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UNIQUENESS OF THE DIFFERENCES OF MEROMORPHIC FUNCTIONS

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Abstract. In this paper, we investigate the uniqueness of difference operators concerning an entire function by using the method of complex difference equations. The results include the difference analogues of the Brück conjecture. We also present some results on difference operators concerning an entire function with positive deficiency, which generalize the results of Heittokangas, Chen, Yi, et al.

1. Introduction

We assume that the reader is familiar with the fundamental results and standard notations of Nevanlinna value distribution theory of meromorphic functions, see, e.g., [17]. In particular, we use $\sigma(f)$ and $\sigma_2(f)$ to denote the order and hyper-order of a meromorphic function respectively.

We use $S(r,f)$ to denote any quantity satisfying $S(r,f)=o(T(r,f))$ as $r\to\infty$, possibly outside an exceptional set E with finite logarithmic measure

$$\lim_{r\to\infty}\int_{E\cap[1,r)} dt/t<\infty.$$

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We specify the notion of small functions as follows: Given a meromorphic function $f(z)$, the family of all functions $a(z)$ such that $T(r,a) = S(r,f)$, possibly outside an exceptional set of finite logarithmic measure, is denoted by $\mathcal{S}(f)$. For convenience, we also include all constant functions in $\mathcal{S}(f)$, and denote $\widehat{\mathcal{S}}(f) = \mathcal{S}(f) \cup \{\infty\}$. The deficiency of $a \in \mathcal{S}(f)$ is defined by

$$\delta(a, f(z)) = \liminf_{r\to\infty} \frac{m(r, \frac{1}{f-a})}{T(r,f)} = 1 - \limsup_{r\to\infty} \frac{N(r, \frac{1}{f-a})}{T(r,f)} .$$

As for the standard notations in the uniqueness theory of meromorphic functions, suppose that $f(z)$ and $g(z)$ are meromorphic and $a \in \widehat{\mathcal{S}}(f)$. Denoting by $E(a,f)$ the set of those points $z \in \mathbb{C}$ where $f(z) = a$, we say that $f(z)$ and $g(z)$ share a IM (ignoring multiplicities), if $E(a,f) = E(a,g)$. Provided that $E(a,f) = E(a,g)$ and the multiplicities of the zeros of $f(z) - a$ and $g(z) - a$ are the same at each $z \in \mathbb{C}$, then $f(z)$ and $g(z)$ share a CM (counting multiplicities), see, e.g., [28].

2. A difference analogue of the Brück conjecture

The classical results in the uniqueness theory of meromorphic functions are the five-point, resp. the four-point, theorems due to Nevanlinna [25]: If two meromorphic functions $f(z)$ and $g(z)$ share five distinct values in $\widehat{\mathbb{C}}$ IM, then $f(z) \equiv g(z)$. Similarly, if two meromorphic functions $f(z)$ and $g(z)$ share four distinct values in $\widehat{\mathbb{C}}$ CM, then $f(z) \equiv g(z)$ or $f(z)$ is a Möbius transformation of $g(z)$. The condition "$f(z)$ and $g(z)$ share four values CM" has been weakened to "$f(z)$ and $g(z)$ share two values CM and two values IM" by Gundersen [10], as well as by Mues [24] and Wang [27] independently. But $1\,\mathrm{CM} + 3\,\mathrm{IM}$ is still an open problem.

What is the relation between $f(z)$ and $f'(z)$ if an entire function $f(z)$ shares one finite value CM with its derivative $f'(z)$? R. Brück [1] raised the following conjecture.

Conjecture. *Let $f(z)$ be a nonconstant entire function such that*

$$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r} < \infty,$$

and $\sigma_2(f)$ is not a positive integer. If $f(z)$ and $f'(z)$ share one finite value a CM, then

$$\frac{f'(z) - a}{f(z) - a} = \tau$$

for some constant $\tau \neq 0$.

The case that $a=0$ has been proved by Brück himself in [1], the case that $f(z)$ is of finite order has been proved by Gundersen and Yang in [11], while the case that $f(z)$ is of hyper-order $\sigma_2(f)<\frac{1}{2}$ has been proved by Chen and Shon in [5,6].

Meromorphic solutions of complex difference equations, and the value distribution and uniqueness of complex differences have become an area of current interest and the study is based on the Nevanlinna value distribution of difference operators established by Halburd and Korhonen [12,13] and by Chiang and Feng [9], respectively. Many results on meromorphic solutions of complex difference equations, see, e.g. [4,9,14,19,20,26], and on value distribution and uniqueness of complex differences were rapidly obtained, see e.g., [3,7,15,16,18,22,23]. Now, we recall the following result, which can be seen as the difference analogue of the Brück conjecture.

Theorem 2.1 [15, Theorem 1]. *Let $f(z)$ be a meromorphic function of order of growth*

$$\sigma(f)=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}<2,$$

and let $c\in\mathbb{C}$. If $f(z)$ and $f(z+c)$ share $a\in\mathbb{C}$ and ∞ CM, then

$$\frac{f(z+c)-a}{f(z)-a}=\tau$$

for some constant τ.

Remark 2.2. In order to illustrate that $\sigma(f)<2$ can not be replaced by $\sigma(f)\le 2$, they supposed that $f(z)=e^{z^2}+1$ and $c\in\mathbb{C}\backslash\{0\}$, and obtained

$$\frac{f(z+c)-1}{f(z)-1}=e^{2cz+c^2}\neq\text{constant}$$

though $f(z)$ and $f(z+c)$ share 1 and ∞ CM.

However, if we assume that $f(z)$ is a finite order transcendental entire function with a Borel exceptional value, we obtain the following result.

Theorem 2.3. *Let $f(z)$ be a finite order transcendental entire function with a Borel exceptional value $\alpha\in\mathbb{C}$, let $c\in\mathbb{C}$ and let $a(z)(\not\equiv\alpha)\in\mathcal{S}(f)$. If $f(z)$ and $f(z+c)$ share $a(z)$ CM, then $f(z+c)\equiv f(z)$.*

Example 2.4. Let $f(z)=e^z+1$. Obviously, $f(z)$ has a Borel exceptional value $\alpha=1$. For any $a\neq 1=\alpha$, $f(z)$ and $f(z+2k\pi i)(k\in\mathbb{Z})$ share the value a CM, and $f(z)\equiv f(z+2k\pi i)$.

Now let $f(z)$ be a meromorphic function. We define the difference operators

$$\Delta_c f(z)=f(z+c)-f(z)\quad\text{and}\quad\Delta_c^k f(z)=\Delta_c^{k-1}(\Delta_c f(z)),\quad k\in\mathbb{N},\ k\ge 2,$$

where c is a nonzero constant. In particular, we write $\Delta_c f(z)$ for $\Delta_c^1 f(z)$.

Based on different assumptions, the uniqueness of $\Delta_c^k f(z)$ and $f(z)$ has been investigated. For more details, we can see Chen [3, Theorem 1.1], Chen and Chen [7, Theorem 1.1], Chen and Yi [8, Theorem 1.1], Li et al. [22, Theorem 1.1], Zhang and Liao [29, Theorem 1.1]. Now, we recall the following results, which show that $f(z)$ and $\Delta_c f(z)$, $f(z)$ and $f(z+c)$ can not have any finite CM sharing value if the growth of order satisfies $\sigma(f) < 1$.

Theorem 2.5 [3, Theorem 1.2]. *Let $f(z)$ be a transcendental entire function with $\sigma(f) < 1$, and $c \in \mathbb{C}\backslash\{0\}$ a constant such that $f(z+c) \not\equiv f(z)$. Then*

(i) *$f(z)$ and $\Delta_c f(z)$ can not have any finite CM sharing value*;

(ii) *$f(z)$ and $f(z+c)$ can not have any finite CM sharing value.*

Thus, a result parallel to Theorem 2.1 for difference operators can be obtained as follows.

Theorem 2.6. *Let $f(z)$ be a transcendental entire function of order of growth*

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} < 2.$$

If $f(z)$ and $\Delta_c^k f(z)$ share 0 CM, where $k \in \mathbb{N}$ and $c \in \mathbb{C}\backslash\{0\}$ are such that $\Delta_c^k f(z) \not\equiv 0$, then

$$\Delta_c^k f(z) \equiv \tau f(z)$$

for some positive constant τ.

Example 2.7. Let

$$f(z) = d(z) \exp\Big\{\frac{\operatorname{Log}(1+\tau)}{c} z\Big\},$$

where Log denotes the principal branch of the logarithm, and $d(z)$ is a periodic entire function with period c such that $\sigma(d) \in [0, 2)$. We have $\Delta_c f(z) \equiv \tau f(z)$.

In order to prove Theorems 2.3 and 2.6, we need the following lemmas.

Lemma 2.8 [28, Theorem 1.51]. *Suppose that $f_j(z)$ $(j = 1, 2, \dots, n)$ $(n \geq 2)$ are meromorphic functions, $g_j(z)$ $(j = 1, 2, \dots, n)$ are entire functions satisfying the following conditions.*

(1) $\sum_{j=1}^n f_j(z) e^{g_j(z)} = 0$.

(2) *$g_j(z) - g_k(z)$ are not constants for $1 \leq j < k \leq n$.*

(3) *For $1 \leq j \leq n$, $1 \leq h < k \leq n$,*

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\} \quad (r \to +\infty,\ r \notin E),$$

where $E \subset (1, +\infty)$ is of finite linear measure or finite logarithmic measure. Then $f_j(z) \equiv 0$ $(j = 1, 2, \dots, n)$.

LEMMA 2.9 [9, Theorem 9.2]. *Let $A_0(z)$, $A_1(z)$, …, $A_n(z)$ be entire functions such that there exists an integer l, $0 \le l \le n$, such that*

$$\sigma(A_l) > \max_{0\le j\le n, j\ne l}\{\sigma(A_j)\}.$$

If $f(z)$ $(\not\equiv 0)$ is a meromorphic solution of

$$A_n(z)f(z+n) + \cdots + A_1(z)f(z+1) + A_0(z)f(z) = 0,$$

then $\sigma(f) \ge \sigma(A_l) + 1$.

PROOF OF THEOREM 2.3. Since $f(z)$ is entire of finite order, $f(z)$ and $f(z+c)$ share $a(z)$ $(\not\equiv \alpha)$ CM, the Hadamard factorization theorem shows that

$$\frac{f(z+c) - a(z)}{f(z) - a(z)} = e^{Q(z)}, \tag{2.1}$$

where $Q(z)$ is a polynomial with $\deg Q(z) = \sigma(e^{Q(z)}) \le \sigma(f)$.

We note that $f(z)$ has a Borel exceptional value α, and so $f(z)$ can be written as

$$f(z) = A(z)e^{H(z)} + \alpha, \tag{2.2}$$

where $A(z)$ is a nonzero entire function, $H(z)$ is a polynomial satisfying

$$\lambda(A) = \sigma(A) = \lambda(f - \alpha) < \sigma(f) = \deg H(z) = n.$$

We now conclude from (2.1) and (2.2) that

$$A(z+c)e^{H(z+c)} - A(z)e^{H(z)}e^{Q(z)} + (a(z) - \alpha)e^{Q(z)} - (a(z) - \alpha) = 0. \tag{2.3}$$

First, we suppose that $Q(z)$ is a nonconstant polynomial. We note that $H(z)$ and $Q(z)$ are polynomials and $1 \le \deg Q(z) \le \sigma(f) = \deg H(z) = n$. Thus, we consider the following two cases.

Case 1: $1 \le \deg Q(z) < \sigma(f) = \deg H(z) = n$. (2.3) can be rewritten as

$$A(z+c)e^{H(z+c)-H(z)} - A(z)e^{Q(z)} = (a(z) - \alpha)\big(1 - e^{Q(z)}\big)e^{-H(z)}. \tag{2.4}$$

We deduce that the order of growth of the left-hand side of (2.4) is less than $\sigma(f)$ and the order of growth of the right-hand side of (2.4) is equal to $\sigma(f)$. This is impossible.

Case 2: $1 \le \deg Q(z) = \sigma(f) = \deg H(z) = n$. Set

$$H(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0,$$
$$Q(z) = b_n z^n + b_{n-1} z^{n-1} + \cdots + b_1 z + b_0,$$

where $c_n, c_{n-1}, \ldots, c_0, b_n, b_{n-1}, \ldots, b_0$ are constants such that $c_n b_n \neq 0$. For the coefficients c_n and b_n, there exist three cases: $c_n = b_n$; $c_n = -b_n$; $c_n \neq b_n$ and $c_n \neq -b_n$. Thus, we will discuss the following three subcases.

Subcase 2.1: $c_n = b_n$. (2.3) can be rewritten as

$$A_{11}(z)e^{-H(z)} + A_{12}(z)e^{Q(z)} + A_{13}(z)e^{Q_0(z)} = 0, \tag{2.5}$$

where $Q_0(z) = 0$ and

$$A_{11}(z) = \alpha - a(z), \quad A_{12}(z) = -A(z),$$
$$A_{13}(z) = A(z+c)e^{H(z+c)-H(z)} + (a(z) - \alpha)e^{Q(z)-H(z)}.$$

Since $c_n = b_n$, $\sigma(A) < n$, we conclude that

$$\deg\{-H(z) - Q(z)\} = \deg\{-H(z) - Q_0(z)\} = \deg\{Q(z) - Q_0(z)\} = n,$$
$$T(r, a(z)) = S(r, f) = S(r, e^{H(z)}).$$

Thus, for all $j = 1, 2, 3$,

$$T(r, A_{1j}) = o\big(T(r, e^{-H(z)-Q(z)})\big), \quad T(r, A_{1j}) = o\big(T(r, e^{-H(z)-Q_0(z)})\big),$$
$$T(r, A_{1j}) = o\big(T(r, e^{Q(z)-Q_0(z)})\big).$$

Therefore, it follows from Lemma 2.8 and (2.5) that

$$A_{1j}(z) \equiv 0 \quad (j = 1, 2, 3),$$

a contradiction.

Subcase 2.2: $c_n = -b_n$. (2.3) can be rewritten as

$$A_{21}(z)e^{H(z+c)} + A_{22}(z)e^{Q(z)} + A_{23}(z)e^{Q_0(z)} = 0, \tag{2.6}$$

where $Q_0(z) = 0$ and

$$A_{21}(z) = A(z+c), \quad A_{22}(z) = a(z) - \alpha,$$
$$A_{23}(z) = -A(z)e^{H(z)+Q(z)} - (a(z) - \alpha).$$

Similarly to the proof of Subcase 2.1, we also have

$$A_{2j}(z) \equiv 0 \quad (j = 1, 2, 3),$$

a contradiction.

Subcase 2.3: $c_n \neq b_n$ and $c_n \neq -b_n$. (2.3) can be rewritten as

$$A_{31}(z)e^{H(z+c)} + A_{32}(z)e^{H(z)+Q(z)} + A_{33}(z)e^{Q(z)} + A_{34}(z)e^{Q_0(z)} = 0, \tag{2.7}$$

where $Q_0(z) = 0$ and

$$A_{31}(z) = A(z+c), \quad A_{32}(z) = -A(z),$$
$$A_{33}(z) = a(z) - \alpha, \quad A_{34}(z) = \alpha - a(z).$$

Similarly to the proof of Subcase 2.1, we still obtain from (2.7) that

$$A_{3j}(z) \equiv 0 \quad (j = 1, 2, 3, 4),$$

a contradiction.

Second, we suppose that $Q(z)$ is a constant, and say $e^{Q(z)} \equiv \tau(\neq 0)$. Then, (2.3) turns into

$$A(z+c)e^{H(z+c)-H(z)} - \tau A(z) = (a(z) - \alpha)(1-\tau)e^{-H(z)}. \tag{2.8}$$

If $\tau \neq 1$, we conclude that

$$\sigma\big((a(z) - \alpha)(1-\tau)e^{-H(z)}\big) = n = \sigma(f),$$

and

$$\sigma\big(A(z+c)e^{H(z+c)-H(z)} - \tau A(z)\big) < n = \sigma(f).$$

This is a contradiction.

Thus, we get $\tau = 1$, and (2.1) turns into

$$\frac{f(z+c) - a(z)}{f(z) - a(z)} = e^{Q(z)} = 1,$$

that is $f(z+c) \equiv f(z)$. Theorem 2.3 is proved. $\square$

Proof of Theorem 2.6. It follows from the assumption that

$$\frac{\Delta_c^k f(z)}{f(z)} = e^{Q(z)}, \tag{2.9}$$

where $Q(z)$ is a polynomial.

If $Q(z)$ is a constant, Theorem 2.6 holds obviously. If $Q(z)$ is a nonconstant polynomial, we will deduce a contradiction.

Since

$$\Delta_c f(z) = f(z+c) - f(z), \quad \Delta_c^k f(z) = \Delta_c^{k-1}(\Delta_c f(z)), \quad k \in \mathbb{N},$$

we deduce that

$$\Delta_c^k f(z) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(z+jc), \tag{2.10}$$

by k ($\in \mathbb{N}$) times iteration of the above difference operators to $f(z)$.

Substituting (2.10) into (2.9), we conclude that

$$f(z+kc) + \sum_{j=1}^{k-1} (-1)^{k-j} \binom{k}{j} f(z+jc) + \big[(-1)^k - e^{Q(z)}\big] f(z) = 0. \tag{2.11}$$

Thus, by applying Lemma 2.9 to (2.11), we obtain that $\sigma(f) \geq \sigma(e^{Q(z)}) + 1 \geq 2$, contradicting to the assumption that $\sigma(f) < 2$. Theorem 2.6 is proved. $\square$

3. Uniqueness on difference operators concerned one function

Heittokangas et al. [15,16] considered the problem of value sharing for shifts of meromorphic functions and related things, by considering the cases of sharing three small functions CM and sharing three small functions 2 CM and 1 IM. The case 1 CM and 2 IM, and the case 3 IM were left open in [15]. But these cases have been recently settled partly by Charak et al. [2]. They showed that partial value sharing of $f(z)$ and $f(z+c)$ involving 3 or 4 values combined with an appropriate deficiency assumption is enough to guarantee that $f(z) \equiv f(z+c)$, provided that $f(z)$ is a meromorphic function of hyper-order strictly less than one and $c \in \mathbb{C}$. Now, we recall the case that $f(z)$ and $\Delta_c f(z)$ share three distinct values a, b, ∞ CM, obtained by Chen and Yi.

THEOREM 3.1 [8, Theorem 1.2]. *Let $f(z)$ be a transcendental meromorphic function such that its order of growth $\sigma(f)$ is not an integer or infinite, and let $c \in \mathbb{C}$ be a constant such that $f(z+c) \not\equiv f(z)$. If $\Delta_c f(z)$ and $f(z)$ share three distinct values a, b, ∞ CM, then $f(z+c) \equiv 2f(z)$.*

REMARK 3.2. The restriction in Theorem 3.1 to the growth of order is not the best, as the following example shows.

EXAMPLE 3.3. Take $f_1(z) = e^z$ and $f_2(z) = e^z e^{s(z)}$, where $s(z)$ is a periodic function with period $\log 2$, such that $\sigma(f_1) = 1$ and $\sigma(f_2) = \infty$, then we get that $f_j(z + \log 2) - f_j(z)$ and $f_j(z)$ share values 1, 2, ∞ CM, and $f_j(z + \log 2) \equiv 2f_j(z)$.

Thus, we establish some improvements of Theorem 3.1, which are stated as follows.

Theorem 3.4. *Let $f(z)$ be a transcendental entire function of finite order, let $a(z)(\not\equiv 0)\in\mathcal{S}(f)$, and let $c\in\mathbb{C}$ be a constant such that $f(z+c)\not\equiv f(z)$. Assume that $f(z)$ and $\Delta_c f(z)$ share $a(z)$ CM and $\delta(0,f)>0$, then $f(z+c)\equiv 2f(z)$.*

Theorem 3.5. *Let $f(z)$ be a transcendental entire function of finite order, let $a(z)(\not\equiv 0)\in\mathcal{S}(f)$, and let $c\in\mathbb{C}\backslash\{0\}$ be a constant. Assume that $f(z)$ and $f(z+c)$ share $a(z)$ CM and $\delta(0,f)>0$, then $f(z+c)\equiv f(z)$.*

Example 3.6. Let $f(z)=e^z+3$ and $c=\pi i$, then $\delta(0,f)=0$, $\Delta_c f(z)=-2e^z$, $f(z)-2=e^z+1$ and $\Delta_c f(z)-2=-2(e^z+1)$. Obviously, $f(z)$ and $\Delta_c f(z)$ share the value 2 CM. But $f(z+c)\not\equiv 2f(z)$. This example shows that the assumption $\delta(0,f)>0$ in Theorem 3.4 cannot be dropped.

Example 3.7. Let $f(z)=e^z+1$ and $c=1$, then $\delta(0,f)=0$ and $f(z+1)=e^{z+1}+1$. Obviously, $f(z)$ and $f(z+1)$ share the value 1 CM. But $f(z+c)\not\equiv f(z)$. This example shows that the assumption $\delta(0,f)>0$ in Theorem 3.5 cannot be dropped.

We now give some lemmas which are required to prove Theorem 3.4 and Theorem 3.5.

Lemma 3.8 [12, Corollary 2.2]. *Let $f(z)$ be a nonconstant meromorphic function of finite order, and let η_1, η_2 be two arbitrary complex numbers. Then we have*

$$m\Big(r,\frac{f(z+\eta_1)}{f(z+\eta_2)}\Big)=S(r,f).$$

Lemma 3.9 [21, Lemma 3]. *Suppose that h is a nonconstant meromorphic function satisfying*

$$\overline{N}(r,h)+\overline{N}(r,1/h)=S(r,h).$$

Let $f=a_0h^p+a_1h^{p-1}+\cdots+a_p$, and $g=b_0h^q+b_1h^{q-1}+\cdots+b_q$ be polynomials in h with coefficients a_0, a_1, $\dots$, a_p, b_0, b_1, $\dots$, b_q being small functions of h and $a_0b_0a_p\not\equiv 0$. If $q\le p$, then $m(r,g/f)=S(r,h)$.

Proof of Theorem 3.4. Since $f(z)$ and $\Delta_c f(z)$ share $a(z)\not\equiv 0$ CM, we obtain

$$\frac{\Delta_c f(z)-a(z)}{f(z)-a(z)}=e^{H(z)}, \tag{3.1}$$

where $H(z)$ is a polynomial.

If $e^{H(z)}\equiv 1$, then we obtain at once from (3.1) that $f(z+c)\equiv 2f(z)$. If $e^{H(z)}\not\equiv 1$, we will deduce a contradiction.

We may apply Nevanlinna's main theorems to conclude that

$$T\Big(r, \frac{1}{f(z)-a(z)}\Big) = T(r, f(z)) + S(r, f)$$

$$\leq N\Big(r, \frac{1}{f(z)}\Big) + N\Big(r, \frac{1}{f(z)-a(z)}\Big) + S(r, f),$$

and so

$$m\Big(r, \frac{1}{f(z)-a(z)}\Big) \leq N\Big(r, \frac{1}{f(z)}\Big) + S(r, f). \tag{3.2}$$

Thus, we conclude from Lemma 3.8, (3.1) and (3.2) that

$$T\big(r, e^{H(z)}\big) = m\big(r, e^{H(z)}\big) = m\Big(r, \frac{\Delta_c f(z) - a(z)}{f(z)-a(z)}\Big) \tag{3.3}$$

$$= m\Big(r, \frac{(f(z+c)-a(z+c)) - (f(z)-a(z)) + a(z+c) - 2a(z)}{f(z)-a(z)}\Big)$$

$$\leq m\Big(r, \frac{f(z+c)-a(z+c)}{f(z)-a(z)}\Big) + m\Big(r, \frac{a(z+c)-2a(z)}{f(z)-a(z)}\Big) + O(1)$$

$$\leq N\Big(r, \frac{1}{f(z)}\Big) + S(r, f) \leq T(r, f) + S(r, f).$$

Now, we rewrite (3.1) as

$$-f(z+c) + (e^{H(z)}+1)f(z) = a(z)(e^{H(z)}-1).$$

Dividing the above equality by $f(z)a(z)(e^{H(z)}-1)$, we obtain

$$-\frac{1}{a(z)(e^{H(z)}-1)}\frac{f(z+c)}{f(z)} + \frac{e^{H(z)}+1}{a(z)(e^{H(z)}-1)} = \frac{1}{f(z)}. \tag{3.4}$$

If $H(z)$ is a constant such that $e^{H(z)} \not\equiv 1$, then we deduce from (3.4) and Lemma 3.8 that

$$m\Big(r, \frac{1}{f(z)}\Big) = m\Big(r, -\frac{1}{a(z)(e^{H(z)}-1)}\frac{f(z+c)}{f(z)} + \frac{e^{H(z)}+1}{a(z)(e^{H(z)}-1)}\Big)$$

$$\leq 2m\Big(r, \frac{1}{a(z)}\Big) + m\Big(r, \frac{f(z+c)}{f(z)}\Big) + O(1) = S(r, f).$$

So

$$N\left(r, \frac{1}{f(z)}\right) = T(r, f) + S(r, f),$$

which gives $\delta(0, f) = 0$, contradicting $\delta(0, f) > 0$.

If $H(z)$ is a nonconstant polynomial, then we deduce from Lemma 3.9 and (3.3) that

$$m\Big(r, \frac{1}{e^{H(z)}-1}\Big) = S(r, e^{H(z)}) = S(r, f). \tag{3.5}$$

By (3.4), (3.5) and Lemma 3.8, we obtain

$$\begin{aligned} m\Big(r, \frac{1}{f(z)}\Big) &= m\Big(r, -\frac{1}{a(z)(e^{H(z)}-1)}\frac{f(z+c)}{f(z)} + \frac{e^{H(z)}+1}{a(z)(e^{H(z)}-1)}\Big) \\ &\le m\Big(r, \frac{1}{e^{H(z)}-1}\Big) + m\Big(r, \frac{f(z+c)}{f(z)}\Big) + m\Big(r, \frac{e^{H(z)}+1}{e^{H(z)}-1}\Big) + S(r, f) \\ &= m\Big(r, 1 + \frac{2}{e^{H(z)}-1}\Big) + S(r, f) = S(r, f). \end{aligned}$$

We also obtain $\delta(0, f) = 0$, contradicting $\delta(0, f) > 0$. Theorem 3.4 is proved. □

Proof of Theorem 3.5. Since $f(z)$ and $f(z+c)$ share $a(z) \not\equiv 0$ CM, we obtain

$$\frac{f(z+c)-a(z)}{f(z)-a(z)} = e^{H(z)}, \tag{3.6}$$

where $H(z)$ is a polynomial. If $e^{H(z)} \equiv 1$, then we obtain at once from (3.6) that $f(z+c) \equiv f(z)$. If $e^{H(z)} \not\equiv 1$, then we obtain from (3.6) that

$$-f(z+c) + e^{H(z)}f(z) = a(z)(e^{H(z)}-1),$$

which gives

$$-\frac{1}{a(z)(e^{H(z)}-1)}\frac{f(z+c)}{f(z)} + \frac{e^{H(z)}}{a(z)(e^{H(z)}-1)} = \frac{1}{f(z)}.$$

Using a proof similar to that of Theorem 3.4, we can obtain $\delta(0, f) = 0$, contradicts $\delta(0, f) > 0$. Theorem 3.5 is proved. □

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