



*Research article*

## Meromorphic solutions of three certain types of non-linear difference equations

Min Feng Chen<sup>1,\*</sup>, Zhi Bo Huang<sup>2</sup> and Zong Sheng Gao<sup>3</sup>

<sup>1</sup> School of Mathematics and Statistics, Guangdong University of Foreign Studies, Guangzhou, 510006, China

<sup>2</sup> School of Mathematical Sciences, South China Normal University, Guangzhou, 510631, China

<sup>3</sup> LMIB & School of Mathematical Sciences, Beihang University, Beijing, 100191, China

\* **Correspondence:** Email: chenminfeng198710@126.com.

**Abstract:** In this paper, the representations of meromorphic solutions for three types of non-linear difference equations of form

$$\begin{aligned} f^n(z) + P_d(z, f) &= u(z)e^{v(z)}, \\ f^n(z) + P_d(z, f) &= p_1e^{\lambda z} + p_2e^{-\lambda z} \end{aligned}$$

and

$$f^n(z) + P_d(z, f) = p_1e^{\alpha_1 z} + p_2e^{\alpha_2 z}$$

are investigated, where  $n \geq 2$  is an integer,  $P_d(z, f)$  is a difference polynomial in  $f$  of degree  $d \leq n - 1$  with small coefficients,  $u(z)$  is a non-zero polynomial,  $v(z)$  is a non-constant polynomial,  $\lambda, p_j, \alpha_j$  ( $j = 1, 2$ ) are non-zero constants. Some examples are also presented to show our results are best in certain sense.

**Keywords:** meromorphic solution; difference equation; difference polynomial

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### 1. Introduction

In 1964, Hayman [6, P69] extended and improved the results of Tumura [16] and Clunie [1], and obtained the following theorem.

**Theorem 1.1.** (See [6]) *Suppose that  $f(z), g(z)$  are non-constant meromorphic function and satisfy the equation*

$$f^n(z) + Q_d(z, f) = g(z), \tag{1.1}$$

where  $Q_d(z, f)$  is a differential polynomial in  $f$  of degree  $d \leq n - 1$ . If

$$N(r, f) + N\left(r, \frac{1}{g}\right) = S(r, f), \quad (1.2)$$

then  $g(z) = (\gamma_0(z) + f(z))^n$ , where  $\gamma_0(z)$  is a small function of  $f(z)$ .

With the establishment of difference analogue of the lemma of the logarithmic derivative, the Clunie and Mohon'ko lemmas, see Halburd and Korhonen [8], Chiang and Feng [5], more and more researchers [2–4, 13, 15, 17, 20] come to study the properties of complex difference equations or complex differential-difference equations. In particular, there has been a renewed interest [11, 12, 14, 17] in solvability and existence for entire or meromorphic solutions of Eq (1.1). Replacing the differential polynomial  $Q_d(z, f)$  in Eq (1.1) by a differential-difference polynomial, and satisfying the condition (1.2), Chen et al. [4] proved the following result.

**Theorem 1.2.** (See [4]) *Let  $n \geq 2$  be an integer. Suppose that  $f(z)$  is a non-constant finite order meromorphic solution of*

$$f^n(z) + Q_d(z, f) = u(z)e^{v(z)}, \quad (1.3)$$

where  $Q_d(z, f)$  is a differential-difference polynomial in  $f$  with degree  $d \leq n - 2$ ,  $u(z)$  is a non-zero rational function,  $v(z)$  is a non-constant polynomial. If  $N(r, f) = S(r, f)$ , then

$$u(z)e^{v(z)} = f^n(z) \quad \text{and} \quad Q_d(z, f) = 0.$$

**Remark 1.1.** *With other conditions fixed, if  $Q_d(z, f)$  is a difference polynomial in  $f$ , the Theorem 1.2 is still valid. If the condition  $d \leq n - 2$  is omitted, then the Theorem 1.2 is impossible. For example,  $f(z) = 1 + e^z$  is an entire solution of equation  $f^2(z) - f(z + \ln 2) = e^{2z}$ , where  $n = 2$ ,  $d = n - 1 = 1$ . So it's natural to ask what will happen to the solutions of Eq (1.3) when  $d = n - 1$ ? In this paper, we study this problem and obtain the following result.*

**Theorem 1.3.** *Let  $n \geq 2$  be an integer,  $P_d(z, f)$  be a difference polynomial in  $f$  of degree  $d \leq n - 1$  with polynomial coefficients,  $u(z)$  be a non-zero polynomial,  $v(z)$  be a non-constant polynomial. If  $f(z)$  is a transcendental meromorphic solution of equation*

$$f^n(z) + P_d(z, f) = u(z)e^{v(z)} \quad (1.4)$$

and  $N(r, f) = S(r, f)$ , then  $\sigma(f) = \deg v(z)$ , and one of the following holds:

(1)  $f^n(z) = u(z)e^{v(z)}$  and  $P_d(z, f) = 0$ ;

(2) If  $\varphi(z) = (u'(z) + u(z)v'(z))f(z) - nu(z)f'(z) \neq 0$ , then  $T(r, f) = N_1\left(r, \frac{1}{f}\right) + S(r, f)$ . Furthermore, if  $\varphi(z)$  is a non-zero polynomial, then  $f(z) = \gamma_0(z) + p(z)e^{q(z)}$ , where  $\gamma_0(z), p(z), q(z)$  are non-zero polynomials satisfying  $p^n(z) = u(z)$ ,  $nq(z) = v(z)$ .

**Example 1.1.**  $f(z) = z + e^z$  is a solution of the following difference equation

$$f^3(z) - zf(z)f(z + \ln 3) + z(z + \ln 3)f(z) - z^3 = e^{3z}.$$

Here,  $u(z) = 1$ ,  $v(z) = 3z$ ,  $n = 3$ ,  $\varphi = (u' + uv')f - 3uf' = 3(z - 1)$ . It implies that the solution satisfying Theorem 1.3 does exist.

If  $g(z) = p_1e^{\lambda z} + p_2e^{-\lambda z}$  and the condition (1.2) is not met, what will happen to the solutions of Eq (1.1)? Li [11] obtained the following result.

**Theorem 1.4.** (See [11]) *Let  $n \geq 2$  be an integer,  $Q_d(z, f)$  be a differential polynomial in  $f$  of degree  $d \leq n - 1$ , and let  $\lambda, p_1, p_2$  be non-zero constants. If  $f(z)$  is a transcendental meromorphic solution of the equation*

$$f^n(z) + Q_d(z, f) = p_1e^{\lambda z} + p_2e^{-\lambda z} \quad (1.5)$$

and  $N(r, f) = S(r, f)$ , then  $f(z) = c_0(z) + c_1e^{\frac{\lambda}{n}z} + c_2e^{-\frac{\lambda}{n}z}$ , where  $c_0(z)$  is small function of  $f(z)$ ,  $c_j$  are constants satisfying  $c_j^n = p_j, j = 1, 2$ .

It's natural to ask: what will happen to the solutions of Eq (1.5) when  $Q_d(z, f)$  is a difference polynomial in  $f$  of degree  $d \leq n - 1$ ? In this paper, we consider this problem and obtain the following result.

**Theorem 1.5.** *Let  $n \geq 2$  be an integer,  $P_d(z, f)$  be a difference polynomial in  $f$  of degree  $d \leq n - 1$  with polynomial coefficients,  $\lambda, p_1, p_2$  be non-zero constants. If  $f(z)$  is a transcendental meromorphic solution of equation*

$$f^n(z) + P_d(z, f) = p_1e^{\lambda z} + p_2e^{-\lambda z} \quad (1.6)$$

and  $N(r, f) = S(r, f)$ , then  $\sigma(f) = 1$ , and one of the following holds:

(1)  $f(z) = c_1e^{\frac{\lambda}{n}z} + c_2e^{-\frac{\lambda}{n}z}$ ;

(2) If  $\varphi(z) = \lambda^2 f - n^2 f'' \not\equiv 0$ , then  $T(r, f) = N_2\left(r, \frac{1}{f}\right) + S(r, f)$ . Furthermore, if  $\varphi(z)$  is a non-zero polynomial, then  $f(z) = \gamma_0(z) + c_1e^{\frac{\lambda}{n}z} + c_2e^{-\frac{\lambda}{n}z}$ , where  $\gamma_0(z)$  is a non-zero polynomial,  $c_j$  are constants satisfying  $c_j^n = p_j, j = 1, 2$ .

**Example 1.2.**  $f(z) = z + e^z + e^{-z}$  is a solution of the following difference equation

$$f^2(z) + 2zf(z + i\pi) - (3z^2 + 2i\pi z + 2) = e^{2z} + e^{-2z}.$$

Here,  $\lambda = 2, n = 2, \varphi = \lambda^2 f - n^2 f'' = 4z$ . It implies that the solution satisfying Theorem 1.5 does exist.

If  $g(z) = p_1e^{\alpha_1 z} + p_2e^{\alpha_2 z}$  and the condition (1.2) is not met, what will happen to the solutions of Eq (1.1)? Liu et al. [14] obtained the following result.

**Theorem 1.6.** (See [14]) *Let  $n \geq 2$  be an integer,  $Q_d(z, f)$  be a differential polynomial in  $f$  of degree  $d \leq n - 1$  with polynomial coefficients, and let  $p_j, \alpha_j (j = 1, 2)$  be non-zero constants satisfying*

$$\frac{\alpha_1}{\alpha_2} \in \left\{ \frac{t}{n}, \frac{n}{t} : 1 \leq t \leq n \right\}. \quad (1.7)$$

If equation

$$f^n(z) + Q_d(z, f) = p_1e^{\alpha_1 z} + p_2e^{\alpha_2 z} \quad (1.8)$$

admits a meromorphic solution  $f(z)$  satisfying  $N(r, f) = S(r, f)$ , then one of the following holds:

(1)  $f(z) = \gamma_1(z) + c_1e^{\alpha_1 z}, \frac{\alpha_1}{\alpha_2} = \frac{n}{t}$ ,

(2)  $f(z) = \gamma_2(z) + c_2e^{\alpha_2 z}, \frac{\alpha_1}{\alpha_2} = \frac{t}{n}$ , where  $\gamma_j(z)$  are small functions of  $f(z)$ ,  $c_j$  are constants satisfying  $c_j^n = p_j, j = 1, 2$ .

What will happen to the solutions of Eq (1.8) when  $Q_d(z, f)$  is a difference polynomial in  $f$  of degree  $d \leq n - 1$ ? In this paper, we study this problem and obtain the following result.

**Theorem 1.7.** *Let  $n \geq 2$  be an integer,  $P_d(z, f)$  be a differential polynomial in  $f$  of degree  $d \leq n - 1$  with polynomial coefficients, and let  $p_j, \alpha_j$  ( $j = 1, 2$ ) be non-zero constants satisfying the condition (1.7). If  $f(z)$  is a meromorphic solution of equation*

$$f^n(z) + P_d(z, f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z} \quad (1.9)$$

and  $N(r, f) = S(r, f)$ , then  $\sigma(f) = 1$  and  $T(r, f) = N_2\left(r, \frac{1}{f}\right) + S(r, f)$ . If  $\varphi(z) = \alpha_1 \alpha_2 f(z) - n(\alpha_1 + \alpha_2)f'(z) + n^2 f''(z)$  is a non-zero polynomial and then one of the following holds:

(1)  $f(z) = \gamma_0(z) + c_1 e^{\frac{\alpha_1}{n} z}$ ,  $\frac{\alpha_1}{\alpha_2} = \frac{n}{l}$ ;

(2)  $f(z) = \gamma_0(z) + c_2 e^{\frac{\alpha_2}{n} z}$ ,  $\frac{\alpha_1}{\alpha_2} = \frac{l}{n}$ , where  $\gamma_0(z)$  is a non-zero polynomial,  $c_j$  are constants satisfying  $c_j^n = p_j$ ,  $j = 1, 2$ .

**Remark 1.2.** *The condition (1.7) in Theorem 1.7 is necessary, see example 1.3.*

**Example 1.3.**  $f(z) = e^z + e^{2z}$  is a solution of the following difference equation

$$f^2(z) - \frac{1}{2}f(z + \ln 2) + f(z) = e^{4z} + 2e^{3z}.$$

Here,  $\alpha_1 = 4$ ,  $\alpha_2 = 3$ ,  $n = 2$ ,  $\frac{\alpha_1}{\alpha_2} = \frac{2n}{n+1}$ .

**Remark 1.3.** *The following examples 1.4 and 1.5 show that the solution satisfying Theorem 1.7 does exist.*

**Example 1.4.**  $f(z) = 1 + e^z$  is a solution of the following difference equation

$$f^3(z) + 2f(z)f(z + i\pi) - 3f(z) = e^{3z} + e^{2z}.$$

Here,  $\alpha_1 = 3$ ,  $\alpha_2 = 2$ ,  $n = 3$ ,  $\frac{\alpha_1}{\alpha_2} = \frac{n}{n-1}$ ,  $\varphi = \alpha_1 \alpha_2 f - n(\alpha_1 + \alpha_2)f' + n^2 f'' = 6$ .

**Example 1.5.**  $f(z) = 1 - e^z$  is a solution of the following difference equation

$$f^3(z) - f(z + \ln 3)f(z) + f(z + i\pi) - 1 = 2e^z - e^{3z}.$$

Here,  $\alpha_1 = 1$ ,  $\alpha_2 = 3$ ,  $n = 3$ ,  $\frac{\alpha_1}{\alpha_2} = \frac{n-2}{n}$ ,  $\varphi = \alpha_1 \alpha_2 f - n(\alpha_1 + \alpha_2)f' + n^2 f'' = 3$ .

In this paper, we assume familiarity with the basic results and standard notations of Nevanlinna theory [6, 9, 19].  $f$  is meromorphic function in the whole complex plane  $\mathbb{C}$ . In addition, we use  $\sigma(f)$  to denote the order of growth of  $f$ . For simplicity, we denote by  $S(r, f)$  any quantity satisfying  $S(r, f) = o(T(r, f))$ , as  $r \rightarrow \infty$ , outside of a possible exceptional set of finite logarithmic measure. A meromorphic function  $\varphi$  defined in  $\mathbb{C}$  is called a small function of  $f$  if  $T(r, \varphi) = S(r, f)$ . Let  $N_k\left(r, \frac{1}{f}\right)$  denote the counting function of those zeros of  $f$  (counting multiplicity) whose multiplicities are not greater than  $k$ , and let  $N_{(k)}\left(r, \frac{1}{f}\right)$  denote the counting function of those zeros of  $f$  (counting multiplicity) whose multiplicities are not less than  $k$ .

## 2. Some Lemmas

**Lemma 2.1.** (See [5, Corollary 2.6]) Let  $\eta_1, \eta_2$  be two complex numbers such that  $\eta_1 \neq \eta_2$  and let  $f(z)$  be a finite order meromorphic function. Let  $\sigma$  be the order of  $f(z)$ , then for each  $\varepsilon > 0$ , we have

$$m\left(r, \frac{f(z + \eta_1)}{f(z + \eta_2)}\right) = O(r^{\sigma-1+\varepsilon}).$$

**Lemma 2.2.** (See [7, Corollary 3.3]) Let  $f$  be a non-constant finite order meromorphic solution of

$$f^n(z)P(z, f) = Q(z, f),$$

where  $P(z, f)$  and  $Q(z, f)$  are difference polynomials in  $f$  with small meromorphic coefficients, and let  $c \in \mathbb{C}$ ,  $\delta < 1$ . If the total degree of  $Q(z, f)$  as a polynomial in  $f$  and its shifts is  $\leq n$ , then

$$m(r, P(z, f)) = o\left(\frac{T(r + |c|, f)}{r^\delta}\right) + o(T(r, f))$$

for all  $r$  outside of a possible exceptional set  $E$  with finite logarithmic measure  $\int_E \frac{dr}{r} < \infty$ .

Using the same methods as in proof of [9, Lemma 2.4.2] and Lemma 2.1, we have a similar conclusion as follows.

**Lemma 2.3.** Let  $f$  be a non-constant finite order meromorphic solution of

$$f^n(z)P(z, f) = Q(z, f),$$

where  $P(z, f)$  and  $Q(z, f)$  are differential-difference polynomials in  $f$  with small meromorphic coefficients. If the total degree of  $Q(z, f)$  as a polynomial in  $f$  and its derivatives and its shifts is  $\leq n$ , then

$$m(r, P(z, f)) = S(r, f)$$

for all  $r$  outside of a possible exceptional set with finite logarithmic measure.

**Lemma 2.4.** Suppose that  $\alpha(z), \beta(z), \varphi(z)$  are non-zero polynomials and satisfy the differential equation

$$\alpha(z)f(z) + \beta(z)f'(z) = \varphi(z), \quad (2.1)$$

where  $\deg \varphi(z) \geq \deg \alpha(z) \geq \deg \beta(z)$ . Then Eq (2.1) has a special solution  $\gamma_0(z)$  which is a non-zero polynomial.

*Proof.* Now we distinguish two cases below. Case 1:  $\deg \alpha(z) = \deg \beta(z)$ ; Case 2:  $\deg \alpha(z) > \deg \beta(z)$ .

**Case 1.** In this case, we consider three subcases. Subcase 1.1. If  $\alpha, \beta, \varphi$  are non-zero constants, then  $\gamma_0 = \frac{\varphi}{\alpha}$ . Subcase 1.2. If  $\alpha, \beta$  are non-zero constants, and  $\varphi$  is non-constant polynomial. Let

$$\varphi(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad n \geq 1,$$

where  $a_n (\neq 0), a_{n-1}, \dots, a_0$  are constants. Assuming

$$\gamma_0(z) = b_n z^n + b_{n-1} z^{n-1} + \cdots + b_0,$$

and using the method of undetermined coefficients, it follows from Eq (2.1) that

$$\begin{cases} b_n = \frac{a_n}{\alpha}; \\ b_j = \frac{a_{j-n} \frac{\beta}{\alpha} a_{j+1}}{\alpha}, \quad j = n-1, n-2, \dots, 0. \end{cases}$$

So, we get the expression of the special solution  $\gamma_0(z)$  and  $\deg \gamma_0(z) = \deg \varphi(z)$ . Subcase 1.3. If  $\alpha, \beta$  are non-constant polynomials, then  $\varphi(z)$  must be non-constant polynomial. Using the method of undetermined coefficients and Eq (2.1), similar to the proof of subcase 1.2, we can also obtain the expression of the special solution  $\gamma_0(z)$  and  $\deg \gamma_0(z) = \deg \varphi(z) - \deg \alpha(z)$ .

**Case 2.** If  $\deg \alpha(z) > \deg \beta(z)$ , then  $\varphi(z)$  must be are non-constant polynomials. By using the method of undetermined coefficients and equation (2.1), similar to the proof of subcase 1.2, we can also obtain the expression of the special solution  $\gamma_0(z)$  and  $\deg \gamma_0(z) = \deg \varphi(z) - \deg \alpha(z)$ .  $\square$

**Lemma 2.5.** (See [3, Lemma 2.6]) *Let  $n \geq 1$  be an integer,  $\lambda$  be a non-zero constant and  $\varphi(z)$  be a non-zero polynomial. Then the differential equation*

$$\lambda^2 f(z) - n^2 f''(z) = \varphi(z)$$

*has a special solution  $\gamma_0(z)$  which is a non-zero polynomial.*

**Lemma 2.6.** *Let  $n \geq 2$  be an integer,  $P_d(z, f)$  be a difference polynomial in  $f$  of degree  $d \leq n-1$  with small meromorphic coefficients, and let  $p_j, \alpha_j$  ( $j = 1, 2$ ) be non-zero constants satisfying  $\alpha_1 \neq \alpha_2$ . If  $f(z)$  is a meromorphic solution of equation*

$$f^n(z) + P_d(z, f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z} \quad (2.2)$$

*and  $N(r, f) = S(r, f)$ , then  $\sigma(f) = 1$ .*

*Proof.* Set

$$P_d(z, f) = \sum_{\mu \in I} a_\mu(z) \prod_{j=1}^{t_\mu} f(z + \delta_{\mu j})^{l_{\mu j}}, \quad (2.3)$$

where  $I$  is a finite set of the index  $\mu$ ,  $t_\mu, l_{\mu j}$  ( $\mu \in I, j = 1, \dots, t_\mu$ ) are natural numbers,  $\delta_{\mu j}$  ( $\mu \in I, j = 1, \dots, t_\mu$ ) are distinct complex constants. Denote  $g_{\mu j}(z) := \frac{f(z + \delta_{\mu j})}{f(z)}$  and substitute this equality into (2.3) yields

$$P_d(z, f) = \sum_{\mu \in I} \left( a_\mu(z) \prod_{j=1}^{t_\mu} g_{\mu j}^{l_{\mu j}}(z) \right) f^{l_\mu}(z) := \sum_{k=0}^d b_k(z) f^k(z), \quad (2.4)$$

where  $l_\mu = \sum_{j=1}^{t_\mu} l_{\mu j}$ ,  $d = \max_{\mu \in I} \{l_\mu\}$ ,  $b_k(z) = \sum_{l_\mu=k} \left( a_\mu(z) \prod_{j=1}^{t_\mu} g_{\mu j}^{l_{\mu j}}(z) \right)$  ( $k = 0, \dots, d$ ). By applying Lemma 2.1, we get  $m(r, b_k(z)) = S(r, f)$  ( $k = 0, \dots, d$ ). Differentiating (2.4) yields

$$P'_d(z, f) = \sum_{\mu \in I} \left( \left( a'_\mu(z) + \sum_{j=1}^{t_\mu} \frac{a_\mu(z) l_{\mu j} f'(z + \delta_{\mu j})}{f(z + \delta_{\mu j})} \right) \prod_{j=1}^{t_\mu} g_{\mu j}^{l_{\mu j}}(z) \right) f^{l_\mu}(z) := \sum_{k=0}^d c_k(z) f^k(z). \quad (2.5)$$

By using Lemma 2.1 and the lemma of the logarithmic derivative, we get  $m(r, c_k(z)) = S(r, f)$  ( $k = 0, \dots, d$ ). Noting that  $m(r, b_k(z)) = S(r, f)$  ( $k = 0, \dots, d$ ) and by induction, we obtain

$$m(r, P_d(z, f)) \leq dm(r, f) + S(r, f). \quad (2.6)$$

Combining (2.2) and (2.6), and noting that  $N(r, f) = S(r, f)$ , we get

$$\begin{aligned} T(r, p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}) &= T(r, f^n + P_d(z, f)) \\ &= m(r, f^n + P_d(z, f)) + S(r, f) \\ &\leq nT(r, f) + S(r, f), \end{aligned}$$

and

$$\begin{aligned} T(r, p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}) &= T(r, f^n + P_d(z, f)) \\ &= m(r, f^n + P_d(z, f)) + S(r, f) \\ &\geq m(r, f^n) - m(r, P_d(z, f)) + S(r, f) \\ &\geq (n - d)T(r, f) + S(r, f). \end{aligned}$$

From the above two inequalities, we have

$$(n - d)T(r, f) + S(r, f) \leq T(r, p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}) \leq nT(r, f) + S(r, f).$$

Then  $\sigma(f) = 1$ . □

**Lemma 2.7.** (See [14, Lemma 2.5]) *Let  $n \geq 1$  be an integer,  $\alpha_1, \alpha_2$  be non-zero constants satisfying  $\alpha_1 \neq \alpha_2$ , and let  $\varphi(z)$  be a non-vanishing polynomial. Then the differential equation*

$$\alpha_1 \alpha_2 f(z) - n(\alpha_1 + \alpha_2) f'(z) + n^2 f''(z) = \varphi(z)$$

*has a special solution  $\gamma_0(z)$  which is a non-vanishing polynomial.*

**Lemma 2.8.** (See [7, p.247]) *Suppose that  $f(z)$  is a transcendental meromorphic function and  $K > 1$ . Then there exists a set  $M(K)$  of upper logarithmic density at most  $\delta(K) = \min\{(2e^{K-1} - 1)^{-1}, (1 + e^{(K-1)}) \exp(e(1 - K))\}$  such that for every positive integer  $k$ , we have*

$$\limsup_{\substack{r \rightarrow \infty \\ r \notin M(K)}} \frac{T(r, f)}{T(r, f^{(k)})} \leq 3eK.$$

**Remark 2.1.** *By Lemma 2.8, we see that if  $f$  is a transcendental meromorphic function, and if  $\varphi$  satisfying  $T(r, \varphi^{(k)}) = S_1(r, f)$ , then  $T(r, \varphi) = S_1(r, f)$ , where  $S_1(r, f)$  is defined to be any quantity such that for any positive number  $\varepsilon$  there exists a set  $E(\varepsilon)$  whose upper logarithmic density is less than  $\varepsilon$ , and  $S_1(r, f) = o(T(r, f))$  as  $r \rightarrow \infty$ ,  $r \notin E(\varepsilon)$ .*

**Lemma 2.9.** (See [18, Theorem 1.51]) *Suppose that  $f_1, f_2, \dots, f_n$  ( $n \geq 2$ ) are meromorphic functions and  $g_1, g_2, \dots, g_n$  are entire functions satisfying the following conditions:*

$$(1) \sum_{j=1}^n f_j e^{g_j} \equiv 0.$$

(2)  $g_j - g_k$  are not constants for  $1 \leq j < k \leq n$ .

(3) For  $1 \leq j \leq n, 1 \leq h < k \leq n$ ,

$$T(r, f_j) = o(T(r, e^{g_h - g_k})) \quad (r \rightarrow \infty, r \notin E),$$

where  $E \subset [1, \infty)$  is finite linear measure  $\int_E dr < \infty$  or finite logarithmic measure  $\int_E \frac{dr}{r} < \infty$ . Then  $f_j \equiv 0$  ( $j = 1, \dots, n$ ).

**Lemma 2.10.** (See [10, Lemma 3]) Suppose that  $h$  is a nonconstant meromorphic function satisfying

$$\bar{N}(r, h) + \bar{N}\left(r, \frac{1}{h}\right) = S(r, h).$$

Let  $f = a_0 h^p + a_1 h^{p-1} + \dots + a_p$ ,  $g = b_0 h^q + b_1 h^{q-1} + \dots + b_q$  be polynomials in  $h$  with coefficients  $a_0, a_1, \dots, a_p, b_0, b_1, \dots, b_q$  being small functions of  $h$  and  $a_0 b_0 a_p \neq 0$ . If  $q \leq p$ , then  $m(r, g/f) = S(r, f)$ .

### 3. Proof of Theorem 1.3

*Proof.* Combining (1.4), (2.6) and  $N(r, f) = S(r, f)$ , we get

$$\begin{aligned} T(r, u(z)e^{v(z)}) &= T(r, f^n + P_d(z, f)) \\ &= m(r, f^n + P_d(z, f)) + S(r, f) \\ &\leq nT(r, f) + S(r, f), \end{aligned}$$

and

$$\begin{aligned} T(r, u(z)e^{v(z)}) &= T(r, f^n + P_d(z, f)) \\ &= m(r, f^n + P_d(z, f)) + S(r, f) \\ &\geq m(r, f^n) - m(r, P_d(z, f)) + S(r, f) \\ &\geq (n - d)T(r, f) + S(r, f). \end{aligned}$$

From the above two inequalities, we get

$$(n - d)T(r, f) + S(r, f) \leq T(r, u(z)e^{v(z)}) \leq nT(r, f) + S(r, f).$$

Then  $\sigma(f) = \deg v(z)$ .

Denote  $P_d(z, f) = P$ . By differentiating (1.4), we have

$$nf^{n-1}f' + P' = (u' + uv')e^v. \quad (3.1)$$

Eliminating  $e^v$  from (1.4) and (3.1), we have

$$f^{n-1}\varphi = uP' - (u' + uv')P, \quad (3.2)$$



where

$$\varphi = (u' + uv')f - nuf'. \quad (3.3)$$

By Lemma 2.3, we have

$$m(r, \varphi) = S(r, f).$$

Noting that  $N(r, f) = S(r, f)$ , then  $T(r, \varphi) = S(r, f)$ . We consider two cases as follows.

**Case 1.** If  $\varphi \equiv 0$ , then  $(u' + uv')f = nuf'$ , that is  $\frac{f'}{f} = \frac{u'+uv'}{nu} = \frac{1}{n}\left(\frac{u'}{u} + v'\right)$ . By integration, we get  $f^n = cue^v$  ( $c \neq 0$ ). Substituting this equality into (1.4) yields  $(1 - \frac{1}{c})f^n = -P$ . If  $c \neq 1$ , by Lemma 2.2, we have  $m(r, f) = S(r, f)$ . Noting that  $N(r, f) = S(r, f)$ , then  $T(r, f) = S(r, f)$ , a contradiction. Therefore  $c = 1$ , and  $f(z) = p(z)e^{q(z)}$ , where  $p(z), q(z)$  are non-zero polynomials satisfying  $p^n(z) = u(z)$ ,  $nq(z) = v(z)$ .

**Case 2.** If  $\varphi \not\equiv 0$ , and  $z_0$  is a multiple zero of  $f$ , it follows from (3.3) that  $z_0$  is a zero of  $\varphi$ . Then

$$N_{(2)}\left(r, \frac{1}{f}\right) = S(r, f). \quad (3.4)$$

We claim that  $u' + uv' \not\equiv 0$ , otherwise  $f' = -\frac{\varphi}{nu}$ , then  $T(r, f') = S_1(r, f)$ , by Lemma 2.8, we have  $T(r, f) = S_1(r, f)$ , which is impossible. Rewrite (3.3) as  $\frac{1}{f} = \frac{1}{\varphi}[(u' + uv') - nu\frac{f'}{f}]$ , by the lemma of the logarithmic derivative and  $T(r, \varphi) = S(r, f)$ , we get

$$m\left(r, \frac{1}{f}\right) \leq m\left(r, \frac{1}{\varphi}\right) + m(r, u' + uv') + m(r, nu) + m\left(r, \frac{f'}{f}\right) + O(1) = S(r, f). \quad (3.5)$$

Combining (3.4) and (3.5), we get

$$T(r, f) = m\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f}\right) + O(1) = N_{(1)}\left(r, \frac{1}{f}\right) + S(r, f).$$

If  $\varphi$  is a non-zero polynomial and  $\deg(u' + uv') \geq \deg(nu)$ , by Lemma 2.4, we have

$$f(z) = \gamma_0(z) + p(z)e^{q(z)}, \quad (3.6)$$

where  $\gamma_0(z), p(z), q(z)$  are non-zero polynomials. Substituting (3.6) into (1.4), and using Lemma 2.9, we get  $p^n(z) = u(z)$ ,  $nq(z) = v(z)$ .

This completes the proof of Theorem 1.3. □

#### 4. Proof of Theorem 1.5

*Proof.* Suppose that  $f$  is a transcendental meromorphic solution of finite order of Eq (1.6) and  $N(r, f) = S(r, f)$ , by Lemma 2.6, we get  $\sigma(f) = 1$ . Denote  $P_d(z, f) = P$ . Differentiating (1.6) gives

$$nf^{n-1}f' + P' = \lambda(p_1e^{\lambda z} - p_2e^{-\lambda z}). \quad (4.1)$$

Differentiating (4.1) yields

$$n(n-1)f^{n-2}(f')^2 + nf^{n-1}f'' + P'' = \lambda^2(p_1e^{\lambda z} + p_2e^{-\lambda z}). \quad (4.2)$$

Combining (1.6) and (4.1), we get

$$\lambda^2 f^{2n} - n^2 f^{2(n-1)} (f')^2 + 2\lambda^2 P f^n - 2n P' f^{n-1} f' + \lambda^2 P^2 - (P')^2 - 4\lambda^2 p_1 p_2 = 0. \quad (4.3)$$

Combining (1.6) and (4.2), we get

$$\lambda^2 f^n - n(n-1) f^{n-2} (f')^2 - n f^{n-1} f'' + \lambda^2 P - P'' = 0. \quad (4.4)$$

Eliminating  $(f')^2$  from (4.3) and (4.4), we have

$$f^{2n-1} \varphi = Q(z, f), \quad (4.5)$$

where

$$\varphi = \lambda^2 f - n^2 f'' \quad (4.6)$$

and

$$Q(z, f) = [(n-2)\lambda^2 P + nP''] f^n - 2n(n-1)P' f^{n-1} f' + (n-1)[\lambda^2 P^2 - (P')^2 - 4\lambda^2 p_1 p_2], \quad (4.7)$$

$Q(z, f)$  is a differential-difference polynomial in  $f$  of degree  $2n-1$ . By Lemma 2.3, we have  $m(r, \varphi) = S(r, f)$ . Noting that  $N(r, f) = S(r, f)$ , then

$$T(r, \varphi) = S(r, f). \quad (4.8)$$

We distinguish two cases below:

**Case 1.** If  $\varphi \equiv 0$ , that is

$$\lambda^2 f - n^2 f'' = 0, \quad (4.9)$$

which has two fundamental solutions  $f_1(z) = e^{\frac{\lambda}{n}z}$ ,  $f_2(z) = e^{-\frac{\lambda}{n}z}$ . Then the general solution of Eq (4.9) can be expressed as

$$f(z) = c_1 e^{\frac{\lambda}{n}z} + c_2 e^{-\frac{\lambda}{n}z}.$$

Substituting the above formula into (1.6), and applying Lemma 2.9, we obtain  $c_j^n = p_j$ ,  $j = 1, 2$ .

**Case 2.** If  $\varphi \not\equiv 0$ , and  $z_0$  is a multiple zero of  $f$  whose multiplicities are not less than 3, it follows from (4.6) that  $z_0$  is a zero of  $\varphi$ . Then

$$N_{(3)}\left(r, \frac{1}{f}\right) = S(r, f). \quad (4.10)$$

Rewriting (4.6) as  $\frac{1}{f} = \frac{1}{\varphi} (\lambda^2 - n^2 \frac{f''}{f})$ , by the lemma of the logarithmic derivative and (4.8), we get

$$m\left(r, \frac{1}{f}\right) \leq m\left(r, \frac{1}{\varphi}\right) + m(r, \lambda^2) + m\left(r, n^2 \frac{f''}{f}\right) + O(1) = S(r, f). \quad (4.11)$$

It follows from (4.10) and (4.11) that

$$T(r, f) = m\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f}\right) + O(1) = N_2\left(r, \frac{1}{f}\right) + S(r, f).$$

If  $\varphi$  is a non-zero polynomial, by Lemma 2.5, we have

$$f(z) = \gamma_0(z) + c_1 e^{\frac{\lambda}{n}z} + c_2 e^{-\frac{\lambda}{n}z}, \quad (4.12)$$

where  $\gamma_0(z)$  is a non-zero polynomial,  $c_j$  ( $j = 1, 2$ ) are constants. Substituting (4.12) into (1.6), and using Lemma 2.9, we get  $c_j^n = p_j$ ,  $j = 1, 2$ .

This completes the proof of Theorem 1.5.

□

## 5. Proof of Theorem 1.7

*Proof.* Suppose that  $f$  is a transcendental meromorphic solution of finite order of Eq (1.9) and  $N(r, f) = S(r, f)$ , by Lemma 2.6, we get  $\sigma(f) = 1$ . Denote  $P_d(z, f) = P$ . Differentiating (1.9) gives

$$nf^{n-1}f' + P' = \alpha_1 p_1 e^{\alpha_1 z} + \alpha_2 p_2 e^{\alpha_2 z}. \quad (5.1)$$

Eliminating  $e^{\alpha_1 z}$  and  $e^{\alpha_2 z}$  from (1.4) and (4.1), respectively, we have

$$\alpha_1 f^n - nf^{n-1}f' + \alpha_1 P - P' = (\alpha_1 - \alpha_2) p_2 e^{\alpha_2 z}, \quad (5.2)$$

$$\alpha_2 f^n - nf^{n-1}f' + \alpha_2 P - P' = (\alpha_2 - \alpha_1) p_1 e^{\alpha_1 z}. \quad (5.3)$$

Differentiating (5.3) yields

$$n\alpha_2 f^{n-1}f' - n(n-1)f^{n-2}(f')^2 - nf^{n-1}f'' + \alpha_2 P' - P'' = \alpha_1(\alpha_2 - \alpha_1) p_1 e^{\alpha_1 z}. \quad (5.4)$$

Eliminating  $e^{\alpha_1 z}$  from (5.3) and (5.4), we have

$$\alpha_1 \alpha_2 f^n - n(\alpha_1 + \alpha_2) f^{n-1} f' + n(n-1) f^{n-2} (f')^2 + n f^{n-1} f'' = -Q(z, f), \quad (5.5)$$

where

$$Q(z, f) = \alpha_1 \alpha_2 P - (\alpha_1 + \alpha_2) P' + P'' \quad (5.6)$$

is a differential-difference polynomial in  $f$  of degree  $n-1$ . It follows from (5.2) and (5.3) that

$$\alpha_1 \alpha_2 f^{2n} - n(\alpha_1 + \alpha_2) f^{2n-1} f' + n^2 f^{2n-2} (f')^2 = -(\alpha_1 - \alpha_2)^2 p_1 p_2 e^{(\alpha_1 + \alpha_2)z} - R(z, f), \quad (5.7)$$

where

$$R(z, f) = (\alpha_1 f^n - nf^{n-1}f')(\alpha_2 P - P') + (\alpha_2 f^n - nf^{n-1}f')(\alpha_1 P - P') + (\alpha_1 P - P')(\alpha_2 P - P') \quad (5.8)$$

is a differential-difference polynomial in  $f$  of degree  $2n-1$ . Eliminating  $(f')^2$  from (5.5) and (5.7), we have

$$f^{2n-1} \varphi = (n-1)(\alpha_1 - \alpha_2)^2 p_1 p_2 e^{(\alpha_1 + \alpha_2)z} + T(z, f), \quad (5.9)$$

where

$$\varphi = \alpha_1 \alpha_2 f - n(\alpha_1 + \alpha_2) f' + n^2 f'', \quad (5.10)$$

and

$$T(z, f) = (n-1)R(z, f) - n f^n Q(z, f) \quad (5.11)$$

is a differential-difference polynomial in  $f$  of degree  $2n-1$ . It follows from (1.9), (2.6) and  $N(r, f) = S(r, f)$  that

$$\begin{aligned} O(T(r, e^z)) &\geq T(r, p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}) = T(r, f^n + P_d(z, f)) \\ &= m(r, f^n + P_d(z, f)) + S(r, f) \\ &\geq m(r, f^n) - m(r, P_d(z, f)) + S(r, f) \\ &\geq (n-d)T(r, f) + S(r, f). \end{aligned} \quad (5.12)$$

It follows from (2.4), (2.5) and (5.2) that

$$\begin{aligned} T(r, e^{\alpha_2 z}) &= m(r, e^{\alpha_2 z}) \\ &\leq m\left(r, \left(\left(\alpha_1 - n \frac{f'}{f}\right) f^n + \sum_{k=0}^d (\alpha_1 b_k - c_k) f^k\right)\right) + O(1) \\ &\leq nT(r, f) + S(r, f). \end{aligned} \quad (5.13)$$

Combining (5.12) and (5.13), we have

$$S(r, f) = S(r, e^z). \quad (5.14)$$

Substituting (2.4) into (1.9) yields

$$\frac{1}{p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z} - b_0} + \sum_{k=1}^d \frac{b_k}{p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z} - b_0} \left(\frac{1}{f}\right)^{n-k} = \left(\frac{1}{f}\right)^n. \quad (5.15)$$

Without loss of generality, we assume that  $b_0 = P_d(z, 0) \neq 0$ . Otherwise, we make the transformation  $\hat{f} = f - c$  for a suitable constant  $c$  satisfying  $c^n + P_d(z, c) \neq 0$ . Then (1.9) is changed to the form  $(\hat{f})^n(z) + P_d^*(z, \hat{f}) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}$ , where  $P_d^*(z, \hat{f})$  is a difference polynomial in  $\hat{f}$  of degree at most  $n - 1$  with polynomial coefficients, and  $P_d^*(z, 0) = c^n + P_d(z, c) \neq 0$ . Noting that  $b_0 = P_d(z, 0) \neq 0$ , by applying Lemma 2.10, it follows from (1.7) that

$$\begin{aligned} m\left(r, \frac{1}{p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z} - b_0}\right) &= S(r, e^z), \\ m\left(r, \frac{e^{\alpha_j z}}{p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z} - b_0}\right) &= S(r, e^z), \quad j = 1, 2. \end{aligned} \quad (5.16)$$

It follows from (5.14) – (5.16) that

$$\begin{aligned} nm\left(r, \frac{1}{f}\right) &\leq (n - 1)m\left(r, \frac{1}{f}\right) + S(r, f), \\ m\left(r, \frac{e^{\alpha_j z}}{f^n}\right) &\leq (n - 1)m\left(r, \frac{1}{f}\right) + S(r, f), \quad j = 1, 2, \end{aligned}$$

then

$$m\left(r, \frac{1}{f}\right) = S(r, f), \quad m\left(r, \frac{e^{\alpha_j z}}{f^n}\right) = S(r, f), \quad j = 1, 2. \quad (5.17)$$

Without loss of generality, if  $\frac{\alpha_1}{\alpha_2} = \frac{t}{n}$ , for some  $t \in \{1, 2, \dots, n - 1\}$ , then

$$\left| \frac{e^{(\alpha_1 + \alpha_2)z}}{f^{2n-1}} \right| = \frac{|e^{\alpha_2 z}|^{\frac{n+t}{n}}}{|f|^{n \frac{n+t}{n} + n-1-t}} = \left| \frac{e^{\alpha_2 z}}{f^n} \right|^{\frac{n+t}{n}} \frac{1}{|f|^{n-1-t}}.$$

From (5.17) and the above equality, we have

$$m\left(r, \frac{e^{(\alpha_1 + \alpha_2)z}}{f^{2n-1}}\right) = S(r, f). \quad (5.18)$$

By applying Lemma 2.1 and the lemma of the logarithmic derivative, from (5.9), (5.17) and (5.18), we get  $m(r, \varphi) = S(r, f)$ . Noting that  $N(r, f) = S(r, f)$ , then  $T(r, \varphi) = S(r, f)$ .

We consider two cases as follows.

**Case 1.** If  $\varphi \equiv 0$ , that is  $\alpha_1\alpha_2f - n(\alpha_1 + \alpha_2)f' + n^2f'' \equiv 0$ . The corresponding characteristic equation  $\alpha_1\alpha_2 - n(\alpha_1 + \alpha_2)\lambda + n^2\lambda^2 = 0$  has two roots  $\frac{\alpha_1}{n}, \frac{\alpha_2}{n}$ , we get

$$f(z) = c_1e^{\frac{\alpha_1}{n}z} + c_2e^{\frac{\alpha_2}{n}z}, \quad (5.19)$$

where  $c_j$  ( $j = 1, 2$ ) are constants. Noting that  $f(z)$  is a transcendental meromorphic solution of equation (1.9) and satisfies  $m(r, 1/f) = S(r, f)$ , then  $c_1c_2 \neq 0$ . Substituting (5.19) into (1.9) yields

$$(c_1^n - p_1)e^{\alpha_1z} + (c_2^n - p_2)e^{\alpha_2z} + \sum_{j=1}^{n-1} \binom{j}{n} c_1^j c_2^{n-j} e^{\left(\frac{j}{n}\alpha_1 + \frac{n-j}{n}\alpha_2\right)z} + P_d(z, f) = 0.$$

From (1.7) and Lemma 2.9, at least one of  $\binom{j}{n} c_1^j c_2^{n-j} e^{\left(\frac{j}{n}\alpha_1 + \frac{n-j}{n}\alpha_2\right)z}$  ( $j = 1, \dots, n-1$ ) is zero, then  $c_1c_2 = 0$ , which is impossible.

**Case 2.** If  $\varphi \neq 0$ , and  $z_0$  is a multiple zero of  $f$  whose multiplicities are not less than 3, it follows from (5.10) that  $z_0$  is a zero of  $\varphi$ . Then

$$N_{(3)}\left(r, \frac{1}{f}\right) = S(r, f). \quad (5.20)$$

It follows from (5.17) and (5.20) that

$$T(r, f) = m\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f}\right) + O(1) = N_{(2)}\left(r, \frac{1}{f}\right) + S(r, f).$$

If  $\varphi$  is a non-zero polynomial, by applying Lemma 2.6, we have

$$f(z) = \gamma_0(z) + c_1e^{\frac{\alpha_1}{n}z} + c_2e^{\frac{\alpha_2}{n}z}, \quad (5.21)$$

where  $\gamma_0(z)$  is a non-zero polynomial,  $c_j$  ( $j = 1, 2$ ) are constants. Noting that  $f(z)$  is a transcendental meromorphic solution of Eq (1.9) and satisfies  $m(r, 1/f) = S(r, f)$ , then  $c_1, c_2$  are not complete zeroes. If  $c_1c_2 \neq 0$ , substituting (5.21) into (1.9) yields

$$(c_1^n - p_1)e^{\alpha_1z} + (c_2^n - p_2)e^{\alpha_2z} + \sum_{j=1}^{n-1} \binom{j}{n} c_1^j c_2^{n-j} e^{\left(\frac{j}{n}\alpha_1 + \frac{n-j}{n}\alpha_2\right)z} + \sum_{j=1}^n \binom{j}{n} \gamma_0^j(z) (c_1e^{\frac{\alpha_1}{n}z} + c_2e^{\frac{\alpha_2}{n}z})^{n-j} + P_d(z, f) = 0.$$

From (1.7) and Lemma 2.9, at least one of  $\binom{j}{n} c_1^j c_2^{n-j} e^{\left(\frac{j}{n}\alpha_1 + \frac{n-j}{n}\alpha_2\right)z}$  ( $j = 1, \dots, n-1$ ) is zero, then  $c_1c_2 = 0$ , which is impossible. Then we get  $f(z) = \gamma_0(z) + c_1e^{\frac{\alpha_1}{n}z}$  or  $f(z) = \gamma_0(z) + c_2e^{\frac{\alpha_2}{n}z}$ .

This completes the proof of Theorem 1.7. □

## 6. Conclusions

Using the Nevalinna theory of meromorphic functions, this paper study the meromorphic solutions of three Clunie-Tumura types of non-linear difference equations and get the exact forms of the meromorphic solutions of these difference equations with some added conditions. Improvements and extensions of some results in the literature are presented.

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## Conflict of interest

The authors declare no conflict of interest.

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