-separated pole pairs of f, which are exactly the c-separated 0-pairs of $\frac{1}{f}$.

A natural difference analogue of $\overline{N}(r,a)$ is

$$\widetilde{N_c}(r,a):=N(r,a)-N_c(r,a), \tag{6.1}$$

which counts the number of those a-points (or poles) of f which are not in c-separated pairs.

A difference analogue of the index of multiplicity $\theta(a,f)$ is called the c-separated pair index, defined as follows:

$$\pi_a(a,f):=\liminf_{r\to\infty}\frac{N_c(r,a)}{T(r,f)}, \tag{6.2}$$

where a is either a slowly moving periodic function with period c, or $a=\infty$. Similarly, we define

$$\Pi_c(a,f):=1-\limsup_{r\to\infty}\frac{\widetilde{N_c}(r,a)}{T(r,f)}, \tag{6.3}$$

which is an analogue of

$$\Theta_c(a,f):=1-\limsup_{r\to\infty}\frac{\overline{N}(r,a)}{T(r,f)}.$$

Lemma 6.1 (see [14]) Let $c\in\mathbf{C}$, and let f be a meromorphic function of finite order such that $f(z+c)-f(z)\not\equiv 0$. Then $\Pi_c(a,f)=0$ except for at most countably many meromorphic periodic functions a with period c such that a satisfies

$$T(r,a)=S(r,f)$$

and

$$\sum_{a}(\delta(a,f)+\pi_c(a,f)) \leq \sum_{a} \Pi_c(a,f) \leq 2. \tag{6.4}$$

Definition 6.1 (see [14]) We say that a is an exceptional paired value of f with the separation c if $f(z)=a$, then also $f(z+c)=a$ with the same or higher multiplicity.

Clearly, for every exceptional paired value of f,

$$N(r,a) \leq N_c(r,a) + O(\log r).$$

Thus, we see that if a is an exceptional paired value of f with the separation c, then

$$1 \leq \Pi_c(a,f) \leq 2.$$

Lemma 6.2 (see [14]) If a finite order meromorphic function f has three exceptional paired values with the separation c, then f is a periodic function with period c.

From Lemma 6.2, we see that, under the meaning of Definition 6.1, a finite order meromorphic function f has at most two exceptional paired values if f is not a periodic function with period c.

By Definition 6.1, we see that all Picard exceptional values of $f(z)$ are also exceptional paired values. But from Definition 6.1 only, we cannot directly see that if all Borel exceptional values of $f(z)$ are also exceptional paired value.

Now, we give the following definition which slightly modifies Definition 6.1.

Definition 6.2 We say that a is an exceptional paired value of finite order meromorphic function f with separation c if

$$\Pi_c(a,f) \geq 1.$$

Thus, by Definition 6.2, we see that, for a finite order meromorphic function f, its Borel exceptional value must be the exceptional paired value with separation c. In fact, if a is the Borel exceptional value of f, then

$$\limsup_{r\to\infty} \frac{N(r,a)}{T(r,f)} = 0,$$

so, $\Pi_c(a,f)=1$ by (6.1), (6.3) and the fact that $N_c(r,a) \geq 0$.

By Definition 6.2 and (6.4), we see that under the meaning of Definition 6.2, a finite order meromorphic function f has at most two exceptional paired values if f is not a periodic function with period c.

From Definition 6.1 or Definition 6.2, we see that ∞ must be an exceptional paired value of every transcendental solution of (1.1) (or (1.2)). Thus, we can see that equations (1.2) and (1.2) has the following property.

For any rational function $R(z) \not\equiv 0$ (or a nonconstant polynomial $P(z)$), every finite order transcendental meromorphic solution $y(z)$ of (1.1) (or (1.2)) has at most one finite exceptional paired value with separation 1.

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