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On Meromorphic Solutions of Non-linear Difference Equations

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Abstract In this paper, using the theory of linear algebra, we investigate the non-linear difference equation of the following form in the complex plane:

$$f(z)^n + p(z)f(z+\eta) = \beta_1 e^{\alpha_1 z} + \beta_2 e^{\alpha_2 z} + \cdots + \beta_s e^{\alpha_s z},$$

where n, s are the positive integers, $p(z) \not\equiv 0$ is a polynomial and $\eta, \beta_1, \ldots, \beta_s, \alpha_1, \ldots, \alpha_s$ are the constants with $\beta_1 \ldots \beta_s \alpha_1 \ldots \alpha_s \neq 0$, and show that this equation just has meromorphic solutions with hyper-order at least one when $n \geq 2+s$. Other cases are also obtained.

Keywords Nevanlinna theory · Meromorphic solution · Entire solution · Difference equation

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1 Introduction and results

Considering a meromorphic function f in the complex plane $\mathbb{C}$, we assume that the reader is familiar with the basic Nevanlinna value distribution theory and its standard

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notation such as the proximity function $m(r, f)$, the unintegrated counting function $n(r, f)$, the counting function $N(r, f)$, the reduced counting function $\overline{N}(r, f)$, and the characteristic function $T(r, f)$, see, e.g., [5,7,12]. We use $\sigma(f)$ to denote the order of growth of f and $\lambda(f)$ to denote the exponent of convergence of zeros of f. The hyper-order of f is defined by

$$\sigma_2(f) = \overline{\lim_{r\to\infty}} \frac{\log\log T(r, f)}{\log r},$$

the hyper-exponent of convergence of poles of f is defined by

$$\lambda_2\left(\frac{1}{f}\right) = \overline{\lim_{r\to\infty}} \frac{\log\log N(r, f)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log\log n(r, f)}{\log r},$$

and the deficiency in which zeros of f are counted only once is defined by

$$\Theta(0, f) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}(r, \frac{1}{f})}{T(r, f)}.$$

We denote by $S(r, f)$ any real function of growth $o(T(r, f))$ as $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure. A meromorphic function α is said to be a small function of f, if $T(r, \alpha) = S(r, f)$. An algebraic differential polynomial $P(f)$ is a polynomial in f and its derivatives, with small functions of f as its coefficients. An algebraic difference polynomial $Q(f)$ is a polynomial in f and its shifts, with small functions of f as its coefficients. And an algebraic differential–difference polynomial $R(f)$ is a polynomial in f, its derivatives, its shifts and derivatives of its shifts, with small functions of f as its coefficients.

It is an interesting and difficult question to study the solvability and existence of entire or meromorphic solution of non-linear differential, or difference, or differential–difference equations in complex domains. Many authors have investigated this question by utilizing the Nevanlinna value distribution theory and its difference counterparts, see, e.g., [1,7,9–11,13,14].

The logarithmic derivative lemma and its difference analogue play a key role in the study of non-linear equations. The logarithmic derivative lemma is valid for all meromorphic functions. While the difference analogue of the logarithmic derivative lemma is valid for meromorphic functions with finite order or hyper-order less than one (see [2,4]). So for non-linear differential equations, there is no need to restrict the order of growth of entire (or meromorphic) solutions, see, e.g., Theorems A and B below. For non-linear difference or differential–difference equations, only the entire (or meromorphic) solutions with finite order or hyper-order less than one were discussed, see, e.g., Theorems C, D and E below.

Theorem A ([11]) *Let $n \geq 4$ be an integer and $P_d(f)$ denote an algebraic differential polynomial in $f(z)$ of degree $d \leq n-3$. If $p_1(z)$, $p_2(z)$ are two non-zero polynomials and α_1, α_2 are two non-zero constants such that $\frac{\alpha_1}{\alpha_2}$ is not rational, then the equation*

$$f(z)^n + P_d(f) = p_1(z)e^{\alpha_1 z} + p_2(z)e^{\alpha_2 z} \tag{1.1}$$

has no transcendental entire solution.

Theorem B ([14]) *Let $n \geq 3$ be an integer and $P_d(f)$ denote an algebraic differential polynomial in f(z) of degree $d \leq n-2$. If $p_1(z)$, $p_2(z)$ are two non-zero polynomials and α_1, α_2 are two non-zero constants such that $\frac{\alpha_1}{\alpha_2} \neq (\frac{\mathrm{d}}{n})^{\pm 1}, 1$. Then any transcendental entire solution $f(z)$ of the Eq. (1.1) satisfies that $\Theta(0, f) = 0$.*

Theorem C ([13]) *Let $p(z)$, $q(z)$ be polynomials. Then a non-linear difference equation*

$$f(z)^2 + q(z)f(z+1) = p(z)$$

has no transcendental entire solution of finite order.

Theorem D ([13]) *A non-linear difference equation*

$$f(z)^3 + q(z)f(z+1) = c \sin bz,$$

where $q(z)$ is a non-constant polynomial and $b, c \in \mathbb{C}$ are the non-zero constants, does not admit entire solutions of finite order. If $q(z) = q$ is a non-zero constant, then this equation possesses three distinct entire solutions of finite order, provided $b = 3\pi n$ and $q^3 = (-1)^{n+1}\frac{27}{4}c^2$ for a non-zero integer n.

Theorem E ([14]) *Let $n \geq 4$ be an integer and $P_d(f)$ denote an algebraic differential–difference polynomial in $f(z)$ of degree $d \leq n-3$. If $p_1(z)$, $p_2(z)$ are two non-zero polynomials and α_1, α_2 are two non-zero constants with $\frac{\alpha_1}{\alpha_2} \neq (\frac{\mathrm{d}}{n})^{\pm 1}, 1$, then the Eq.(1.1) does not have any transcendental entire solution of finite order.*

In Theorems A, B and E, the right-hand side of the Eq. (1.1) has only two terms. Thus, a natural question is: What can be said if the right-hand side of (1.1) is replaced by $s(\geq 1)$ terms? For the Eq. (1.1), the basic idea is to eliminate $e^{\alpha_1 z}$ and $e^{\alpha_2 z}$ by differentiating both sides of (1.1). When $p_1(z)e^{\alpha_1 z}$ and $p_2(z)e^{\alpha_2 z}$ are replaced by $\beta_1 e^{\alpha_1 z}, \beta_2 e^{\alpha_2 z}, \ldots, \beta_s e^{\alpha_s z}$, if we wish to use the same idea, we will be faced with complicated calculations, which will make the investigations of this problem difficult. In this paper, by combining the Nevanlinna value distribution theory and the theory of linear algebra, we investigate a certain type of non-linear difference equations, where the difference polynomials take the special form $p(z)f(z+\eta)$ as in Theorems C and D. We discuss meromorphic solutions instead of entire solutions.

Theorem 1.1 *Let $n \geq 2+s$ be an integer, $p(z) \not\equiv 0$ be a polynomial, η be a constant, $\beta_1, \beta_2, \ldots, \beta_s$ be non-zero constants and $\alpha_1, \alpha_2, \ldots, \alpha_s$ be distinct non-zero constants. Suppose that $\frac{\alpha_i}{\alpha_j} \neq n$ for all $i, j \in \{1, 2, \ldots, s\}$. And when $s \geq 5$, suppose further that $n\alpha_k \neq l_{k1}\alpha_1 + l_{k2}\alpha_2 + \cdots + l_{ks}\alpha_s$ for $k = 5, 6, \ldots, s$, where $l_{k1}, l_{k2}, \ldots, l_{ks} \in \{0, 1, \ldots, n-1\}$ and $l_{k1} + l_{k2} + \cdots + l_{ks} = n$. Then any meromorphic solution $f(z)$ of the equation*

$$f(z)^n + p(z)f(z+\eta) = \beta_1 e^{\alpha_1 z} + \beta_2 e^{\alpha_2 z} + \cdots + \beta_s e^{\alpha_s z} \tag{1.2}$$

must satisfy $\sigma_2(f) \geq 1$.

Remark 1.1 (1) Similar to the proof of Theorem 1.1, using the Clunie lemma of differential polynomials, we can easily show that (1.2) has no meromorphic solutions if $\eta = 0$.
(2) Example 1.1 below shows that the condition "$n \geq 2 + s$" is necessary.

Example 1.1 The difference equation

$$f(z)^5 - 10f(z + 10\pi i) = 5e^{\frac{3}{5}z} + 5e^{-\frac{3}{5}z} + e^z + e^{-z}$$

has an entire solution $f(z) = e^{\frac{1}{5}z} + e^{-\frac{1}{5}z}$.

From Theorem 1.1, we can easily get the following corollary.

Corollary 1.1 *Let* $1 \leq t \leq 4$ *be an integer,* $n \geq 2 + t$ *be an integer,* $p(z) \not\equiv 0$ *be a polynomial,* η *be a constant,* $\beta_1, \beta_2, \ldots, \beta_t$ *be non-zero constants and* $\alpha_1, \alpha_2, \ldots, \alpha_t$ *be distinct non-zero constants. Suppose that* $\frac{\alpha_i}{\alpha_j} \neq n$ *for all* $i, j \in \{1, 2, \ldots, t\}$. *Then any meromorphic solution, namely* $f(z)$ *of the equation*

$$f(z)^n + p(z)f(z+\eta) = \beta_1 e^{\alpha_1 z} + \beta_2 e^{\alpha_2 z} + \cdots + \beta_t e^{\alpha_t z}$$

must satisfy $\sigma_2(f) \geq 1$.

In Theorem 1.1, we discuss the case $n \geq 2+s$. Now, a natural question is: What can be said if $n \leq 1 + s$? We investigate this problem and get Theorem 1.2 and Remark 1.3.

Theorem 1.2 *Let* $n = 1 + s$ *be an integer,* $p(z) \not\equiv 0$ *be a polynomial,* η *be a constant,* $\beta_1, \beta_2, \ldots, \beta_s$ *be non-zero constants and* $\alpha_1, \alpha_2, \ldots, \alpha_s$ *be distinct non-zero constants. Suppose that* $\frac{\alpha_i}{\alpha_j} \neq n$ *for all* $i, j \in \{1, 2, \ldots, s\}$. *And when* $s \geq 5$, *suppose further that* $n\alpha_k \neq l_{k1}\alpha_1 + l_{k2}\alpha_2 + \cdots + l_{ks}\alpha_s$ *for* $k = 5, 6, \ldots, s$, *where* $l_{k1}, l_{k2}, \ldots, l_{ks} \in \{0, 1, \ldots, n-1\}$ *and* $l_{k1} + l_{k2} + \cdots + l_{ks} = n$. *Then any meromorphic solution* $f(z)$ *with* $\sigma_2(f) < 1$ *of the Eq.* (1.2) *must be an entire function and satisfy* $\Theta(0, f) = 0$ *and* $\sigma(f) = 1$.

From Theorem 1.2, we can easily get the following corollary.

Corollary 1.2 *Let* $1 \leq t \leq 4$ *be an integer,* $p(z) \not\equiv 0$ *be a polynomial,* η *be a constant,* $\beta_1, \beta_2, \ldots, \beta_t$ *be non-zero constants and* $\alpha_1, \alpha_2, \ldots, \alpha_t$ *be distinct non-zero constants. Suppose that* $\frac{\alpha_i}{\alpha_j} \neq n$ *for all* $i, j \in \{1, 2, \ldots, t\}$. *Then any meromorphic solution* $f(z)$ *with* $\sigma_2(f) < 1$ *of the equation*

$$f(z)^{t+1} + p(z)f(z+\eta) = \beta_1 e^{\alpha_1 z} + \beta_2 e^{\alpha_2 z} + \cdots + \beta_t e^{\alpha_t z}$$

must be an entire function and satisfy $\Theta(0, f) = 0$ *and* $\sigma(f) = 1$.

Example 1.2 Consider the difference equation

$$f(z)^3 - 3f(z+6\pi i) = e^z + e^{-z},$$

where $n = 3, s = 2, \alpha_1 = 1$ and $\alpha_2 = -1$. We see that this equation satisfies all hypotheses of Theorem 1.2. A simple calculation shows that $f(z) = e^{\frac{z}{3}} + e^{-\frac{z}{3}}$ is a solution of this equation and $\Theta(0, f) = 0$ and $\sigma(f) = 1$.

Remark 1.2 Example 1.2 shows that the case $\Theta(0, f) = 0$ and $\sigma(f) = 1$ in Theorem 1.2 does exist. Example 1.3 below is one more example, where $s = 6$.

Example 1.3 Consider the difference equation

$$f(z)^7 + 35f(z+\pi i) = 21e^{3z} + 21e^{-3z} + 7e^{5z} + 7e^{-5z} + e^{7z} + e^{-7z},$$

where $n = 7, s = 6, \alpha_1 = 3, \alpha_2 = -3, \alpha_3 = 5, \alpha_4 = -5, \alpha_5 = 7$ and $\alpha_6 = -7$. We see that this equation satisfies all hypotheses of Theorem 1.2. A simple calculation shows that $f(z) = e^z + e^{-z}$ is a solution of this equation and $\Theta(0, f) = 0$ and $\sigma(f) = 1$.

Remark 1.3 The following example shows that the conclusions in Theorem 1.2 may not hold, if $n < s + 1$.

Example 1.4 The difference equation

$$f(z)^2 - f(z+\pi i) = e^z + 2e^{3z} + e^{4z}$$

has an entire solution $f(z) = e^{2z} + e^z$. We see that $\Theta(0, f) = \frac{1}{2} \neq 0$.

From Example 1.4, we see that $f(z) = e^{2z} + e^z$ has infinitely many zeros and $\lambda(f) = \sigma(f) = 1$, though $\Theta(0, f) \neq 0$. In this direction, we prove the following theorem.

Theorem 1.3 *Let $n \geq 2$ be an integer, $p(z) \not\equiv 0$ be a polynomial, η be a constant, $\beta_1, \beta_2, \ldots, \beta_s$ be non-zero constants and $\alpha_1, \alpha_2, \ldots, \alpha_s$ be distinct non-zero constants. Suppose that $\frac{\alpha_i}{\alpha_j} \neq n$ for all $i, j \in \{1, 2, \ldots, s\}$. Then any meromorphic solution $f(z)$ with $\sigma_2(f) < 1$ of the Eq. (1.2) must be an entire function and satisfy $\lambda(f) = \sigma(f) = 1$.*

2 Proof of Theorem 1.1

To prove Theorem 1.1, we need the following lemmas. The first of these lemmas is a version of the difference analogue of the logarithmic derivative lemma.

Lemma 2.1 ([4]) *Let $f(z)$ be a non-constant meromorphic function and $c \in \mathbb{C}$. If $\sigma_2(f) < 1$ and $\varepsilon > 0$, then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r, f)}{r^{1-\sigma_2(f)-\varepsilon}}\right)$$

for all r outside a set of finite logarithmic measure.

Laine-Yang [8] gave a difference analogue of Clunie lemma as follows.

Lemma 2.2 ([8]) *Let $f(z)$ be a transcendental meromorphic solution of finite order ρ of a difference equation of the form*

$$U(z, f)P(z, f) = Q(z, f),$$

where $U(z, f)$, $P(z, f)$, $Q(z, f)$ are difference polynomials such that the total degree of $U(z, f)$ in $f(z)$ and its shifts is n, and that the total degree of $Q(z, f)$ is at most n. Moreover, we assume that $U(z, f)$ contains just one term of maximal total degree in $f(z)$ and its shifts. Then, for each $\varepsilon > 0$,

$$m(r, P(z, f)) = O(r^{\rho-1+\varepsilon}) + o(T(r, f)),$$

possibly outside an exceptional set of finite logarithmic measure.

Remark 2.1 In the proof of Lemma 2.2, Laine-Yang used a version of difference analogue of the logarithmic derivative lemma due to Chiang-Feng [2]: Let η_1, η_2 be two complex numbers such that $\eta_1 \neq \eta_2$, let $f(z)$ be a finite order meromorphic function, and let ρ be the order of $f(z)$, then for each $\varepsilon > 0$,

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O(r^{\rho-1+\varepsilon}).$$

Applying Lemma 2.1, logarithmic derivative lemma and $\frac{f^{(k)}(z+\delta)}{f(z)} = \frac{f^{(k)}(z+\delta)}{f^{(k)}(z)}\frac{f^{(k)}(z)}{f(z)}$ $(\delta \in \mathbb{C}/\{0\})$ to the proof of Lemma 2.2, we can get

$$m\big(r, P(z, f)\big) = S(r, f)$$

when the hyper-order of $f(z)$ is less than 1 and $P(z, f)$ and $Q(z, f)$ are the differential–difference polynomials in $f(z)$.

The following lemma is a generalization of Borel's theorem on linear combinations of entire functions.

Lemma 2.3 ([3, pp. 69–70] or [12, p. 82]) *Suppose that $f_1(z), f_2(z), \ldots, f_n(z)$ are meromorphic functions and that $g_1(z), g_2(z), \ldots, g_n(z)$ are entire functions satisfying the following conditions.*

(1) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;
(2) $g_j(z) - g_k(z)$ *are not constants for* $1 \leq j < k \leq n$;
(3) *for* $1 \leq j \leq n$, $1 \leq h < k \leq n$,

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\} \quad (r \to \infty,\ r \notin E),$$

where $E \subset (1, \infty)$ is of finite linear measure or finite logarithmic measure.

Then $f_j(z) \equiv 0$ $(j = 1, 2, \ldots, n)$.

To state the following lemma, we introduce some notation. The determinant

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ \cdots & \cdots & \cdots & \cdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{vmatrix}$$

is called the principal Vandermondian and is denoted by V_{n0}. For every $k = 1, 2, \ldots, n-1$, the determinant

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ \cdots & \cdots & \cdots & \cdots \\ a_1^{n-k-1} & a_2^{n-k-1} & \cdots & a_n^{n-k-1} \\ a_1^{n-k+1} & a_2^{n-k+1} & \cdots & a_n^{n-k+1} \\ \cdots & \cdots & \cdots & \cdots \\ a_1^{n} & a_2^{n} & \cdots & a_n^{n} \end{vmatrix}$$

is called the secondary Vandermondian and is denoted by V_{nk}. For V_{n0}, we have

$$V_{n0} = \prod_{1\le j<i\le n} (a_i - a_j).$$

For the relationship of V_{n0} and V_{nk}, we have the following lemma.

Lemma 2.4 ([6]) *The elementary symmetric function $E_i \equiv \sum a_1 a_2 \ldots a_i$ of the n variables $a_1, a_2, \ldots, a_n$ is equal to the quotient of the secondary Vandermondian V_{ni} by the principal Vandermondian V_{n0}.*

In the next lemma, the elementary row transformations consist of the following: (i) switch two rows; (ii) multiply a row by a non-zero number; (iii) replace a row by a multiple of another row added to it.

Lemma 2.5 *Let $n \ge 2$, $s \ge 1$ be integers, $\alpha_1, \alpha_2, \ldots, \alpha_s$ be non-zero constants, d_1, $d_2, \ldots, d_s$ be constants and c_1, c_2, c_3, c_4 be rational functions. For all $i = 1, 2, 3, 4$, suppose that α_i are distinct non-zero constants and that $n\alpha_i \ne \alpha_p$ ($p = 1, 2, \ldots, s$). If*

$$(c_1e^{\alpha_1 z} + c_2e^{\alpha_2 z} + c_3e^{\alpha_3 z} + c_4e^{\alpha_4 z})^n = d_1e^{\alpha_1 z} + d_2e^{\alpha_2 z} + \cdots + d_se^{\alpha_s z}, \quad (2.1)$$

then $c_1 \equiv c_2 \equiv c_3 \equiv c_4 \equiv 0$.

Proof of Lemma 2.5. We deduce from (2.1) that

$$\begin{aligned} d_1e^{\alpha_1 z} &+ d_2e^{\alpha_2 z} + \cdots + d_se^{\alpha_s z} \\ &= c_1^n e^{n\alpha_1 z} + c_2^n e^{n\alpha_2 z} + c_3^n e^{n\alpha_3 z} + c_4^n e^{n\alpha_4 z} \\ &\quad + \sum_{(m_1,m_2,m_3,m_4)} c_{m_1,m_2,m_3,m_4} e^{(m_1\alpha_1+m_2\alpha_2+m_3\alpha_3+m_4\alpha_4)z}, \end{aligned} \quad (2.2)$$

where c_{m_1,m_2,m_3,m_4} are rational functions and the sum $\sum_{(m_1,m_2,m_3,m_4)}$ is carried out such that $m_j \in \{0, 1, \dots, n-1\}(j = 1, 2, 3, 4)$ and $m_1+m_2+m_3+m_4 = n$. Suppose that there exists $b_{ij} \in \{0, 1, \dots, n-1\}(i, j = 1, 2, 3, 4)$ with $b_{i1}+b_{i2}+b_{i3}+b_{i4} = n(i = 1, 2, 3, 4)$, such that

$$\begin{cases} n\alpha_1 = b_{11}\alpha_1 + b_{12}\alpha_2 + b_{13}\alpha_3 + b_{14}\alpha_4, \\ n\alpha_2 = b_{21}\alpha_1 + b_{22}\alpha_2 + b_{23}\alpha_3 + b_{24}\alpha_4, \\ n\alpha_3 = b_{31}\alpha_1 + b_{32}\alpha_2 + b_{33}\alpha_3 + b_{34}\alpha_4, \\ n\alpha_4 = b_{41}\alpha_1 + b_{42}\alpha_2 + b_{43}\alpha_3 + b_{44}\alpha_4. \end{cases} \tag{2.3}$$

(2.3) can be seen as a system of linear equations of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. We will deduce a contradiction. □

Obviously, $b_{ij} \geq 0(i, j = 1, 2, 3, 4)$. For $i = 1, 2, 3, 4$, if only one of $b_{ij}(j = 1, 2, 3, 4$ and $j \neq i)$ is greater than zero, we deduce a contradiction immediately. In fact, without loss of generality, we may assume that $b_{21} > 0$ and $b_{23} = b_{24} = 0$. Then by the second equation of the system (2.3), we get $(n - b_{22})\alpha_2 = b_{21}\alpha_1$. Since $b_{21} + b_{22} = n$ and $b_{21} \neq 0$, we get $\alpha_1 = \alpha_2$, a contradiction.

Now we assume that, for all $i = 1, 2, 3, 4$, at least two of $b_{ij}(j = 1, 2, 3, 4$ and $j \neq i)$ are greater than zero. Then for all $i = 1, 2, 3, 4$, we have $b_{ij} < n - b_{ii}(j \neq i)$. The system (2.3) can be written as

$$\begin{cases} (b_{11} - n)\alpha_1 + b_{12}\alpha_2 + b_{13}\alpha_3 + b_{14}\alpha_4 = 0, \\ b_{21}\alpha_1 + (b_{22} - n)\alpha_2 + b_{23}\alpha_3 + b_{24}\alpha_4 = 0, \\ b_{31}\alpha_1 + b_{32}\alpha_2 + (b_{33} - n)\alpha_3 + b_{34}\alpha_4 = 0, \\ b_{41}\alpha_1 + b_{42}\alpha_2 + b_{43}\alpha_3 + (b_{44} - n)\alpha_4 = 0. \end{cases} \tag{2.4}$$

Denote the coefficient matrix of the system (2.4) by

$$A = \begin{pmatrix} b_{11} - n & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} - n & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} - n & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} - n \end{pmatrix}.$$

Next we discuss the rank of the matrix A. Adding columns 2, 3 and 4 to column 1, and noting that $b_{i1} + b_{i2} + b_{i3} + b_{i4} = n(i = 1, 2, 3, 4)$, we see that $\det(A) = 0$. Now we discuss the determinants of 3×3 submatrices of the matrix A. To this end, we divide our discussion into two cases.

Case 1. $b_{13} = b_{23} = b_{43} = 0$. Noting that for all $i = 1, 2, 3, 4$, at least two of $b_{ij}(j = 1, 2, 3, 4$ and $j \neq i)$ are greater than zero, we see that $b_{12} > 0$, $b_{14} > 0$, $b_{21} > 0$, $b_{24} > 0$, $b_{41} > 0$, $b_{42} > 0$. For the 3×3 submatrix

$$A_1 = \begin{pmatrix} b_{11} - n & b_{12} & b_{14} \\ b_{31} & b_{32} & b_{34} \\ b_{41} & b_{42} & b_{44} - n \end{pmatrix}$$

of the matrix A, we have

$$\det(A_1) = (b_{11} - n)b_{32}(b_{44} - n) + b_{12}b_{34}b_{41} + b_{14}b_{31}b_{42} \\ - b_{14}b_{32}b_{41} - b_{12}b_{31}(b_{44} - n) - (b_{11} - n)b_{34}b_{42}.$$

Since for all $i = 1, 2, 3, 4$, $b_{ij} < n - b_{ii} (j \neq i)$, we have $b_{14} < n - b_{11}$ and $b_{41} < n - b_{44}$. If $b_{32} > 0$, then $b_{14}b_{32}b_{41} < (n - b_{11})(n - b_{44})b_{32}$. So

$$\det(A_1) > b_{12}b_{34}b_{41} + b_{14}b_{31}b_{42} - b_{12}b_{31}(b_{44} - n) - (b_{11} - n)b_{34}b_{42} \geq 0.$$

If $b_{32} = 0$, then $b_{31} > 0$, $b_{34} > 0$ and so

$$\det(A_1) = b_{12}b_{34}b_{41} + b_{14}b_{31}b_{42} - b_{12}b_{31}(b_{44} - n) - (b_{11} - n)b_{34}b_{42} > 0.$$

Thus, we proved $\det(A_1) > 0$ in this case.

Case 2. At least one of b_{13}, b_{23}, b_{43} is greater than zero. For the 3×3 submatrix

$$A_2 = \begin{pmatrix} b_{11} - n & b_{12} & b_{13} \\ b_{21} & b_{22} - n & b_{23} \\ b_{41} & b_{42} & b_{43} \end{pmatrix}$$

of the matrix A, we have

$$\det(A_2) = (b_{11} - n)(b_{22} - n)b_{43} + b_{12}b_{23}b_{41} + b_{13}b_{21}b_{42} \\ - b_{13}(b_{22} - n)b_{41} - b_{12}b_{21}b_{43} - (b_{11} - n)b_{23}b_{42}.$$

If $b_{43} > 0$, then by $b_{12} < n - b_{11}$ and $b_{21} < n - b_{22}$, we have $b_{12}b_{21}b_{43} < (n - b_{11})(n - b_{22})b_{43}$. So

$$\det(A_2) > b_{12}b_{23}b_{41} + b_{13}b_{21}b_{42} - b_{13}(b_{22} - n)b_{41} - (b_{11} - n)b_{23}b_{42} \geq 0.$$

If $b_{43} = 0$, then $b_{41} > 0$, $b_{42} > 0$ and at least one of b_{13}, b_{23} is greater than zero. So

$$\det(A_2) = b_{12}b_{23}b_{41} + b_{13}b_{21}b_{42} - b_{13}(b_{22} - n)b_{41} - (b_{11} - n)b_{23}b_{42} > 0.$$

Thus, we proved $\det(A_2) > 0$ in this case.

Since $\det(A) = 0$ and there exists a 3×3 submatrix of matrix A with non-zero determinant, we see that the rank of the matrix A is 3. So by elementary row transformations, we deduce that the matrix A becomes

$$B = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (2.5)$$

(2.4) and (2.5) give $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$, a contradiction. So (2.3) does not hold.
Without loss of generality, we may assume that

$$n\alpha_4 \neq m_1\alpha_1 + m_2\alpha_2 + m_3\alpha_3 + m_4\alpha_4$$

for all $m_1, m_2, m_3, m_4 \in \{0, 1, \ldots, n-1\}$ such that $m_1 + m_2 + m_3 + m_4 = n$. Since $n\alpha_4 \neq \alpha_p (p = 1, 2, \ldots, s)$ and $n\alpha_4 \neq n\alpha_q (q = 1, 2, 3)$, by (2.2) and Lemma 2.3, we get $c_4 \equiv 0$. So the Eq. (2.1) becomes

$$(c_1 e^{\alpha_1 z} + c_2 e^{\alpha_2 z} + c_3 e^{\alpha_3 z})^n = d_1 e^{\alpha_1 z} + d_2 e^{\alpha_2 z} + \cdots + d_s e^{\alpha_s z}.$$

Using a similar proof as in (2.2)–(2.5), we get $c_3 \equiv 0$ and the equation (2.1) becomes

$$(c_1 e^{\alpha_1 z} + c_2 e^{\alpha_2 z})^n = d_1 e^{\alpha_1 z} + d_2 e^{\alpha_2 z} + \cdots + d_s e^{\alpha_s z}. \tag{2.6}$$

By (2.6), we get

$$\begin{aligned} d_1 e^{\alpha_1 z} + d_2 e^{\alpha_2 z} + \cdots + d_s e^{\alpha_s z} = c_1^n e^{n\alpha_1 z} + c_2^n e^{n\alpha_2 z} \\ + \sum_{j=1}^{n-1} \binom{n}{j} c_1^j c_2^{n-j} e^{(j\alpha_1 + (n-j)\alpha_2)z}, \end{aligned} \tag{2.7}$$

where $\binom{n}{j}$ are the binomial coefficients. Since $\alpha_1 \neq \alpha_2$, we see that for $j = 1, 2, \ldots, n-1$,

$$n\alpha_1 \neq j\alpha_1 + (n-j)\alpha_2, \quad n\alpha_2 \neq j\alpha_1 + (n-j)\alpha_2.$$

So by (2.7) and Lemma 2.3, we get $c_1 \equiv c_2 \equiv 0$.

Proof of Theorem 1.1. Suppose that the Eq. (1.2) has a meromorphic solution $f(z)$ with $\sigma_2(f) < 1$. We will deduce a contradiction for the case $f(z)$ has at least one pole and the case $f(z)$ is an entire function, respectively. □

Case 1. $f(z)$ has at least one pole. In this case, if $\eta = 0$, then comparing the orders of poles of both sides of (1.2), we immediately get a contradiction. If $\eta \neq 0$, then suppose that z_0 is a pole of $f(z)$ with order q. We deduce from (1.2) that $z_0 + \eta$ is a pole of $f(z)$ with order at least nq. Substituting $z_0 + \eta$ for z in (1.2), we obtain

$$\begin{aligned} f(z_0+\eta)^n + p(z_0+\eta) f(z_0+2\eta) = \beta_1 e^{\alpha_1 (z_0+\eta)} + \beta_2 e^{\alpha_2 (z_0+\eta)} \\ + \cdots + \beta_s e^{\alpha_s (z_0+\eta)}. \end{aligned} \tag{2.8}$$

Since $z_0 + \eta$ is a pole of $f(z)^n$ with order at least $n^2 q$, we see from (2.8) that $z_0 + 2\eta$ is a pole of $f(z)$ with order at least $n^2 q$. Following the steps above, we will find a sequence $\{z_0 + j\eta\}_{j=0}^{\infty}$ of poles of $f(z)$ with order at least $n^j q$, respectively. So for $m = 1, 2, \ldots$, we have

$$n(m|\eta| + |z_0| + 1, f(z)) \geq q + nq + \cdots + n^m q.$$

Furthermore, $n \geq 2 + s \geq 3$. Thus,

$$\begin{aligned}\lambda_2\left(\frac{1}{f(z)}\right) &= \overline{\lim_{r\to\infty}} \frac{\log\log n(r, f(z))}{\log r} \\ &\geq \overline{\lim_{m\to\infty}} \frac{\log\log n(m|\eta| + |z_0| + 1, f(z))}{\log(m|\eta| + |z_0| + 1)} \\ &\geq \overline{\lim_{m\to\infty}} \frac{\log\log n^m}{\log m} = 1.\end{aligned}$$

This contradicts $\sigma_2(f) < 1$. So the Eq. (1.2) does not have any meromorphic solution of hyper-order less than one with at least one pole.

Case 2. $f(z)$ is an entire function. If $f(z)$ is a polynomial, then comparing both sides of equation (1.2), we obtain that the order of growth of the left side is 0, while the order of growth of the right side is 1. This is impossible. So $f(z)$ is transcendental. Since $s \geq 1$, we divide our discussion into two subcases: $s = 1$ and $s > 1$.

Subcase 2.1. $s = 1$. The Eq. (1.2) becomes

$$f(z)^n + p(z)f(z+\eta) = \beta_1 e^{\alpha_1 z}. \tag{2.9}$$

Differentiating both sides of (2.9), we get

$$nf(z)^{n-1}f'(z) + (p(z)f(z+\eta))' = \alpha_1\beta_1 e^{\alpha_1 z}.$$

Combining this equation with (2.9), we get

$$f(z)^{n-1}(nf'(z) - \alpha_1 f(z)) = \alpha_1 p(z)f(z+\eta) - (p(z)f(z+\eta))'. \tag{2.10}$$

If $nf'(z) - \alpha_1 f(z) \not\equiv 0$, then we deduce from (2.10), $n \geq 2 + s = 3$, Lemma 2.2 and Remark 2.1 that

$$T(r, nf'(z) - \alpha_1 f(z)) = m(r, nf'(z) - \alpha_1 f(z)) = S(r, f), \tag{2.11}$$

$$T(r, f(z)(nf'(z) - \alpha_1 f(z))) = m(r, f(z)(nf'(z) - \alpha_1 f(z))) = S(r, f). \tag{2.12}$$

Combining (2.11) with (2.12), we get

$$T(r, f(z)) \leq T(r, f(z)(nf'(z) - \alpha_1 f(z))) + T\left(r, \frac{1}{nf'(z) - \alpha_1 f(z)}\right) = S(r, f),$$

a contradiction.

If $nf'(z) - \alpha_1 f(z) \equiv 0$, then $f(z) = ce^{\frac{\alpha_1}{n}z}$, where c is a constant. Substituting $f(z) = ce^{\frac{\alpha_1}{n}z}$ into the Eq. (1.2), we get

$$c^n e^{\alpha_1 z} + p(z)ce^{\frac{\alpha_1}{n}\eta}e^{\frac{\alpha_1}{n}z} = \beta_1 e^{\alpha_1 z}. \tag{2.13}$$

By (2.13) and Lemma 2.3, we get $p(z)ce^{\frac{\alpha_1}{n}\eta} \equiv 0$. So $c = 0$ and $f(z) \equiv 0$, a contradiction. Therefore, we proved that the Eq. (1.2) does not have any entire solution of hyper-order less than one when $s = 1$.

Subcase 2.2. $s > 1$. Set $F = f(z)^n + p(z)f(z+\eta)$. Then the equation (1.2) becomes

$$F = \beta_1 e^{\alpha_1 z} + \beta_2 e^{\alpha_2 z} + \cdots + \beta_s e^{\alpha_s z}. \tag{2.14}$$

Differentiating both sides of the Eq. (2.14) $s-1$ times, we get

$$\begin{aligned} F' &= \alpha_1\beta_1 e^{\alpha_1 z} + \alpha_2\beta_2 e^{\alpha_2 z} + \cdots + \alpha_s\beta_s e^{\alpha_s z}, \\ F'' &= \alpha_1^2\beta_1 e^{\alpha_1 z} + \alpha_2^2\beta_2 e^{\alpha_2 z} + \cdots + \alpha_s^2\beta_s e^{\alpha_s z}, \\ &\cdots, \\ F^{(s-1)} &= \alpha_1^{s-1}\beta_1 e^{\alpha_1 z} + \alpha_2^{s-1}\beta_2 e^{\alpha_2 z} + \cdots + \alpha_s^{s-1}\beta_s e^{\alpha_s z}. \end{aligned}$$

Combining these equations with (2.14) and using Cramer's Rule, we get

$$\beta_1 e^{\alpha_1 z} = \frac{D_1}{D},$$

where

$$D_1 = \begin{vmatrix} F & 1 & \cdots & 1 \\ F' & \alpha_2 & \cdots & \alpha_s \\ F'' & \alpha_2^2 & \cdots & \alpha_s^2 \\ \cdots & \cdots & \cdots & \cdots \\ F^{(s-1)} & \alpha_2^{s-1} & \cdots & \alpha_s^{s-1} \end{vmatrix},$$

$$D = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_s \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_s^2 \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_1^{s-1} & \alpha_2^{s-1} & \cdots & \alpha_s^{s-1} \end{vmatrix}. \tag{2.15}$$

Obviously, D is a principal Vandermondian with order s. Since $\alpha_1, \alpha_2, \ldots, \alpha_s$ are distinct constants, we get

$$D = \prod_{1 \le j < i \le s} (\alpha_i - \alpha_j) \neq 0.$$

By expanding determinant (2.15) along column 1, we get

$$\beta_1 e^{\alpha_1 z} = \frac{1}{D}((-1)^{s+1} M_{s1} F^{(s-1)} + \cdots + (-1)^{s-j+1} M_{s-j,1} F^{(s-j-1)} + \cdots + M_{11} F), \tag{2.16}$$

where $M_{s-j,1} (j = 0, 1, \ldots, s-1)$ is the determinant formed by throwing away column 1 and row $s-j$ from the determinant (2.15), i.e.,

$$M_{s1} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha_2 & \alpha_3 & \cdots & \alpha_s \\ \alpha_2^2 & \alpha_3^2 & \cdots & \alpha_s^2 \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_2^{s-2} & \alpha_3^{s-2} & \cdots & \alpha_s^{s-2} \end{vmatrix}, \tag{2.17}$$

$$M_{s-j,1} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha_2 & \alpha_3 & \cdots & \alpha_s \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_2^{s-j-2} & \alpha_3^{s-j-2} & \cdots & \alpha_s^{s-j-2} \\ \alpha_2^{s-j} & \alpha_3^{s-j} & \cdots & \alpha_s^{s-j} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_2^{s-1} & \alpha_3^{s-1} & \cdots & \alpha_s^{s-1} \end{vmatrix}, \ (j = 1, \cdots, s-1). \tag{2.18}$$

From (2.17) and (2.18), we see that M_{s1} is the principal Vandermondian with variables $\alpha_2, \alpha_3, \ldots, \alpha_s$, and $M_{s-j,1} (j = 1, 2, \ldots, s-1)$ is the secondary Vandermondian with variables $\alpha_2, \alpha_3, \ldots, \alpha_s$. For $j = 1, 2, \ldots, s-1$, let

$$\sigma_j \equiv \sum \alpha_2 \alpha_3 \cdots \alpha_{j+1} \tag{2.19}$$

be the elementary symmetric function of $s-1$ variables $\alpha_2, \alpha_3, \ldots, \alpha_s$. By (2.17), (2.18) and Lemma 2.4, we get

$$\sigma_j = \frac{M_{s-j,1}}{M_{s1}}, \ (j = 1, 2, \ldots, s-1). \tag{2.20}$$

Differentiating both sides of (2.16), we get

$$\alpha_1 \beta_1 e^{\alpha_1 z} = \frac{1}{D}((-1)^{s+1} M_{s1} F^{(s)} + \cdots + (-1)^{s-j+1} M_{s-j,1} F^{(s-j)} + \cdots + M_{11} F'). \tag{2.21}$$

By eliminating $e^{\alpha_1 z}$ from (2.16) and (2.21), we get

$$(-1)^{s+1} M_{s1} F^{(s)} + ((-1)^s M_{s-1,1} - (-1)^{s+1} \alpha_1 M_{s1}) F^{(s-1)} + \cdots + ((-1)^{s-j+1} M_{s-j,1} - (-1)^{s-j} \alpha_1 M_{s-j+1,1}) F^{(s-j)} + \cdots - \alpha_1 M_{11} F = 0. \tag{2.22}$$

Let

$$\begin{aligned} L(w) = w^{(s)} &+ \frac{(-1)^s M_{s-1,1} - (-1)^{s+1}\alpha_1 M_{s1}}{(-1)^{s+1}M_{s1}} w^{(s-1)} + \cdots \\ &+ \frac{(-1)^{s-j+1} M_{s-j,1} - (-1)^{s-j}\alpha_1 M_{s-j+1,1}}{(-1)^{s+1}M_{s1}} w^{(s-j)} \\ &+ \cdots - \frac{\alpha_1 M_{11}}{(-1)^{s+1}M_{s1}} w \end{aligned} \tag{2.23}$$

be a linear differential operator. Since $F = f(z)^n + p(z)f(z+\eta)$, we deduce from (2.22) and (2.23) that

$$L(f(z)^n) = -L(p(z)f(z+\eta)). \tag{2.24}$$

For $j = 1, 2, \ldots, s$, let

$$\tau_j \equiv \sum \alpha_1\alpha_2 \ldots \alpha_j \tag{2.25}$$

be the elementary symmetric function of s variables $\alpha_1, \alpha_2, \ldots, \alpha_s$. We deduce from (2.19), (2.20) and (2.25) that

$$\begin{aligned} \frac{(-1)^s M_{s-1,1} - (-1)^{s+1}\alpha_1 M_{s1}}{(-1)^{s+1}M_{s1}} &= -\sigma_1 - \alpha_1 \\ &= -(\alpha_1 + \alpha_2 + \cdots + \alpha_s) = -\tau_1, \\ \frac{(-1)^{s-j+1} M_{s-j,1} - (-1)^{s-j}\alpha_1 M_{s-j+1,1}}{(-1)^{s+1}M_{s1}} &= (-1)^j\sigma_j + (-1)^j\alpha_1\sigma_{j-1} \\ &= (-1)^j\tau_j, (j = 2, 3, \ldots, s-1), \end{aligned}$$

and

$$-\frac{\alpha_1 M_{11}}{(-1)^{s+1}M_{s1}} = (-1)^s\alpha_1\sigma_{s-1} = (-1)^s(\alpha_1\alpha_2 \ldots \alpha_s) = (-1)^s\tau_s.$$

So $L(w)$ becomes

$$L(w) = w^{(s)} - \tau_1 w^{(s-1)} + \cdots + (-1)^j\tau_j w^{(s-j)} + \cdots + (-1)^s\tau_s w. \tag{2.26}$$

Since

$$\begin{aligned} &(f(z)^n)\prime = nf(z)^{n-1}f'(z), \\ &(f(z)^n)'' = n(n-1)f(z)^{n-2}(f'(z))^2 + nf(z)^{n-1}f''(z), \end{aligned}$$

we deduce inductively that, for $m = 1, 2, \ldots, s$,

$$\begin{aligned}(f(z)^n)^{(m)} &= n(n-1)\ldots(n-(m-1))f(z)^{n-m}(f'(z))^m \\ &\quad + \sum_{j=2}^{m-1}\sum_{\lambda}\gamma_{j\lambda}f(z)^{n-j}(f'(z))^{\lambda_{j1}}(f''(z))^{\lambda_{j2}}\cdots(f^{(m-1)}(z))^{\lambda_{j,m-1}} \\ &\quad + nf(z)^{n-1}f^{(m)}(z), \end{aligned} \tag{2.27}$$

where $\gamma_{j\lambda}$ are the positive integers, $\lambda_{j1}, \lambda_{j2}, \ldots, \lambda_{j,m-1}$ are the non-negative integers and the sum $\sum_{\lambda}$ is carried out such that $\lambda_{j1}+\lambda_{j2}+\cdots+\lambda_{j,m-1} = j$ and $\lambda_{j1}+2\lambda_{j2}+\cdots+(m-1)\lambda_{j,m-1} = m$. By (2.26) and (2.27), we get

$$L(f(z)^n) = f(z)^{n-s}\phi, \tag{2.28}$$

where ϕ is a differential polynomial in $f(z)$ of degree s with constant coefficients. By (2.24), (2.26) and (2.28), we get

$$f(z)^{n-s}\phi = -L(p(z)f(z+\eta)), \tag{2.29}$$

where $L(p(z)f(z+\eta))$ is a differential–difference polynomial in $f(z)$ of degree 1 with polynomial coefficients.

If $\phi \not\equiv 0$, then by (2.29), $n \geq s+2$, Lemma 2.2 and Remark 2.1, we get

$$\begin{aligned} T(r,\phi) &= m(r,\phi) = S(r,f), \\ T(r,f(z)\phi) &= m(r,f(z)\phi) = S(r,f). \end{aligned} \tag{2.30}$$

The above two equalities give

$$T(r,f(z)) \leq T(r,f(z)\phi) + T\left(r,\frac{1}{\phi}\right) = S(r,f),$$

a contradiction. So we must have $\phi \equiv 0$, which yields $L(f(z)^n) \equiv 0$ and $L(p(z)f(z+\eta)) \equiv 0$. By $L(p(z)f(z+\eta)) \equiv 0$ and (2.26), we get

$$\begin{aligned}(p(z)f(z+\eta))^{(s)} - \tau_1(p(z)f(z+\eta))^{(s-1)} + \cdots + (-1)^j\tau_j(p(z)f(z+\eta))^{(s-j)} \\ + \cdots + (-1)^s\tau_s p(z)f(z+\eta) \equiv 0.\end{aligned}$$

The characteristic equation of this equation is

$$\lambda^s - \tau_1\lambda^{s-1} + \cdots + (-1)^j\tau_j\lambda^{s-j} + \cdots + (-1)^s\tau_s = 0. \tag{2.31}$$

Since (2.31) has s distinct roots $\alpha_1, \alpha_2, \ldots, \alpha_s$, we see that $p(z)f(z+\eta)$ has the form

$$p(z)f(z+\eta) = \tilde{c_1}e^{\alpha_1 z} + \tilde{c_2}e^{\alpha_2 z} + \cdots + \tilde{c_s}e^{\alpha_s z},$$

where $\tilde{c_j}(j = 1, 2 \ldots, s)$ are the constants. So

$$f(z) = c_1 e^{\alpha_1 z} + c_2 e^{\alpha_2 z} + \cdots + c_s e^{\alpha_s z}, \tag{2.32}$$

where $c_j = \frac{\tilde{c_j} e^{-\alpha_j \eta}}{p(z-\eta)} (j = 1, 2 \cdots, s)$ are the rational functions.

Similarly, we deduce from $L(f(z)^n) \equiv 0$ that

$$f(z)^n = d_1 e^{\alpha_1 z} + d_2 e^{\alpha_2 z} + \cdots + d_s e^{\alpha_s z}, \tag{2.33}$$

where $d_j (j = 1, 2 \cdots, s)$ are the constants. By (2.32) and (2.33), we get

$$d_1 e^{\alpha_1 z} + d_2 e^{\alpha_2 z} + \cdots + d_s e^{\alpha_s z} = c_1^n e^{n\alpha_1 z} + c_2^n e^{n\alpha_2 z} + \cdots + c_s^n e^{n\alpha_s z} + \sum_{(m_1, \ldots, m_s)} c_{m_1, \ldots, m_s} e^{(m_1\alpha_1 + m_2\alpha_2 + \cdots + m_s\alpha_s)z}, \tag{2.34}$$

where $c_{m_1, \ldots, m_s}$ are the rational functions and the sum $\sum_{(m_1, \ldots, m_s)}$ is carried out such that $m_j \in \{0, 1, \ldots, n-1\} (j = 1, 2, \ldots, s)$ and $m_1 + m_2 + \cdots + m_s = n$. Since $\frac{\alpha_i}{\alpha_j} \neq n$ for all $i, j \in \{1, 2, \ldots, s\}$ and $n\alpha_k \neq l_{k1}\alpha_1 + l_{k2}\alpha_2 + \cdots + l_{ks}\alpha_s$ for $k = 5, 6, \ldots, s$, where $l_{k1}, l_{k2}, \ldots, l_{ks} \in \{0, 1, \ldots, n-1\}$ and $l_{k1} + l_{k2} + \cdots + l_{ks} = n$, by (2.34) and Lemma 2.3, we get

$$c_5 \equiv c_6 \equiv \cdots \equiv c_s \equiv 0.$$

So $f(z)$ becomes

$$f(z) = c_1 e^{\alpha_1 z} + c_2 e^{\alpha_2 z} + c_3 e^{\alpha_4 z} + c_4 e^{\alpha_4 z}. \tag{2.35}$$

By (2.33), (2.35) and Lemma 2.5, we obtain that $c_1 \equiv c_2 \equiv c_3 \equiv c_4 \equiv 0$, which gives $f(z) \equiv 0$, a contradiction. Therefore, we proved that the Eq. (1.2) does not have any entire solution of hyper-order less than one when $s > 1$.

From the above discussion, we see that any meromorphic solution $f(z)$ of the equation (1.2) must satisfy $\sigma_2(f) \geq 1$.

3 Proof of Theorem 1.2

To prove Theorems 1.2 and 1.3, we also need the following lemma.

Lemma 3.1 ([15]) *Let c be a non-zero constant, let $H(z)$ be a meromorphic function and let $h(z)$ be a polynomial with* $\deg h(z) \geq 1$. *If $\sigma(H(z)) < \sigma(e^{h(z)})$, then*

$$T(r, H(z)) = S(r, e^{h(z)}), \quad T(r, H(z+c)) = S(r, e^{h(z)}).$$

Proof of Theorem 1.2. Suppose that (1.2) has a meromorphic solution $f(z)$ with $\sigma_2(f) < 1$. From the proof of Theorem 1.1, we see that $f(z)$ is a transcendental entire function. □

By Lemma 2.1, we get

$$\begin{aligned} T(r, f(z)^n + p(z)f(z+\eta)) &= m(r, f(z)^n + p(z)f(z+\eta)) \\ &\geq m(r, f(z)^n) - m\left(r, p(z)\frac{f(z+\eta)}{f(z)}f(z)\right) - \log 2 \\ &\geq nm(r, f(z)) - m(r, f(z)) + S(r, f) \\ &= (n-1)m(r, f(z)) + S(r, f) \\ &= (n-1)T(r, f(z)) + S(r, f). \end{aligned} \tag{3.1}$$

On the other hand,

$$\begin{aligned} &T(r, \beta_1 e^{\alpha_1 z} + \beta_2 e^{\alpha_2 z} + \cdots + \beta_s e^{\alpha_s z}) \\ &\leq T(r, \beta_1 e^{\alpha_1 z}) + T(r, \beta_2 e^{\alpha_2 z}) + \cdots + T(r, \beta_s e^{\alpha_s z}) + O(1) \\ &= \frac{(|\alpha_1| + |\alpha_2| + \cdots + |\alpha_s|)r}{\pi}(1 + o(1)). \end{aligned} \tag{3.2}$$

By (1.2), (3.1) and (3.2), we get

$$(n-1)T(r, f(z)) + S(r, f) \leq \frac{(|\alpha_1| + |\alpha_2| + \cdots + |\alpha_s|)r}{\pi}(1 + o(1)).$$

So $\sigma(f) \leq 1$. If $\sigma(f) < 1$, then by Lemma 3.1, we see that for $1 \leq h < k \leq s$,

$$T(r, f(z)^n + p(z)f(z+\eta)) = S(r, e^{(\alpha_h - \alpha_k)z}).$$

By (1.2) and Lemma 2.3, we get $\beta_1 = \beta_2 = \cdots = \beta_s = 0$, which contradicts the hypotheses. So $\sigma(f) = 1$.

To prove $\Theta(0, f) = 0$, we divide our discussion into two cases.

Case 1. $s = 1$. As in the proof of Theorem 1.1, we get (2.10) and (2.11). Furthermore, we have

$$nf'(z) - \alpha_1 f(z) = f(z)\left(n\frac{f'(z)}{f(z)} - \alpha_1\right). \tag{3.3}$$

If $nf'(z) - \alpha_1 f(z) \equiv 0$, as in the proof of Theorem 1.1, we get a contradiction. So $nf'(z) - \alpha_1 f(z) \not\equiv 0$. By logarithmic derivative lemma, we get

$$m\left(r, n\frac{f'(z)}{f(z)} - \alpha_1\right) = S(r, f).$$

Combining this equality with

$$N\left(r, n\frac{f'(z)}{f(z)} - \alpha_1\right) = \overline{N}\left(r, \frac{1}{f(z)}\right),$$

we get

$$T\left(r, n\frac{f'(z)}{f(z)} - \alpha_1\right) = \overline{N}\left(r, \frac{1}{f(z)}\right) + S(r, f). \tag{3.4}$$

(2.11), (3.3) and (3.4) yield

$$\begin{aligned} T(r, f(z)) &= T\left(r, \frac{nf'(z) - \alpha_1 f(z)}{n\frac{f'(z)}{f(z)} - \alpha_1}\right) \\ &\leq T(r, nf'(z) - \alpha_1 f(z)) + T\left(r, n\frac{f'(z)}{f(z)} - \alpha_1\right) + O(1) \\ &= \overline{N}\left(r, \frac{1}{f(z)}\right) + S(r, f), \end{aligned}$$

which gives $\Theta(0, f) = 0$.

Case 2. $s > 1$. As in the proof of Theorem 1.1, we get (2.14)–(2.29). If $\phi \equiv 0$, then as in the proof of Theorem 1.1, we get a contradiction. So $\phi \not\equiv 0$ and (2.30) holds. By (2.28), we have

$$f(z)^n \frac{L(f(z)^n)}{f(z)^n} = L(f(z)^n) = f(z)^{n-s}\phi = f(z)^n \frac{\phi}{f(z)^s},$$

which gives

$$\frac{L(f(z)^n)}{f(z)^n} = \frac{\phi}{f(z)^s}. \tag{3.5}$$

By logarithmic derivative lemma, we get for $m = 1, 2, \ldots, s$,

$$m\left(r, \frac{(f(z)^n)^{(m)}}{f(z)^n}\right) = S(r, f(z)^n) = S(r, f). \tag{3.6}$$

Combining (3.6) with (2.26) and (3.5), we get

$$m\left(r, \frac{\phi}{f(z)^s}\right) = m\left(r, \frac{L(f(z)^n)}{f(z)^n}\right) = S(r, f). \tag{3.7}$$

For $m = 1, 2, \ldots, s$, we have

$$N\left(r, \frac{(f(z)^n)^{(m)}}{f(z)^n}\right) \leq m\overline{N}\left(r, \frac{1}{f(z)^n}\right) = m\overline{N}\left(r, \frac{1}{f(z)}\right). \tag{3.8}$$

Combining (3.8) with (2.26) and (3.5), we get

$$N\left(r, \frac{\phi}{f(z)^s}\right) = N\left(r, \frac{L(f(z)^n)}{f(z)^n}\right) \leq s\overline{N}\left(r, \frac{1}{f(z)}\right). \tag{3.9}$$

(3.7) and (3.9) yield

$$T\left(r, \frac{\phi}{f(z)^s}\right) \leq s\overline{N}\left(r, \frac{1}{f(z)}\right) + S(r, f). \tag{3.10}$$

By (2.30) and (3.10), we get

$$\begin{aligned} sT(r, f(z)) &= T(r, f(z)^s) \\ &= T\left(r, \frac{f(z)^s}{\phi}\phi\right) \\ &\leq T\left(r, \frac{f(z)^s}{\phi}\right) + T(r, \phi) + O(1) \\ &\leq s\overline{N}\left(r, \frac{1}{f(z)}\right) + S(r, f), \end{aligned}$$

which gives $\Theta(0, f) = 0$.

4 Proof of Theorem 1.3

Suppose that (1.2) has a meromorphic solution $f(z)$ with $\sigma_2(f) < 1$. From the proof of Theorem 1.1, we see that $f(z)$ is a transcendental entire function. From the proof of Theorem 1.2, we see that $\sigma(f) = 1$. If $\lambda(f) < 1$, then by Hadamard factorization theorem, $f(z)$ can be written as

$$f(z) = H(z)e^{az}, \tag{4.1}$$

where a is a non-zero constant and $H(z)$ is an entire function with $\sigma(H) < 1$. We deduce from Lemma 3.1 that for $1 \leq h < k \leq s$,

$$T(r, H(z)) = S(r, e^{(\alpha_h - \alpha_k)z}), \quad T(r, H(z+\eta)) = S(r, e^{(\alpha_h - \alpha_k)z}). \tag{4.2}$$

Substituting (4.1) into the Eq. (1.2), we get

$$H(z)^n e^{naz} + p(z)H(z+\eta)e^{a\eta}e^{az} = \beta_1 e^{\alpha_1 z} + \beta_2 e^{\alpha_2 z} + \cdots + \beta_s e^{\alpha_s z}. \tag{4.3}$$

Since $\alpha_1, \alpha_2, \ldots, \alpha_s$ are distinct, we deduce from (4.2), (4.3) and Lemma 2.3 that $s = 2$ and

$$\alpha_1 = na, \quad \alpha_2 = a,$$

or

$$\alpha_2 = na, \quad \alpha_1 = a.$$

These contradict our hypotheses. So $\lambda(f) = \sigma(f) = 1$.

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