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ON PROPERTIES OF DIFFERENCE POLYNOMIALS*

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Abstract We study the value distribution of difference polynomials of meromorphic functions, and extend classical theorems of Tumura-Clunie type to difference polynomials. We also consider the value distribution of $f(z)f(z+c)$.

Key words Difference polynomial; Tumura-Clunie theorem; value distribution.

2000 MR Subject Classification 30D35; 39B12

1 Introduction and Results

In this article, we use the basic notions of Nevanlinna's theory [1–3].

The Clunie lemma and Tumura-Clunie type theorems were powerful tools in the field of complex differential equations , difference polynomials, and related field.

Clunie [4] obtained

Lemma A Let f be a transcendental meromorphic solution of

$$f^n P(z,f) = Q(z,f),$$

where $P(z,f),\ Q(z,f)$ are differential polynomials in f and its derivatives with small meromorphic coefficients $a_\lambda,\ \lambda\in I$, in the sense of $m(r,a_\lambda)=S(r,f)$ for all $\lambda\in I$. If total degree of $Q(r,f)$ as a polynomial in f and its derivatives is $\le n$, then,

$$m(r,P)=S(r,f).$$

Mues and Steinmets [5] proved that

Theorem B Let f be a meromorphic function, and suppose that

$$\Psi(z)=a_n(z)f(z)^n+\cdots+a_0(z)$$

has small meromorphic coefficients $a_j(z),\ a_n\ne 0$, in the sense of $T(r,a_j)=S(r,f)$. Moreover, assume that

$$\overline{N}\left(r,\frac{1}{\Psi}\right)+\overline{N}(r,f)=S(r,f).$$

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Then,

$$\Psi = a_n \left(f + \frac{a_{n-1}}{n a_n} \right).$$

Recently, many articles [6–15] focused on complex difference. Halburd and Korhonen obtained a difference counterpart of the Clunie lemma, see [10, Corollary 3.3].

Theorem C Let f be a non-constant finite-order meromorphic solution of

$$f^n P_1(z,f) = Q_1(z,f),$$

where $P_1(z,f),\ Q_1(z,f)$ are difference polynomials in f with small meromorphic coefficients, and let $\delta < 1$. If the degree of $Q_1(r,f)$ as a polynomial in f and its shifts is at most n, then,

$$m(r, P_1(z,f)) = o\left(\frac{T(r+|c|, f)}{r^{\delta}} + o(T(r,f) \right).$$

for all r outside of a possible exceptional set with finite logarithmic measure.

Lanie and Yang obtained a more general version, see [15, Theorem 2.3].

Throughout this article, we consider that a difference polynomial

$$G(z) = \sum_{\lambda \in J} b_\lambda(z) \prod_{j=1}^{\tau_\lambda} f(z+\delta_{\lambda,j})^{\mu_{\lambda,j}}, \tag{1.1}$$

where meromorphic coefficients b_λ $(\lambda \in J)$ (where J is a finite index set) satisfy

$$T(r, b_\lambda) = o\{T(r,f)\}, \tag{1.2}$$

$\mu_{\lambda,j}$ (≥ 0) $(\lambda \in J,\ j = 1, \cdots, \tau_\lambda)$ are integers satisfying

$$\max_{\lambda \in J} \sum_{j=1}^{\tau_\lambda} \mu_{\lambda,j} = n, \tag{1.3}$$

and at least one of the shift arguments $\delta_{\lambda,j}$ is non-zero.

In [14], Laine and Yang obtained

Theorem D If f is a meromorphic function of finite order ρ, such that

$$N(r, 1/f) + N(r,f) = O(r^{\rho-1+\varepsilon}) + S(r,f),$$

then, the difference polynomial (1.1) in $f(z)$ and its shifts, with maximal total degree n, must have sufficiently many zeros satisfying

$$N(r, 1/G) \neq O(r^{\rho-1+\varepsilon}) + S(r,f).$$

Theorem E Let $f(z)$ be a transcendental entire function of finite order, and c be a non-zero complex constant. Then, for $n \geq 2$, $f(z)^n f(z+c)$ assumes every non-zero value $a \in \mathbf{C}$ infinitely often.

Remark 1.1 Function $f(z) = e^z$ and its difference polynomial

$$G(z) = f(z)f(z+1) + f(z)f(z+2) = e(e+1)e^{2z},$$

satisfy the conditions of Theorem D, but $N(r, 1/G) = 0$. This shows that Theorem D was incomplete.

To complete Theorem D, we revise it and obtain the following Theorem 1.1. In Theorem 1.1, we add the following condition (1.5), and Remark 1.1 also shows that the condition (1.5) cannot be omitted.

Theorem 1.1 Let f be a meromorphic function of finite order ρ such that, for any given ε $(0 < \varepsilon < 1)$,

$$N(r, 1/f) + N(r, f) = O(r^{\rho-1+\varepsilon}) + S(r, f). \tag{1.4}$$

Suppose that the difference polynomial (1.1) in $f(z)$ and its shifts with small meromorphic coefficients is of maximal total degree n. If $G(z)$ also satisfies

$$\sum_{\lambda \in J_{n-1}} b_\lambda(z) \prod_{j=1}^{\tau_\lambda} f(z + \delta_{\lambda,j})^{\mu_{\lambda,j}} \not\equiv 0, \tag{1.5}$$

where $J_{n-1} = \{\lambda \in J \mid \sum_{j=1}^{\tau_\lambda} \mu_{\lambda,j} = n-1\}$, then, G must have sufficiently many zeros satisfying

$$N(r, 1/G) \neq O(r^{\rho-1+\varepsilon}) + S(r, f).$$

Remark 1.2 In the following example [14], let $f(z) = e^z + 1$ and

$$H(z) = f(z)f(z + \pi \mathrm{i}) - 1 = (1 + e^z)(1 - e^z) - 1 = -e^{2z},$$

where $H(z) \neq 0$. This example shows that the condition of Theorem E, $n \geq 2$, cannot be omitted.

From the above example, we see that $f(z)$ has Borel exceptional value 1. Thus, it is natural to ask what can be said about the value distribution of $f(z)f(z+c)$ if $f(z)$ has a Borel exceptional value d?.

In this article, we consider this problem and obtain the following Theorem 1.2.

Theorem 1.2 Let $f(z)$ be a transcendental entire function of finite order with a Borel exceptional value d, and $c \in \mathbf{C}\backslash\{0\}$ be a complex constant.

Set $H(z) = f(z)f(z + c)$. Then, for every $b(\neq d^2) \in \mathbf{C}$, $\lambda(H - b) = \sigma(f)$.

From Theorem 1.2, we obtain the following corollary.

Corollary 1.3 Let $f(z)$ be a transcendental entire function of finite order, and $c \in \mathbf{C}\backslash\{0\}$ be a complex constant. If $f(z)$ has the Borel exceptional value 0, then, $H(z) = f(z)f(z + c)$ takes every non-zero value $a \in \mathbf{C}$ infinitely often.

Theorem 1.4 Let $f(z)$ be a transcendental entire function of finite order and $c \in \mathbf{C}\backslash\{0\}$ be a complex constant. If $f(z)$ has infinitely many multi-order zeros, then, $H(z) = f(z)f(z+c)$ takes every value $a \in \mathbf{C}$ infinitely often.

Example 1.1 The entire function $f(z) = e^z + 2$ has the Borel exceptional value 2. For any given value $b \neq 4$, we have

$$H(z) - b = f(z)f(z + \pi i) - b = -e^{2z} + 4 - b,$$

which satisfies

$$\lambda(H - b) = \sigma(f) = 1.$$

Example 1.2 The entire function $f(z)=e^z$ has the Borel exceptional value 0. For any given values $b\neq 0$ and $c\in \mathbf{C}\backslash\{0\}$, we have

$$H(z)-b=f(z)f(z+c)-b=e^ce^{2z}-b,$$

which satisfies

$$\lambda(H-b)=\sigma(f)=1.$$

Example 1.3 The entire function $f(z)=(e^z+1)^2$ has infinitely many multi-order zeros. For any given values a (no matter $a=0$ or $a\neq 0$),

$$H(z)-a=f(z)f(z+\pi)-a=e^{4z}-2e^{2z}+1-a$$

has infinitely many zeros.

2 Proof of Theorem 1.1

We need the following lemma for the proof of Theorem 1.1.

Lemma 2.1 [9, 14] Given two distinct complex constants η_1,η_2. Let f be a meromorphic function of finite order ρ. Then, for each $\varepsilon>0$, we have

$$m\left(r,\frac{f(z+\eta_1)}{f(z+\eta_2)}\right)=O(r^{\rho-1+\varepsilon}).$$

Proof of Theorem 1.1

Contrary to the assertion, suppose that

$$N(r,1/G)=O(r^{\rho-1+\varepsilon})+S(r,f). \tag{2.1}$$

We use the same method as in the proof of Theorem D (see the proof of Theorem1.1 in [14]) and obtain

$$G(z)=\widetilde{b_n}(z)(f(z)+\alpha(z))^n, \tag{2.2}$$

where $\alpha(z)=\frac{\widetilde{b_{n-1}}(z)}{n\widetilde{b_n}(z)}$, and for $s=n,n-1$,

$$\widetilde{b_s}(z)=\sum_{\lambda\in J_s}b_\lambda(z)\prod_{j=1}^{\tau_\lambda}\left(\frac{f(z+\delta_{\lambda,j})}{f(z)}\right)^{\mu_{\lambda,j}},\quad J_s=\{\lambda\in J\mid \sum_{j=1}^{\tau_\lambda}\mu_{\lambda,j}=s\}. \tag{2.3}$$

To apply the second main theorem, we need to prove $\alpha(z)\not\equiv 0,\infty$. By (1.5) and (2.3), we see $\widetilde{b_{n-1}}(z)\not\equiv 0$. By (2.3) and the assumption of the theorem that $G(z)$ is of maximal total degree n, we see $\widetilde{b_n}(z)\not\equiv 0$. Clearly, $\widetilde{b_{n-1}}(z)\not\equiv\infty$ and $\widetilde{b_n}(z)\not\equiv\infty$. Thus,

$$\alpha(z)\not\equiv 0,\infty. \tag{2.4}$$

By (1.4), we have

$$N\left(r,\frac{f(z+\delta_{\lambda,j})}{f(z)}\right)=O(r^{\rho-1+\varepsilon})+S(r,f).$$

By Lemma 2.1, we have

$$m\left(r,\frac{f(z+\delta_{\lambda,j})}{f(z)}\right)=O(r^{\rho-1+\varepsilon})+S(r,f).$$

So that
$$T\left(r, \frac{f(z+\delta_{\lambda,j})}{f(z)}\right) = O(r^{\rho-1+\varepsilon}) + S(r,f). \tag{2.5}$$
Combining (2.5) with the assumption that $T(r, b_\lambda) = o\{T(r,f)\}$, by (2.3), we obtain
$$T(r, \widetilde{b_s}(z) = O(r^{\rho-1+\varepsilon}) + S(r,f) \quad s = n-1,\ n.$$
Hence, we have
$$T(r, \alpha(z)) = O(r^{\rho-1+\varepsilon}) + S(r,f). \tag{2.6}$$
Thus, (2.1), (2.2), and (2.6) give
$$N\left(r, \frac{1}{f+\alpha}\right) = O(r^{\rho-1+\varepsilon}) + S(r,f). \tag{2.7}$$
At the end, applying the second main theorem, we have
$$T(r,f) \le N\left(r, \frac{1}{f}\right) + N(r,f) + N\left(r, \frac{1}{f+\alpha}\right) + S(r,f).$$
Combing this with (1.4), (2.4), (2.7), we obtain
$$T(r,f) = O(r^{\rho-1+\varepsilon}) + S(r,f).$$
This is a contradiction. Hence, $N(r, 1/G) \neq O(r^{\rho-1+\varepsilon}) + S(r,f)$.

3 Proof of Theorem 1.2

We need the following lemmas for the proof of Theorem 1.2.

Lemma 3.1 Let $f(z)$ be a transcendental entire function of finite order ρ and $c \in \mathbf{C}\backslash\{0\}$ be a complex constant. Set $H(z) = f(z)f(z+c)$. Then, $\sigma(H) = \sigma(f)$.

Proof We can rewrite $H(z)$ as the form
$$H(z) = f(z)^2 \frac{f(z+c)}{f(z)}. \tag{3.1}$$
For each $\varepsilon > 0$, by Lemma 2.1 and (3.1), we obtain
$$m(r,H) \le 2m(r,f) + m\left(r, \frac{f(z+c)}{f(z)}\right) = 2m(r,f) + O(r^{\rho-1+\varepsilon}) \tag{3.2}$$
and
$$2m(r,f) = m(r,f^2) \le m(r,H) + m\left(r, \frac{f(z)}{f(z+c)}\right) = m(r,H) + O(r^{\rho-1+\varepsilon}). \tag{3.3}$$
Thus, (3.2) and (3.3) give $\sigma(H) = \sigma(f)$.

Lemma 3.2 [16, p.79–80 or 17] Let $f_j(z)$ $(j = 1, \cdots, n)$ $(n \ge 2)$ be meromorphic functions, $g_j(z), j = 1, \cdots, n$, be entire functions satisfying

(i) $\sum\limits_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;

(ii) when $1 \le j < k \le n$, $g_j(z) - g_k(z)$ is not a constant;

(iii) when $1 \le j \le n,\ 1 \le h < k \le n$,

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\}\ (r \to \infty, r \notin E),$$

where $E \subset (1, \infty)$ is of finite linear measure or finite logarithmic measure. Then, $f_j(z) \equiv 0$ $(j = 1, \cdots, n)$.

Proof of Theorem 1.2

As d is the Borel exceptional value of $f(z)$, we can rewrite $f(z)$ in the form

$$f(z) = d + P(z)e^{sz^k}, \tag{3.1}$$

where $P(z)$ is an entire function with $\sigma(P) < \sigma(f) = k,\ s(\neq 0)$ is a constant and k is a positive integer. Thus,

$$f(z+c) = d + P(z+c)P_1(z)e^{sz^k}, \tag{3.2}$$

where

$$P_1(z) = e^{skz^{k-1} + \cdots + sc^k}, \quad \sigma(P_1) = k - 1.$$

Now, suppose that $\lambda(H - b) < \sigma(f)$. By Lemma 3.1, we see that $\sigma(H) = \sigma(f) = \sigma(H - b)$, so that $\lambda(H - b) < \sigma(H - b) = \sigma(f) = k$ and $H(z) - b$ can be rewritten in the form

$$H(z) - b = q(z)e^{\beta z^k}, \tag{3.3}$$

where $\beta\ (\neq 0)$ is a constant, $q(z)$ is an entire function of

$$\sigma(q) \le \max\{\lambda(H - b),\ k - 1\}.$$

By (3.1)–(3.3), we obtain

$$P(z)P(z+c)P_1(z)e^{2sz^k} + (dP(z+c)P_1(z) + dP(z))e^{sz^k} + d^2 - b = q(z)e^{\beta z^k}. \tag{3.4}$$

As $P(z)P(z+c)P_1(z) \not\equiv 0$ and $q(z) \not\equiv 0$, by comparing growths of both sides of (3.4), we see that $\beta = 2s$. Thus, by (3.4), we have

$$[P(z)P(z+c)P_1(z) - q(z)]e^{2sz^k} + (dP(z+c)P_1(z) + dP(z))e^{sz^k} + d^2 - b = 0. \tag{3.5}$$

By Lemma 3.2 and (3.5), we obtain $b = d^2$. This contradicts our assumption that $b \neq d^2$. Hence, $\lambda(H - b) = \sigma(f)$.

4 Proof of Theorem 1.4

We suppose that $f(z)$ has infinitely many multi-order zeros. If $a = 0$, then, $H(z)$ has obviously infinitely many zeros. Now, suppose that $a \neq 0$. If $H(z) - a$ has only finitely many zeros, then, $H(z) - a$ can be rewritten in the form

$$H(z) - a = f(z)f(z+c) - a = p(z)e^{q(z)}, \tag{4.1}$$

where $p(z),\ q(z)$ are polynomials, and $p(z) \not\equiv 0,\ \deg q(z) \ge 1$.

Differentiating (4.1) and eliminating $e^{q(z)}$, we obtain

$$\frac{(f(z)f(z+c))'}{f(z)f(z+c)}=\frac{p'(z)+p(z)q'(z)}{p(z)}-a\frac{p'(z)+p(z)q'(z)}{p(z)}\frac{1}{f(z)f(z+c)}. \tag{4.2}$$

As $f(z)$ has infinitely many multi-order zeros, there is a multi-order zero z_0, such that $|z_0|$ is sufficiently large and $p(z_0)\neq 0$, $p'(z_0)+p(z_0)q'(z_0)\neq 0$. Thus, the right side of (4.2) has a multi-order pole at z_0, but the left side of (4.2) has only a simple pole at z_0. This is a contradiction.

Hence, $H(z)$ takes any value $a\in\mathbf{C}$ infinitely often.

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