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ENTIRE SOLUTIONS OF DELAY DIFFERENTIAL EQUATIONS OF MALMQUIST TYPE*

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Abstract The celebrated Malmquist theorem states that a differential equation, which admits a transcendental meromorphic solution, reduces into a Riccati differential equation. Motivated by the integrability of difference equations, this paper investigates the delay differential equations of form $w(z+1) - w(z-1) + a(z)\frac{w'(z)}{w(z)} = R(z, w(z))(*)$, where $R(z, w(z))$ is an irreducible rational function in $w(z)$ with rational coefficients and $a(z)$ is a rational function. We characterize all reduced forms when the equation $(*)$ admits a transcendental entire solution with hyper-order less than one. When we compare with the results obtained by Halburd and Korhonen[Proc. Amer. Math. Soc. 145, no.6 (2017)], we obtain the reduced forms without the assumptions that the denominator of rational function $R(z, w(z))$ has roots that are nonzero rational functions in z. The value distribution and forms of transcendental entire solutions for the reduced delay differential equations are studied. The existence of finite iterated order entire solutions of the Kac-van Moerbeke delay differential equation is also detected.

Keywords Delay differential equation, entire solution, Nevanlinna theory.

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1. Introduction

We assume that the reader is familiar with the standard notations and basic results of the Nevanlinna theory, see e.g. [12]. Let w be a meromorphic function in the complex plane. The order of growth of w is denoted by $\sigma(w)$ and the hyper-order of w is defined by

$$\sigma_2(w) = \limsup_{r\to\infty} \frac{\log\log T(r, w)}{\log r}.$$

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For $a \in \mathbb{C}$, the deficiency in which zeros of $w - a$ are counted only once is defined by

$$\Theta(a, w) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, \frac{1}{w-a})}{T(r, w)}.$$

Moreover, we say that a meromorphic function α is a small function of w if $T(r, \alpha) = S(r, w)$, where $S(r, w) = o(T(r, w))$ as $r \to \infty$, possibly outside of an exceptional set of finite logarithmic measure.

The Malmquist type theorems concentrate upon necessary conditions for certain types of differential equations to admit a meromorphic solution growing rapidly with respect to the coefficients. The following result is the celebrated Malmquist theorem.

Theorem 1.1 ([16, 18]). *Let $R(z, y)$ be rational in both arguments. If the differential equation*

$$y' = R(z, y)$$

admits a transcendental meromorphic solution, then $y' = R(z, y)$ reduces into a Riccati differential equation

$$y' = a_0(z) + a_1(z)y + a_2(z)y^2$$

with rational coefficients.

N. Steinmetz [21] generalized the Malmquist-Yosida theorem into the birational cases of $(y^{'})^n = R(z, y)$, which actually admit transcendental meromorphic solutions, and then reduce into $(y^{'})^n = \sum_{i=0}^{2n} \alpha_i(z)y^i$, where at least one of the coefficients $\alpha_i(z)$ does not vanish.

Motivated by the integrability of difference equations, Halburd and Korhonen [9] obtained the following result, which indicates that the existence of a finite-order meromorphic solution of a difference equation is a strong indicator of integrability of the equation.

Theorem 1.2 ([9, Theorem 1.1]). *If the equation*

$$w(z + 1) + w(z - 1) = R(z, w(z)) \tag{1.1}$$

where $R(z, w(z))$ is rational in $w(z)$ with meromorphic coefficients in z, has an admissible meromorphic solution of finite order, then either $w(z)$ satisfies a difference Riccati equation

$$w(z + 1) = \frac{p(z + 1)w(z) + q(z)}{w(z) + p(z)},$$

where $p, q \in \mathcal{S}(w) = \{f \text{ meromorphic} : T(r, f) = o(T(r, w))\}$, or equation (1.1) can be transformed by a linear change in $w(z)$ to one of the following equations:

$$w(z + 1) + w(z) + w(z - 1) = \frac{\pi_1(z)z + \pi_2(z)}{w(z)} + \kappa_1(z),$$

$$w(z + 1) - w(z) + w(z - 1) = \frac{\pi_1(z)z + \pi_2(z)}{w(z)} + (-1)^z\kappa_1(z),$$

$$w(z+1)+w(z-1)=\frac{\pi_1(z)z+\pi_3(z)}{w(z)}+\pi_2(z),$$
$$w(z+1)+w(z-1)=\frac{\pi_1(z)z+\kappa_1(z)}{w(z)}+\frac{\pi_2(z)}{w(z)^2},$$
$$w(z+1)+w(z-1)=\frac{(\pi_1(z)z+\kappa_1(z))w(z)+\pi_2(z)}{(-1)^{-z}-w(z)^2},$$
$$w(z+1)+w(z-1)=\frac{(\pi_1(z)z+\kappa_1(z))w(z)+\pi_2(z)}{1-w(z)^2},$$
$$w(z+1)w(z)+w(z)w(z-1)=p(z),$$
$$w(z+1)+w(z-1)=p(z)w(z)+q(z),$$

where $\pi_k(z), \kappa_k(z) \in \mathcal{S}(w)$ *are arbitrary finite-order periodic functions with period* k*.*

Theorem 1.2 is a Malmquist type theorem for difference equations. Furthermore, many other researchers (see, e.g. [1,14,15,20,26,27]) discussed the complex difference equations of Malmquist type, and mainly presented the value distribution of their meromorphic solutions.

Some reductions of integrable differential-difference equations are known to yield delay differential equations with formal continuum limits to differential Painlevé equations. Painlevé-type delay differential equations were also considered in Grammaticos, Ramani and Moreira [6] from the point of view of a kind of singularity confinement. Viallet [23] has introduced a notion of algebraic entropy for such equations. Halburd and Korhonen [11] discussed a delay differential equation and obtained

Theorem 1.3 ([11, Theorem 1.1])**.** *Let* $w(z)$ *be a non-rational meromorphic solution of*

$$w(z+1)-w(z-1)+a(z)\frac{w'(z)}{w(z)}=R(z,w(z))=\frac{P(z,w(z))}{Q(z,w(z))}, \tag{1.2}$$

where $a(z)$ is rational, $P(z,w)$ is a polynomial in w having rational coefficients in z, and $Q(z,w)$ is a polynomial in $w(z)$ with roots that are nonzero rational functions of z and not roots of $P(z,w)$. If $\sigma_2(w)<1$, then

$$\deg_w(P)=\deg_w(Q)+1\le 3 \ \mathit{or}\ \deg_w(R)\le 1.$$

The notation $\deg_w(P)=\deg_w(P(z,w))$ is used to denote the degree of P as a polynomial in w and $\deg_w(R)=\max\{\deg_w(P),\deg_w(Q)\}$ is used to denote the degree of R as a rational function in w.

In Theorem 1.3, Halburd and Korhonen obtained necessary conditions for the equation (1.2) to admit a non-rational meromorphic solution of hyper-order less than one, under the assumption that "$Q(z,w)$ has roots that are nonzero rational functions of z". Here, we pose two questions related to Theorem 1.3.

Question 1. *Is it possible to obtain some reduction results for the equation (1.2) if the assumption that "*$Q(z,w)$ *has roots that are nonzero rational functions of* z*" of Theorem 1.3 is dropped?*

Question 2. *Is it possible to say something about the properties, including the growth order , the distribution of a-values and the existence of solutions of reduced forms of the equation (1.2)?*

In this paper, we first answer the above Question 1 when the equation (1.2) has a transcendental entire solution. We use different technique, which is different from the method and assumption used by Halburd and Korhonen [11], to get the all reduced forms of certain delay differential equation (1.2). We second focus on the above Question 2. We present the value distribution and forms of transcendental entire solutions for the reduced delay differential equations, and prove that the Kac-van Moerbeke delay differential equation, which is a special reduced form of equation (1.2), has no finite iterated order entire solutions.

The remainders of the paper are organized as follows. In Section 2, we mainly focus our interesting on Question 1 and show how to detect the reduced forms of delay differential equation (1.2). In Section 3, the value distribution and forms of transcendental entire solutions for the reduced delay differential equations have been investigated. The existence of finite iterated order entire solutions of the Kac-van Moerbeke delay differential equation is verified in Section 4. Some examples are listed to show our results occur indeed in Section 5.

2. Reduced forms of delay differential equation

In this section, we mainly try to answer the above Question 1, and characterize all reduced forms when delay differential equation (1.2) admits a transcendental entire solution with hyper-order less than one. We drop the assumption that "$Q(z,w)$ has roots that are nonzero rational functions of z" in Theorem 1.3, and obtain the following results.

Theorem 2.1. *Let $R(z,w(z)) \not\equiv 0$ be an irreducible rational function in $w(z)$ with rational coefficients and let $a(z)$ be a rational function. If the equation (1.2) admits a transcendental entire solution $w(z)$ with $\sigma_2(w) < 1$, then (1.2) reduces into*

$$w(z+1) - w(z-1) + a(z)\frac{w'(z)}{w(z)} = a_1(z)w(z) + a_0(z), \tag{2.1}$$

or

$$w(z+1) - w(z-1) + a(z)\frac{w'(z)}{w(z)} = \frac{a_2(z)w(z)^2 + a_1(z)w(z) + a_0(z)}{w(z)}, \tag{2.2}$$

where $a_j(z), j = 0, 1, 2$ are rational functions.

Remark 2.1. Theorem 2.1 is a reduction result which characterizes all cases actually may appear, in which the equation (1.2) has transcendental entire solutions with hyper-order less than one. So Theorem 2.1 can be viewed as a weaker form of delay differential analogue of Malmquist theorem.

By Theorem 2.1, we easily get the following corollary.

Corollary 2.1. *Let $a(z)$ be a rational function, $P(z,w)$ be a polynomial in w having rational coefficients in z, and $Q(z,w)$ be a polynomial in $w(z)$ with roots that are nonzero rational functions of z and not roots of $P(z,w)$. Then the equation (1.2) has no transcendental entire solutions with $\sigma_2(w) < 1$.*

Remark 2.2. Theorem 1.3 has an assumption "$Q(z,w)$ has roots that are nonzero rational functions of z", under which the equation (1.2) has no transcendental entire solutions with $\sigma_2(w) < 1$. So Theorem 2.1 is independent of Theorem 1.3, though

Theorem 1.3 focuses on the case that (1.2) has a non-rational meromorphic solution with $\sigma_2(w) < 1$ and Theorem 2.1 focuses on the case that (1.2) has a transcendental entire solution with $\sigma_2(w) < 1$.

We now give some lemmas to prove Theorem 2.1. The first of these lemmas is a version of the difference analogue of the logarithmic derivative lemma.

Lemma 2.1 ([10, Theorem 5.1])**.** *Let $f(z)$ be a nonconstant meromorphic function and $c \in \mathbb{C}$. If $\sigma_2(f) < 1$ and $\varepsilon > 0$, then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r,f)}{r^{1-\sigma_2(f)-\varepsilon}}\right)$$

for all r outside of a set of finite logarithmic measure.

Applying logarithmic derivative lemma and Lemma 2.1 to Theorem 2.3 in [17], we get the following lemma, which is a version of the difference analogue of the Clunie lemma.

Lemma 2.2. *Let $f(z)$ be a transcendental meromorphic solution of hyper order $\sigma_2(f) < 1$ of a differential-difference equation of the form*

$$U(z,f)P(z,f) = Q(z,f),$$

where $U(z,f)$ is a difference polynomial in $f(z)$ with small meromorphic coefficients, $P(z,f)$, $Q(z,f)$ are differential-difference polynomials in $f(z)$ with small meromorphic coefficients, $\deg_f(U) = n$ and $\deg_f(Q) \leq n$. Moreover, we assume that $U(z,f)$ contains just one term of maximal total degree in $f(z)$ and its shifts. Then

$$m\big(r, P(z,f)\big) = S(r,f).$$

The following lemma is a generalisation of Borel's theorem on linear combinations of entire functions.

Lemma 2.3 ([7, pp.69–70] or [25, p.82])**.** *Suppose that $f_1(z), f_2(z), \cdots, f_n(z)$ are meromorphic functions and that $g_1(z), g_2(z), \cdots, g_n(z)$ are entire functions satisfying the following conditions.*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;

(ii) $g_j(z) - g_k(z)$ *are not constants for* $1 \leq j < k \leq n$;

(iii) *for* $1 \leq j \leq n, 1 \leq h < k \leq n$,

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\} \quad (r \to \infty,\ r \notin E),$$

where $E \subset (1, \infty)$ is of finite linear measure or finite logarithmic measure.

Then $f_j(z) \equiv 0$ $(j = 1, 2, \cdots, n)$.

The following Lemma 2.4, due to Valiron and Mohon'ko, is of essential importance in the theory of complex differential, difference and differential-difference equations.

Lemma 2.4 ([16, p.29]). *Let f be a meromorphic function. Then for all irreducible rational functions in f*

$$R(z,f)=\frac{\sum_{i=0}^{p}a_i(z)f^i}{\sum_{j=0}^{q}b_j(z)f^j}$$

with meromorphic coefficients $a_i(z)$, $b_j(z)$ such that

$$\begin{cases} T(r,a_i)=S(r,f), & i=0,\cdots,p\\ T(r,b_j)=S(r,f), & j=0,\cdots,q,\end{cases}$$

the characteristic function of $R(z,f)$ satisfies

$$T(r,R(z,f))=\max\{p,q\}T(r,f)+S(r,f).$$

Next we prove the following lemma related to the reduced form (2.1).

Lemma 2.5. *Let $R(z,w(z))\not\equiv 0$ be an irreducible rational function in $w(z)$ with rational coefficients, let $a(z)\not\equiv 0$ be a rational function and let $w(z)$ be a transcendental entire solution of the equation (1.2). If $\sigma_2(w)<1$ and $w(z)$ has finitely many zeros, then (1.2) is of the form (2.1), where $a_1(z),a_0(z)$ are rational functions with $a_1(z)\not\equiv 0$ or $a_0(z)\not\equiv 0$.*

Proof. By applying Hadamard factorization theorem, we see that $w(z)$ takes the form

$$w(z)=H(z)e^{g(z)}, \tag{2.3}$$

where $H(z)$ is a non-zero polynomial, $g(z)$ is a non-constant entire function such that $\sigma_2(w(z))=\sigma_2(e^{g(z)})=\sigma(g(z))<1$. Substituting (2.3) into the equation (1.2) and setting

$$s(z)=H(z+1)e^{g(z+1)-g(z)}-H(z-1)e^{g(z-1)-g(z)},$$

we get

$$s(z)e^{g(z)}+a(z)\left(\frac{H'(z)}{H(z)}+g'(z)\right)=\frac{P(z,w(z))}{Q(z,w(z))}. \tag{2.4}$$

If $s(z)\equiv 0$, then by (2.4), we obtain

$$T\left(r,\frac{P(z,w(z))}{Q(z,w(z))}\right)=S(r,e^g)=S(r,w).$$

By Lemma 2.4, we have $\deg_w(Q)=\deg_w(P)=0$. Thus, the equation (1.2) is of the form (2.1), where $a_0(z)\not\equiv 0$ is a rational function and $a_1(z)\equiv 0$.

If $s(z)\not\equiv 0$, then we deduce from $\sigma_2(e^{g(z)})<1$ and Lemma 2.1 that

$$T(r,s(z))=m(r,s(z))\le m\left(r,\frac{e^{g(z+1)}}{e^{g(z)}}\right)+m\left(r,\frac{e^{g(z-1)}}{e^{g(z)}}\right)+O(\log r)=S(r,e^g).$$

So we get from (2.4) that

$$T\left(r,\frac{P(z,w(z))}{Q(z,w(z))}\right)\le T(r,e^{g(z)})+S(r,e^g)=T(r,w(z))+S(r,w).$$

The above inequality and Lemma 2.4 show that $\max\{\deg_w(P), \deg_w(Q)\} \leq 1$. Thus, (2.4) is of the form

$$s(z)e^{g(z)} + a(z)\left(\frac{H'(z)}{H(z)} + g'(z)\right) = \frac{\widetilde{a_1}(z)H(z)e^{g(z)} + \widetilde{a_0}(z)}{\widetilde{b_1}(z)H(z)e^{g(z)} + \widetilde{b_0}(z)}, \tag{2.5}$$

where $\widetilde{a_1}(z), \widetilde{a_0}(z), \widetilde{b_1}(z), \widetilde{b_0}(z)$ are rational functions.

It follows from (2.5) that

$$\begin{aligned}
&s(z)\widetilde{b_1}(z)H(z)e^{2g(z)}\\
&+\left(\widetilde{b_0}(z)s(z) + \widetilde{b_1}(z)H(z)a(z)\left(\frac{H'(z)}{H(z)} + g'(z)\right) - \widetilde{a_1}(z)H(z)\right)e^{g(z)}\\
&+\widetilde{b_0}(z)a(z)\left(\frac{H'(z)}{H(z)} + g'(z))\right) - \widetilde{a_0}(z) = 0.
\end{aligned}$$

By this equality and Lemma 2.3, we obtain $\widetilde{b_1}(z) \equiv 0$, and then we deduce from (2.5) that the equation (1.2) is of the form (2.1), where $a_1(z) \not\equiv 0$ and $a_0(z)$ are rational functions. □

Finally, we consider the case where the equation (1.2) reduces into (2.2), and get the following Lemma.

Lemma 2.6. *Let $R(z, w(z)) \not\equiv 0$ be an irreducible rational function in $w(z)$ with rational coefficients, let $a(z) \not\equiv 0$ be a rational function and let $w(z)$ be a transcendental entire solution of the equation (1.2). If $\sigma_2(w) < 1$ and there exists a rational function $r(z) \not\equiv 0$ such that $w(z) + r(z)$ has finitely many zeros, then (1.2) is of the form (2.2), where $a_2(z), a_1(z), a_0(z)$ are rational functions with $a_0(z) \not\equiv 0$.*

Proof. It follows from Hadamard factorization theorem that $w(z)$ takes the form

$$w(z) = H(z)e^{g(z)} - r(z),$$

where $H(z)$ is a non-zero polynomial, $g(z)$ is a non-constant entire function such that $\sigma_2(w(z)) = \sigma_2(e^{g(z)}) = \sigma(g(z)) < 1$. Setting

$$s(z) = H(z+1)e^{g(z+1)-g(z)} - H(z-1)e^{g(z-1)-g(z)},$$

we have

$$w(z+1) - w(z-1) = s(z)\frac{w(z) + r(z)}{H(z)} - r(z+1) + r(z-1), \tag{2.6}$$

$$w'(z) = \left(\frac{H'(z)}{H(z)} + g'(z)\right)w(z) + r(z)\left(\frac{H'(z)}{H(z)} + g'(z) - \frac{r'(z)}{r(z)}\right). \tag{2.7}$$

We deduce from (1.2) and Lemmas 2.1 and 2.4 that

$$\begin{aligned}
&\max\{\deg_w(P), \deg_w(Q)\}T(r, w(z)) + O(\log r)\\
=&T\left(r, \frac{P(z, w(z))}{Q(z, w(z))}\right)\\
=&m\left(r, w(z+1) - w(z-1) + a(z)\frac{w'(z)}{w(z)}\right)
\end{aligned}$$

$$
\begin{aligned}
&+N\left(r, w(z+1)-w(z-1)+a(z)\frac{w'(z)}{w(z)}\right)\\
\leq& m(r,w(z))+m\left(r,\frac{w(z+1)}{w(z)}\right)+m\left(r,\frac{w(z-1)}{w(z)}\right)+m\left(r,\frac{w'(z)}{w(z)}\right)\\
&+N\left(r,\frac{1}{w(z)}\right)+S(r,w)\leq 2T(r,w(z))+S(r,w),
\end{aligned}
$$

which gives $\max\{\deg_w(P),\deg_w(Q)\}\leq 2$. Thus, substituting (2.6) and (2.7) into (1.2), we conclude

$$
\begin{aligned}
&\frac{\frac{s(z)}{H(z)}w(z)^2+t(z)w(z)+a(z)r(z)\left(\frac{H'(z)}{H(z)}+g'(z)-\frac{r'(z)}{r(z)}\right)}{w(z)}\\
=&\frac{\widetilde{a_2}(z)w(z)^2+\widetilde{a_1}(z)w(z)+\widetilde{a_0}(z)}{\widetilde{b_2}(z)w(z)^2+\widetilde{b_1}(z)w(z)+\widetilde{b_0}(z)},
\end{aligned}
\tag{2.8}
$$

where

$$
t(z)=\frac{s(z)r(z)}{H(z)}-r(z+1)+r(z-1)+a(z)\left(\frac{H'(z)}{H(z)}+g'(z)\right),
$$

and $\widetilde{a_j}(z)$ and $\widetilde{b_j}(z)(j=0,1,2)$ are rational functions. We futher deduce from Lemma 2.1 that $T(r,s(z))=S(r,w)$ since $\sigma_2(e^{g(z)})<1$. Thus all coefficients in (2.8) are small functions relative to $w(z)$.

Since $H(z)$ is a polynomial, $r(z)$ is a rational function and $g(z)$ is a non-constant entire function, we deduce that $g'(z)\not\equiv\frac{r'(z)}{r(z)}-\frac{H'(z)}{H(z)}$, which gives

$$
a(z)r(z)\left(\frac{H'(z)}{H(z)}+g'(z)-\frac{r'(z)}{r(z)}\right)\not\equiv 0. \tag{2.9}
$$

If $s(z)\not\equiv 0$, then multiplying both sides of (2.8) by $w(z)(\widetilde{b_2}(z)w(z)^2+\widetilde{b_1}(z)w(z)+\widetilde{b_0}(z))$, we conclude

$$
\begin{aligned}
&\widetilde{b_2}(z)\frac{s(z)}{H(z)}w(z)^4+t_3(z)w(z)^3+t_2(z)w(z)^2+t_1(z)w(z)\\
&+\widetilde{b_0}(z)a(z)r(z)\left(\frac{H'(z)}{H(z)}+g'(z)-\frac{r'(z)}{r(z)}\right)=0,
\end{aligned}
\tag{2.10}
$$

where $t_j(z)(j=1,2,3)$ are all small functions relative to $w(z)$. By (2.9), (2.10) and Lemma 2.4, we have $\widetilde{b_2}(z)\equiv 0$ and $\widetilde{b_0}(z)\equiv 0$. So the equation (1.2) is of the form (2.2), where $a_j(z)(j=0,1,2)$ are rational functions. Recalling that $R(z,w(z))$ is irreducible, we get $a_0(z)\not\equiv 0$.

If $s(z)\equiv 0$, then by (2.8) and Lemma 2.4, we have $\max\{\deg_w(P),\deg_w(Q)\}=1$. This shows that (2.8) reduces into

$$
\frac{t(z)w(z)+a(z)r(z)\left(\frac{H'(z)}{H(z)}+g'(z)-\frac{r'(z)}{r(z)}\right)}{w(z)}=\frac{\widetilde{a_1}(z)w(z)+\widetilde{a_0}(z)}{\widetilde{b_1}(z)w(z)+\widetilde{b_0}(z)}. \tag{2.11}
$$

Using the same reasoning as above, we see from (2.11) that the equation (1.2) is of the form (2.2), where $a_2(z)\equiv 0$ and $a_j(z)(j=0,1)$ are rational functions with $a_0(z)\not\equiv 0$. $\square$

We now devote to a complete proof of Theorem 2.1.

Proof of Theorem 2.1. We firstly discuss the case $a(z) \equiv 0$. We deduce from (1.2) and Lemma 2.1 that

$$\begin{aligned} T\left(r, \frac{P(z,w(z))}{Q(z,w(z))}\right) &= T(r, w(z+1) - w(z-1)) = m(r, w(z+1) - w(z-1)) \\ &\leq m(r, w(z)) + m\left(r, \frac{w(z+1)}{w(z)}\right) + m\left(r, \frac{w(z-1)}{w(z)}\right) + O(1) \\ &= m(r, w(z)) + S(r, w). \end{aligned} \tag{2.12}$$

Lemma 2.4 and (2.12) yield that $\max\{\deg_w(P), \deg_w(Q)\} \leq 1$. Thus, the equation (1.2) has the form

$$w(z+1) - w(z-1) = \frac{\widetilde{a_1}(z)w(z) + \widetilde{a_0}(z)}{\widetilde{b_1}(z)w(z) + \widetilde{b_0}(z)}, \tag{2.13}$$

where $\widetilde{a_j}(z)$ and $\widetilde{b_j}(z)(j = 0, 1)$ are rational functions. We affirm that $\widetilde{b_1}(z) \equiv 0$. Otherwise, if $\widetilde{b_1}(z) \not\equiv 0$, we deduce from Lemma 2.4 and (2.13) that

$$T(r, w(z+1) - w(z-1)) = T\left(r, \frac{\widetilde{a_1}(z)w(z) + \widetilde{a_0}(z)}{\widetilde{b_1}(z)w(z) + \widetilde{b_0}(z)}\right) = T(r, w(z)) + S(r, w). \tag{2.14}$$

On the other hand, we get from (2.13) that

$$w(z)(w(z+1) - w(z-1)) = -\frac{\widetilde{b_0}(z)}{\widetilde{b_1}(z)}(w(z+1) - w(z-1)) + \frac{\widetilde{a_1}(z)}{\widetilde{b_1}(z)}w(z) + \frac{\widetilde{a_0}(z)}{\widetilde{b_1}(z)}. \tag{2.15}$$

Applying Lemma 2.2 to (2.15), we obtain

$$T(r, w(z+1) - w(z-1)) = m(r, w(z+1) - w(z-1)) = S(r, w).$$

This contradicts (2.14). So $\widetilde{b_1}(z) \equiv 0$ and (1.2) is of the form (2.1), where $a_j(z)(j = 0, 1)$ are rational functions.

We secondly discuss the case $a(z) \not\equiv 0$. We deduce from (1.2) and Lemmas 2.1 and 2.4 that

$$\begin{aligned} &\max\{\deg_w(P), \deg_w(Q)\}T(r, w(z)) + O(\log r) \\ =&T\left(r, \frac{P(z,w(z))}{Q(z,w(z))}\right) \\ =&T\left(r, w(z+1) - w(z-1) + a(z)\frac{w'(z)}{w(z)}\right) \\ \leq&2T(r, w(z)) + S(r, w). \end{aligned}$$

So $\max\{\deg_w(P), \deg_w(Q)\} \leq 2$, and

$$\frac{P(z,w(z))}{Q(z,w(z))} = \frac{\widetilde{a_2}(z)w(z)^2 + \widetilde{a_1}(z)w(z) + \widetilde{a_0}(z)}{\widetilde{b_2}(z)w(z)^2 + \widetilde{b_1}(z)w(z) + \widetilde{b_0}(z)}, \tag{2.16}$$

where $\widetilde{a_j}(z)$ and $\widetilde{b_j}(z)(j = 0, 1, 2)$ are rational functions.

If $w(z)$ has finitely many zeros, then Lemma 2.5 shows that the equation (1.2) is of the form (2.1). If there exists a rational function $r(z) \not\equiv 0$ such that $w(z) + r(z)$ has finitely many zeros, then lemma 2.6 shows that the equation (1.2) is of the form (2.2).

Thus, we assume that $w(z)$ has infinitely many zeros and $w(z) + r(z)$ also has infinitely many zeros for any rational function $r(z) \not\equiv 0$.

Suppose that z_0 is a zero of $w(z)$ and that neither $a(z)$ nor any of the coefficients in $\frac{P(z,w(z))}{Q(z,w(z))}$ has a zero or a pole at z_0. If $\widetilde{b_0}(z) \not\equiv 0$, then z_0 is a simple pole of $w(z+1) - w(z-1) + a(z)\frac{w'(z)}{w(z)}$ and a finite value of $\frac{P(z,w(z))}{Q(z,w(z))}$, a contradiction. So $\widetilde{b_0}(z) \equiv 0$.

If $\widetilde{b_2}(z) \not\equiv 0$ and $\widetilde{b_1}(z) \equiv 0$, then we deduce from (1.2) and (2.16) that

$$w(z+1) - w(z-1) + a(z)\frac{w'(z)}{w(z)} = \frac{\widetilde{a_2}(z)w(z)^2 + \widetilde{a_1}(z)w(z) + \widetilde{a_0}(z)}{\widetilde{b_2}(z)w(z)^2}. \tag{2.17}$$

Since the right hand side of (2.17) is irreducible in $w(z)$, we see that $\widetilde{a_0}(z) \not\equiv 0$. Choose a zero z_0 of $w(z)$ as above. Then we see that z_0 is a simple pole of the left hand side of (2.17) and a multiple pole of the right hand side of (2.17), a contradiction.

If $\widetilde{b_2}(z) \not\equiv 0$ and $\widetilde{b_1}(z) \not\equiv 0$, then we conclude from (1.2) and (2.16) that

$$w(z+1)w(z) - w(z-1)w(z) + a(z)w'(z) = \frac{\frac{\widetilde{a_2}(z)}{\widetilde{b_2}(z)}w(z)^2 + \frac{\widetilde{a_1}(z)}{\widetilde{b_2}(z)}w(z) + \frac{\widetilde{a_0}(z)}{\widetilde{b_2}(z)}}{w(z) + \frac{\widetilde{b_1}(z)}{\widetilde{b_2}(z)}}. \tag{2.18}$$

Since the right hand side of (2.18) is irreducible in $w(z)$, we see that $\frac{\widetilde{a_2}(z)}{\widetilde{b_2}(z)}w(z)^2 + \frac{\widetilde{a_1}(z)}{\widetilde{b_2}(z)}w(z) + \frac{\widetilde{a_0}(z)}{\widetilde{b_2}(z)}$ and $w(z) + \frac{\widetilde{b_1}(z)}{\widetilde{b_2}(z)}$ have at most finitely many common zeros. Furthermore, $w(z) + \frac{\widetilde{b_1}(z)}{\widetilde{b_2}(z)}$ has infinitely many zeros. So we can choose a zero z_1 of $w(z) + \frac{\widetilde{b_1}(z)}{\widetilde{b_2}(z)}$ such that neither $a(z)$ nor $\frac{\widetilde{a_2}(z)}{\widetilde{b_2}(z)}w(z)^2 + \frac{\widetilde{a_1}(z)}{\widetilde{b_2}(z)}w(z) + \frac{\widetilde{a_0}(z)}{\widetilde{b_2}(z)}$ has a zero or a pole at z_1. So z_1 is a pole of the right hand side of (2.18) and a finite value of the left hand side of (2.18), a contradiction.

From the above discussion, we see that $\widetilde{b_0}(z) \equiv 0$ and $\widetilde{b_2}(z) \equiv 0$. So (1.2) is of the form (2.2), where $a_j(z)(j = 0, 1, 2)$ are rational functions with $a_0(z) \not\equiv 0$. □

3. Value distribution and forms of entire solution for reduced delay differential equations

In this section, we emphasize the growth order and value distribution of transcendental entire solutions for the reduced forms (2.1) and (2.2). The forms of entire solutions for the reduced forms (2.1) and (2.2) are also presented. We first give our results as follows.

Theorem 3.1. *Let $a(z)$, $a_0(z)$ and $a_1(z)$ be rational functions with $a_1(z) \not\equiv 0$ or $a_0(z) \not\equiv 0$, and let $w(z)$ be a transcendental entire solution of the equation (2.1) with $\sigma_2(w) < 1$.*

(i) If $a(z) \equiv 0$, then $\sigma(w) \geq 1$.

(ii) If $a(z) \not\equiv 0$, then $w(z) = H(z)e^{dz}$, where $H(z) \not\equiv 0$ is a polynomial and $d \neq 0$ is some complex number. Especially, if $a_1(z)$ is a polynomial with $a_1(z) \not\equiv \pm 2i$, then $w(z) = Ce^{dz}$, where $C \in \mathbb{C}/\{0\}$; if $a_1(z) \equiv \pm 2i$, then $w(z) = (C_1 z + C_0)e^{(2k\pm\frac{1}{2})\pi i z}$, where k is an integer and $C_1, C_0 \in \mathbb{C}$ with $|C_1| + |C_0| \neq 0$.

Theorem 3.2. *Let $a(z)$, $a_2(z)$, $a_1(z)$ and $a_0(z) \not\equiv 0$ be rational functions, and let $w(z)$ be a transcendental entire solution of the equation (2.2) with $\sigma_2(w) < 1$. Then*

(i) $\sigma(w) \geq 1$;

(ii) $\Theta(b, w) = 0$ provided that $b \in \mathbb{C}$ and $a_2(z)b^2 + a_1(z)b + a_0(z) \not\equiv 0$.

It is trivial for us to get from Theorems 2.1, 3.1, 3.2 and Remark 2.1 that

Corollary 3.1. *Let $R(z, w(z)) \not\equiv 0$ be an irreducible rational function in $w(z)$ with rational coefficients, let $a(z)$ be a rational function, and let $w(z)$ be a transcendental entire solution of the equation (1.2) with $\sigma_2(w) < 1$. Then*

(i) $\sigma(w) \geq 1$;

(ii) If $\deg_w(Q) = 0$ and $a(z) \not\equiv 0$, then $w(z)$ has the form $w(z) = H(z)e^{dz}$, where $H(z) \not\equiv 0$ is a polynomial and $d \neq 0$ is some complex number;

(iii) If $\deg_w(Q) > 0$, then $\Theta(0, w) = 0$.

We now give some lemmas to prove Theorems 3.1 and 3.2. Lemmas 3.1 and 3.2 are concerned with the growth order of meromorphic solutions for linear difference equations.

Lemma 3.1 ([3, Theorem 3]). *Let $P_n(z), \cdots, P_0(z)$ be polynomials such that $P_n(z)P_0(z) \not\equiv 0$ and satisfy $P_n(z) + \cdots + P_0(z) \not\equiv 0$. Then every transcendental meromorphic solution $f(z)$ of the equation*

$$P_n(z)f(z+n) + P_{n-1}(z)f(z+n-1) + \cdots + P_0(z)f(z) = 0$$

satisfies $\sigma(f) \geq 1$.

Lemma 3.2 ([3, Theorem 4]). *Let $F(z), P_n(z), \cdots, P_0(z)$ be polynomials such that $F(z)P_n(z)P_0(z) \not\equiv 0$. Then every transcendental meromorphic solution $f(z)$ of the equation*

$$P_n(z)f(z+n) + P_{n-1}(z)f(z+n-1) + \cdots + P_0(z)f(z) = F(z)$$

satisfies $\sigma(f) \geq 1$.

The following lemma is another version of difference analogue of the logarithmic derivative lemma.

Lemma 3.3 ([4, Corollary 2.6]). *Let η_1, η_2 be two complex numbers such that $\eta_1 \neq \eta_2$ and let $f(z)$ be a finite order meromorphic function. Let σ be the order of $f(z)$, then for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O(r^{\sigma-1+\varepsilon}).$$

By a careful inspection of the proof of Theorem 2.3 in [17] and using Lemma 3.3, we easily get the following lemma, which is another version of the difference analogue of the Clunie lemma.

Lemma 3.4. *Let $f(z)$ be a transcendental meromorphic solution of finite order σ of a differential-difference equation of the form*

$$f(z)^n P(z,f) = Q(z,f),$$

where $P(z,f)$, $Q(z,f)$ are differential-difference polynomials in $f(z)$ with rational coefficients and $\deg_f(Q) \le n$. Then for each $\varepsilon > 0$, we have

$$m\big(r, P(z,f)\big) = O(r^{\sigma-1+\varepsilon}) + O(\log r).$$

Applying Lemma 2.1 to Corollary 3.4 of [8], we easily get the following lemma, which is a version of the difference analogue of the Mohon'ko-Mohon'ko lemma.

Lemma 3.5. *Let $f(z)$ be a transcendental meromorphic solution with $\sigma_2(f) < 1$ of*

$$P(z,f) = 0,$$

where $P(z,f)$ is a differential-difference polynomial in $f(z)$ with small meromorphic coefficients. If $P(z,a) \not\equiv 0$ for a small function $a(z)$, then

$$m\left(r, \frac{1}{f-a}\right) = S(r,f).$$

We now give the proof of Theorem 3.1.

Proof of Theorem 3.1. (i) Since $a(z) \equiv 0$, we get from (2.1) that

$$w(z+2) - a_1(z+1)w(z+1) - w(z) = a_0(z+1).$$

If $a_0(z) \not\equiv 0$, then by Lemma 3.2, we have $\sigma(w) \ge 1$. If $a_0(z) \equiv 0$, then $a_1(z) \not\equiv 0$ by assumption. We also have $\sigma(w) \ge 1$ from Lemma 3.1.

(ii) Since $a(z) \not\equiv 0$, we see from (2.1) that $w(z)$ has only finitely many zeros. By Hadamard factorization theorem, $w(z)$ takes the form

$$w(z) = H(z)e^{g(z)}, \tag{3.1}$$

where $H(z)$ is a non-zero polynomial, $g(z)$ is a non-constant entire function such that $\sigma_2(w(z)) = \sigma_2(e^{g(z)}) = \sigma(g(z)) < 1$. Substituting (3.1) into (2.1), we get

$$\begin{aligned} & - a_1(z)H(z)e^{g(z)} + H(z+1)e^{g(z+1)} - H(z-1)e^{g(z-1)} \\ =& a_0(z) - a(z)\left(\frac{H'(z)}{H(z)} + g'(z)\right). \end{aligned} \tag{3.2}$$

Since $\sigma(g(z)) = \sigma_2(e^{g(z)}) < 1$, we have $\liminf\limits_{r\to\infty} \frac{T(r,g)}{r} = 0$. If $g(z)$ is a transcendental entire function, we see from [24, p. 101] that $g(z+1)-g(z)$, $g(z+1)-g(z-1)$ and $g(z-1)-g(z)$ are all transcendental entire functions. Applying Lemma 2.3 to (3.2), we get $H(z+1) \equiv 0$, a contradiction, and so $g(z)$ must be a polynomial.

If $\deg g(z) \geq 2$, then $\deg(g(z+1)-g(z)) = \deg(g(z+1)-g(z-1)) = \deg(g(z-1)-g(z)) \geq 1$. Using Lemma 2.3 again, we also get a contradiction. Thus, this yields $\deg g(z) = 1$, and so $w(z)$ has the form

$$w(z) = H(z)e^{dz}, \tag{3.3}$$

where $d \neq 0$ is some complex number.

Substituting (3.3) into (2.1), we conclude

$$e^{dz}(H(z+1)e^{d} - H(z-1)e^{-d}) = a_1(z)H(z)e^{dz} + a_0(z) - a(z)\left(\frac{H'(z)}{H(z)} + d\right). \tag{3.4}$$

Applying Lemma 2.3 to (3.4), we have

$$H(z+1)e^{d} - H(z-1)e^{-d} = a_1(z)H(z). \tag{3.5}$$

If $a_1(z)$ is a polynomial, we deduce from (3.5) that $a_1(z)$ must be a constant. Let

$$H(z) = c_n z^n + c_{n-1}z^{n-1} + \cdots,\ (c_n \neq 0). \tag{3.6}$$

If $n \geq 2$, then we deduce from (3.5) and (3.6) that

$$\begin{cases} e^{d} - e^{-d} = a_1(z), \\ c_{n-1}(e^{d} - e^{-d}) + nc_n(e^{d} + e^{-d}) = a_1(z)c_{n-1}, \\ c_{n-2}(e^{d} - e^{-d}) + \frac{n(n-1)}{2}c_n(e^{d} - e^{-d}) + (n-1)c_{n-1}(e^{d} + e^{-d}) = a_1(z)c_{n-2}, \end{cases} \tag{3.7}$$

which yield $e^{d} = e^{-d} = 0$. This is impossible. So $n \leq 1$.

If $a_1(z) \not\equiv \pm 2i$, we must have $n = 0$. Otherwise, if $n = 1$, then by (3.5) and (3.6), we get

$$\begin{cases} e^{d} - e^{-d} = a_1(z), \\ e^{d} + e^{-d} = 0, \end{cases}$$

which yield $e^{d} = \pm i$ and $a_1(z) = \pm 2i$, contradicts $a_1(z) \not\equiv \pm 2i$. So $H(z)$ must be a constant and $w(z)$ has the form $w(z) = Ce^{dz}$, where $C \neq 0, C \in \mathbb{C}$.

If $a_1(z) \equiv \pm 2i$, then by (3.5) and (3.6), we get

$$e^{d} - e^{-d} = \pm 2i.$$

From above, we conclude that $d = (2k \pm \frac{1}{2})\pi i$ and so $w(z) = (C_1 z + C_0)e^{(2k\pm\frac{1}{2})\pi i z}$, where k is an integer and $C_1, C_0 \in \mathbb{C}$ with $|C_1| + |C_0| \neq 0$. □

We then give the proof of Theorem 3.2.

Proof of Theorem 3.2. (i) We first affirm that $a(z) \not\equiv 0$. Otherwise, we deduce from (2.2) and Lemma 2.1 that

$$T\left(r, \frac{a_2(z)w(z)^2 + a_1(z)w(z) + a_0(z)}{w(z)}\right) = T(r, w(z+1) - w(z-1))$$

$$= m(r, w(z+1) - w(z-1))$$
$$\leq m(r, w(z)) + S(r, w).$$

We see from Lemma 2.4 that $a_2(z) \equiv 0$, and so (2.2) can be written as

$$w(z)(w(z+1) - w(z-1)) = a_1(z)w(z) + a_0(z). \tag{3.8}$$

By (3.8), Lemma 2.2 and Lemma 2.4, we have

$$\begin{aligned} T(r, w(z)) + S(r, w) &= T\left(r, \frac{a_1(z)w(z) + a_0(z)}{w(z)}\right) \\ &= T(r, w(z+1) - w(z-1)) \\ &= m(r, w(z+1) - w(z-1)) = S(r, w). \end{aligned}$$

This is a contradiction, and the affirmation is proved.

Assume now that $\sigma(w(z)) < 1$, and we will deduce a contradiction. Equation (2.2) yields

$$w(z)(w(z+1) - w(z-1) - a_2(z)w(z) - a_1(z)) = -a(z)w'(z) + a_0(z). \tag{3.9}$$

Since $w(z)$ is transcendental and $a(z) \not\equiv 0$, we see that $-a(z)w'(z) + a_0(z) \not\equiv 0$. So $w(z+1) - w(z-1) - a_2(z)w(z) - a_1(z) \not\equiv 0$. Since $w(z)$ is an entire function, and $a_1(z)$, $a_2(z)$ are rational functions, we have

$$N(r, w(z+1) - w(z-1) - a_2(z)w(z) - a_1(z)) = O(\log r).$$

Applying Lemma 3.4 to (3.9), we get

$$\begin{aligned} &T(r, w(z+1) - w(z-1) - a_2(z)w(z) - a_1(z)) \\ =&m(r, w(z+1) - w(z-1) - a_2(z)w(z) - a_1(z)) \\ &+ N(r, w(z+1) - w(z-1) - a_2(z)w(z) - a_1(z)) = O(\log r). \end{aligned} \tag{3.10}$$

Rewrite (3.9) as

$$-\frac{w(z+1) - w(z-1) - a_2(z)w(z) - a_1(z)}{a(z)} = \frac{w'(z)}{w(z)} - \frac{a_0(z)}{a(z)w(z)}. \tag{3.11}$$

Since $a(z)$ is rational, we see from (3.10) that $-\frac{w(z+1)-w(z-1)-a_2(z)w(z)-a_1(z)}{a(z)} (\not\equiv 0)$ is also rational. Thus,

$$-\frac{w(z+1) - w(z-1) - a_2(z)w(z) - a_1(z)}{a(z)} = Az^n(1 + o(1)), \tag{3.12}$$

where $z \to \infty$, $A \neq 0$ is a constant and n is an integer.

From the Wiman-Valiron theory (see [13, pp.28-32], [16, p.51] or [22, pp.103-105]), we obtain

$$\frac{w'(z)}{w(z)} = \frac{\nu(r)}{z}(1 + o(1)), \tag{3.13}$$

where $|z| = r \notin [0, 1] \bigcup E, E \subset (1, \infty)$ is of finite logarithmic measure such that $|w(z)| = M(r, w)$ and $\nu(r)$ denotes the central index of $w(z)$.

Substituting (3.12) and (3.13) into (3.11), we conclude

$$Az^{n+1}(1+o(1)) = \nu(r)(1+o(1)) - \frac{za_0(z)}{a(z)w(z)}, \tag{3.14}$$

where $|z| = r \notin [0,1] \bigcup E$ such that $|w(z)| = M(r,w)$. Since $w(z)$ is transcendental, we have

$$\frac{|za_0(z)|}{|a(z)|M(r,w)} \to 0, \ (r \to \infty). \tag{3.15}$$

Thus, (3.14) and (3.15) yield

$$\nu(r) = |A|r^{n+1}(1+o(1)), \ (r \notin [0,1] \bigcup E, \ r \to \infty).$$

This shows that $\sigma(w(z)) = \limsup\limits_{r\to\infty} \frac{\log \nu(r)}{\log r} = n+1 \leq 0$, a contradiction. Thus, $\sigma(w(z)) \geq 1$.

(ii) Let

$$P(z,w) = w(z)(w(z+1) - w(z-1)) + a(z)w'(z) - a_2(z)w(z)^2 - a_1(z)w(z) - a_0(z).$$

Obviously, $P(z,b) = -a_2(z)b^2 - a_1(z)b - a_0(z) \not\equiv 0$. By applying lemma 3.5, we have

$$m\left(r, \frac{1}{w(z)-b}\right) = S(r,w),$$

and then

$$N\left(r, \frac{1}{w(z)-b}\right) = T(r, w(z)) + S(r,w). \tag{3.16}$$

First, we suppose that $b = 0$. Equation (2.2) shows that $w(z)$ has at most finitely many multiple zeros. Thus, we get from (3.16) that $\Theta(0, w(z)) = 0$.

Second, we suppose that $b \neq 0$. Let $g(z) = w(z) - b$, then $g(z)$ has infinitely many zeros. Substituting $w(z) = g(z) + b$ into (2.2), we obtain

$$\begin{aligned} &g(z)(g(z+1) - g(z-1)) + b(g(z+1) - g(z-1)) + a(z)g'(z) \\ =&a_2(z)g(z)^2 + 2ba_2(z)g(z) + a_1(z)g(z) + \psi(z), \end{aligned} \tag{3.17}$$

where $\psi(z) = a_2(z)b^2 + a_1(z)b + a_0(z)$. Now we divide our discussion into two cases.

Case 1. $g(z+1) - g(z-1) - a_2(z)g(z) \not\equiv 0$.

Rewriting (3.17) as

$$\begin{aligned} &g(z)(g(z+1) - g(z-1) - a_2(z)g(z)) \\ =&-a(z)g'(z) - b(g(z+1) - g(z-1)) + 2ba_2(z)g(z) + a_1(z)g(z) + \psi(z). \end{aligned} \tag{3.18}$$

Applying Lemma 2.2 to (3.18), we have

$$\begin{aligned} &T(r, g(z+1) - g(z-1) - a_2(z)g(z)) \\ =&m(r, g(z+1) - g(z-1) - a_2(z)g(z)) + O(\log r) = S(r,g). \end{aligned}$$

If $g(z+1)-g(z-1)-a_2(z)g(z) \equiv \frac{\psi(z)}{b}$, then by (3.17), we get

$$g(z)(g(z+1)-g(z-1))+a(z)g'(z)=a_2(z)g(z)^2+ba_2(z)g(z)+a_1(z)g(z).$$

Comparing the orders of zeros of both sides of the above equality, we have a contradiction. So $g(z+1)-g(z-1)-a_2(z)g(z) \not\equiv \frac{\psi(z)}{b}$ and

$$N\left(r, \frac{1}{g(z+1)-g(z-1)-a_2(z)g(z)-\frac{\psi(z)}{b}}\right)=S(r,g). \tag{3.19}$$

We denote by $N_1\left(r,\frac{1}{g(z)}\right)$ the counting function of those simple zeros of $g(z)$ in $|z|<r$, and denote by $N_{>1}\left(r,\frac{1}{g(z)}\right)$ the counting function of those multiple zeros of $g(z)$ in $|z|<r$. If z_0 is a multiple zero of $g(z)$ and that none of the coefficients in (3.17) has a zero or a pole at z_0, then by (3.17), we have

$$b(g(z_0+1)-g(z_0-1))=\psi(z_0),$$

and so

$$g(z_0+1)-g(z_0-1)-a_2(z_0)g(z_0)-\frac{\psi(z_0)}{b}=0. \tag{3.20}$$

We get from (3.19) and (3.20) that

$$\begin{aligned}
&N\left(r,\frac{1}{g(z)}\right)=N_1\left(r,\frac{1}{g(z)}\right)+N_{>1}\left(r,\frac{1}{g(z)}\right)\\
\le& N_1\left(r,\frac{1}{g(z)}\right)+N\left(r,\frac{1}{g(z+1)-g(z-1)-a_2(z)g(z)-\frac{\psi(z)}{b}}\right)+O(\log r)\\
\le& \overline{N}\left(r,\frac{1}{g(z)}\right)+S(r,g),
\end{aligned}$$

which gives $\Theta(b,w(z))=\Theta(0,g(z))=0$.

Case 2. $g(z+1)-g(z-1)-a_2(z)g(z)\equiv 0$.

Substituting $b(g(z+1)-g(z-1))=ba_2(z)g(z)$ into (3.17), we have

$$g(z)(g(z+1)-g(z-1))+a(z)g'(z)=a_2(z)g(z)^2+ba_2(z)g(z)+a_1(z)g(z)+\psi(z).$$

Since $\psi(z)\not\equiv 0$, we see from the above equality that $g(z)$ has at most finitely many multiple zeros. So $\Theta(b,w(z))=\Theta(0,g(z))=0$. □

4. Existence of entire solutions of the Kac-van Moerbeke delay differential equation

In section 2, we show that some reductions of integrable differential-difference equation (1.2) are known to reduce into delay differential equation (2.1). If $a_1(z)\equiv 0$, then equation (2.1) becomes

$$w(z+1)-w(z-1)+a(z)\frac{w'(z)}{w(z)}=a_0(z), \tag{4.1}$$

where $a(z), a_0(z)$ are rational. The equation (4.1) can be seen as Kac-van Moerbeke delay equation, which is also the similarity reductions of Kac-van Moerbeke delay partial differential difference equation.

Quispel, Capel and Sahadevan [19] showed that equation (4.1) has a formal continuum limit to the first Painlevé equation

$$\frac{d^2y}{dt^2} = 6y^2 + t,$$

if $a(z), a_0(z)$ are constants. They also showed that equation (4.1) is unusual, which makes its integration difficult. Hence, they restrict themselves to exhibiting two particular solutions for the cases $a(z) \equiv a$ and $a_0(z) = 0$ as follows.

(i) Soliton solution

$$w(z) = \frac{a\kappa(1 + e^{\kappa(z+1)+\delta})(1 + e^{\kappa(z-2)+\delta})}{(e^{-\kappa} - e^{\kappa})(1 + e^{\kappa z+\delta})(1 + e^{\kappa(z-1)+\delta})},$$

where κ and δ are arbitrary parameters.

(ii) Rational solution

$$w(z) = -\frac{a(z + 1 + \delta)(z - 2 + \delta)}{2(z + \delta)(z - 1 + \delta)},$$

where δ is an arbitrary parameter.

After that, Halburd and Korhonen [11] indicated that if $a_0(z) \equiv p\pi i a(z)$, where $p \in \mathbb{N}$, then $w(z) = C\exp(p\pi i z)$, $C \neq 0$, is a one-parameter family of zero-free entire transcendental finite-order solution of (4.1) for any rational function $a(z)$. This shows that (4.1) has transcendental entire solutions with finite order.

Question 3. Dose the equation (4.1) have entire solutions of infinite order?

For a meromorphic function w of infinite order, we use the notation of iterated order (see, e.g. [2]) to express its rate of growth. The iterated i-order of w is defined by

$$\sigma_i(w) = \limsup_{r\to\infty} \frac{\log_i T(r, w)}{\log r}, \ (i = 2, 3, 4, \cdots).$$

Obviously, the iterated 2-order of w is the hyper-order of w.

In general, it is difficult to study the existence of meromorphic or entire solutions with $\sigma_2(w) \geq 1$ of equations involving shifts. Here, we answer the above Question 3 and show that equation (4.1) has no entire solutions with finite iterated order.

Theorem 4.1. *Let $a(z)$ and $a_0(z)$ be rational functions such that $a(z) \not\equiv 0$. Then the equation (4.1) has no entire solutions with finite iterated order.*

We first prepare one lemma which relates to the estimate of characteristic function of shifts of a meromorphic function.

Lemma 4.1 ([1, 5]). *Let $f(z)$ be a meromorphic function. For an arbitrary $c \neq 0$, the following inequalities*

$$\big(1 + o(1)\big)T\big(r - |c|, f(z)\big) \leq T\big(r, f(z + c)\big) \leq \big(1 + o(1)\big)T\big(r + |c|, f(z)\big)$$

hold as $r \to \infty$.

We second give a complete proof of Theorem 4.1.

Proof of Theorem 4.1. Suppose that $w(z)$ is an entire solution of (4.1) with finite iterated order. Then there exists an integer $p \geq 1$ such that $\sigma_p(w) = \infty$ and $\sigma_{p+1}(w) < \infty$. We get from (4.1) that

$$a_0(z) - a(z)\frac{w'(z)}{w(z)} = w(z+1) - w(z-1). \tag{4.2}$$

By applying Hadamard factorization theorem, we see that $w(z)$ takes the form

$$w(z) = H(z)e^{g(z)}, \tag{4.3}$$

where $H(z)$ is a non-zero polynomial and $g(z)$ is a transcendental entire function such that $\sigma_p(w(z)) = \sigma_p(e^{g(z)}) = \infty$ and $\sigma_{p+1}(w(z)) = \sigma_p(g(z)) < \infty$.

If $w(z+1) - w(z-1) \equiv 0$, then (4.2) and (4.3) give

$$\frac{a_0(z)}{a(z)} - \frac{H'(z)}{H(z)} = g'(z),$$

a contradiction. So $w(z+1)-w(z-1) \not\equiv 0$. Substituting (4.3) into (4.2) and letting

$$F = a_0(z) - a(z)\left(\frac{H'(z)}{H(z)} + g'(z)\right), \tag{4.4}$$

we get

$$F = H(z+1)e^{g(z+1)} - H(z-1)e^{g(z-1)} \tag{4.5}$$

and

$$\begin{aligned} F' =& (H'(z+1) + H(z+1)g'(z+1))e^{g(z+1)} \\ &- (H'(z-1) + H(z-1)g'(z-1))e^{g(z-1)}. \end{aligned} \tag{4.6}$$

Set

$$\begin{cases} \widetilde{H_1}(z) = H(z+1)(H'(z-1) + H(z-1)g'(z-1)), \\ \widetilde{H_2}(z) = H(z-1)(H'(z+1) + H(z+1)g'(z+1)). \end{cases}$$

If $\widetilde{H_1}(z) \not\equiv \widetilde{H_2}(z)$, then by (4.5) and (4.6), we obtain

$$e^{g(z-1)} = \frac{(H'(z+1) + H(z+1)g'(z+1))F - H(z+1)F'}{\widetilde{H_1}(z) - \widetilde{H_2}(z)}. \tag{4.7}$$

By Lemma 4.1, we have $\sigma_p(e^{g(z-1)}) = \sigma_p(e^{g(z)}) = \infty$, but the iterated p-order of the right hand side of (4.7) is no more than $\sigma_p(g(z)) < \infty$. This is a contradiction.

If $H(z+1)(H'(z-1)+H(z-1)g'(z-1)) \equiv H(z-1)(H'(z+1)+H(z+1)g'(z+1))$, then by (4.6), we get

$$H(z+1)F' = (H'(z+1)+H(z+1)g'(z+1))(H(z+1)e^{g(z+1)} - H(z-1)e^{g(z-1)}). \tag{4.8}$$

Obviously, $H'(z+1)+H(z+1)g'(z+1)\not\equiv 0$. By (4.5) and (4.8), we obtian

$$\frac{F'}{F}=\frac{H'(z+1)}{H(z+1)}+g'(z+1).$$

So $F=CH(z+1)e^{g(z+1)}$, where C is a non-zero constant. By (4.4), we get

$$CH(z+1)e^{g(z+1)}=a_0(z)-a(z)\left(\frac{H'(z)}{H(z)}+g'(z)\right). \tag{4.9}$$

Comparing the iterated p-order of both sides of (4.9), we again get a contradiction.

Thus, the equation (4.1) has no entire solutions with finite iterated order. □

5. Examples

In this section, we give some examples to show that our results are possible. The following examples 5.1 and 5.2 show that the form (2.1) in Theorem 2.1 does exist.

Example 5.1. The equation

$$w(z+1)-w(z-1)+a(z)\frac{w'(z)}{w(z)}=\frac{e(z+1)-e^{-1}(z-1)}{z}w(z)+a(z)\frac{1+z}{z}$$

has an entire solution $w(z)=ze^z$, where $a(z)$ is any rational function.

Example 5.2. The equation

$$w(z+1)-w(z-1)+a(z)\frac{w'(z)}{w(z)}=2\pi i a(z)$$

has an entire solution $w(z)=e^{2\pi i z}$, where $a(z)$ is any rational function.

Examples 5.3−5.5 below show that the form (2.2) in Theorem 2.1 are valid.

Example 5.3. The equation

$$\begin{aligned}&w(z+1)-w(z-1)+a(z)\frac{w'(z)}{w(z)}\\=&\frac{(e-e^{-1})w(z)^2+(-z(e-e^{-1})+2+a(z))w(z)+a(z)(1-z)}{w(z)}\end{aligned}$$

has an entire solution $w(z)=e^z+z$, where $a(z)$ is any rational function.

Example 5.4. The equation

$$w(z+1)-w(z-1)+a(z)\frac{w'(z)}{w(z)}=\frac{2\pi i a(z)w(z)-2\pi i a(z)}{w(z)}$$

has an entire solution $w(z)=e^{2\pi i z}+1$, where $a(z)$ is any rational function.

Example 5.5. The equation

$$w(z+1)-w(z-1)-\frac{1}{\pi i}\frac{w'(z)}{w(z)}=\frac{2z-\frac{1}{\pi i}}{w(z)}$$

has an entire solution $w(z)=e^{2\pi i z}+z$.

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References

[1] M. Ablowitz, R. G. Halburd and B. Herbst, *On the extension of Painlevé property to difference equations*, Nonlinearity, 2000, 13(3), 889–905.

[2] L. G. Bernal, *On growth k-order of solutions of a complex homogeneous linear differential equations*, Proc. Amer. Math. Soc., 1987, 101(2), 317–322.

[3] Z. Chen and K. H. Shon, *On growth of meromorphic solutions for linear difference equations*, Abstr. Appl. Anal., 2013. DOI:10.1155/2013/619296.

[4] Y. M. Chiang and S. Feng, *On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane*, Ramanujan J., 2008, 16(1), 105–129.

[5] A. A. Gol'dberg and I. V. Ostrovskii, *Distribution of Values of Meromorphic Functions*, Nauka,Moscow, 1970.

[6] B. Grammaticos, A. Ramani and I. C. Moreira, *Dealy differential equations and the Painlevé transcendents*, Phy. A, 1993, 196(4), 574–590.

[7] F. Gross, *Factorization of Meromorphic Functions,* U.S. Government Printing Office, Washington D. C.,1972.

[8] R. G. Halburd and R. J. Korhonen, *Difference analogue of the lemma on the logarithmic derivative with applications to difference equations*, J. Math. Anal. Appl., 2006, 314(2), 477–487.

[9] R. G. Halburd and R. J. Korhonen, *Finite-order meromorphic solutions and the discrete Painlevé equations*, Proc. London Math. Soc., 2007, 94(2), 443–474.

[10] R. G. Halburd, R. J. Korhonen and K. Tohge, *Holomorphic curves with shift-invariant hyperplane preimages*, Trans. Amer. Math. Soc., 2014, 366(8), 4267–4298.

[11] R. G. Halburd and R. J. Korhonen, *Growth of meromorphic solutions of delay differential equations*, Proc. Amer. Math. Soc., 2017, 145(6), 2513–2526.

[12] W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford,1964.

[13] Y. He and X. Xiao, *Algebroid Functions and Ordinary Differential Equations*, Science Press(in Chinese), Beijing, 1988.

[14] J. Heittokangas, R. Korhonen, I. Laine, J. Rieppo and K. Tohge, *Complex difference equations of Malmquist type*, Comput. Methods Funct. Theory, 2001, 1(1), 27–39.

[15] Z. Huang, Z. Chen and Q. Li, *On properties of meromorphic solutions for complex difference equation of Malmquist type*, Acta Math. Sci., Ser. B, 2013, 33(4), 1141–1152.

[16] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, Walter de Gruyter, Berlin, 1993.

[17] I. Laine and C. Yang, *Clunie theorems for difference and q-difference polynomials*, J. Lond. Math. Soc., 2007, 76(3), 556–566.

[18] J. Malmquist, *Sur les fonctions a un nombre fini de braches définies par les équations différentielles du premier ordre (French)*, Acta Math., 1913, 36(1), 297–343.

[19] G. R. W. Quispel, H. W. Capel and R. Sahadevan, *Continuous symmetries of differential-difference equations:the Kac-van Moerbeke equation and Painlevé reduction*, Phys. Lett. A, 1992, 170(5), 379–383.

[20] J. Rieppo, *Malmquist-type results for difference equations with periodic coefficients*, Comput.Methods Funct. Theory, 2015, 15(3), 449–457.

[21] N. Steinmetz, *Eigenschaften eindeutiger Lösungen gewöhnlicher Differentialgleichungen im Komplexen*, Karlsruhe, 1978.

[22] G. Valiron, *Lectures on the General Theory of Integral Functions*, Chelsea, New York, 1949.

[23] C. M. Viallet, *Algebraic entropy for differential-delay equations*, 2014, arXiv:1408:6161.

[24] J. M. Whittaker, *Interpolatory Function Theorey. Cambridge Tracts in Mathematics and Mathematical Physics,* Cambridge University Press, No. 33, 1935.

[25] C. Yang and H. Yi, *Uniqueness Theory of Meromorphic Functions*, Kluwer Academic Publishers Group, Dordrecht, 2003.

[26] J. Zhang and J. Zhang, *Meromorphic solutions to complex difference and q-difference equations of Malmquist type*, Electron. J. Differential Equ., 2014, 2014(16), 1–11.

[27] R. Zhang and Z. Huang, *Meromorphic solutions of a complex difference equation of Malmquist type*, Bull. Korean Math. Soc., 2014, 51(6), 1735–1748.