Southeast Asian Bulletin of Mathematics (2010) 34: 429–438

Southeast Asian Bulletin of Mathematics

The Zeros Distribution of Hyper Solutions of Higher Order Differential Equations in Angular Domain*

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Received 11 June 2007
Accepted 16 June 2008

Communicated by S.S. Cheng

AMS Mathematics Subject Classification (2000): 30D35

Abstract. In this paper, we investigate the location of zeros and Borel direction for the solutions of equations

$$f^{(n)} + A_{n-2}(z)f^{(n-2)} + \cdots + A_1(z)f^{'} + A_0(z)f = 0 (n \geq 2) \qquad (*)$$

in an angular domain and obtain a sufficient and necessary condition between Borel direction and the hyper order exponent of convergence of zero sequence of $E = f_1 f_2 \cdots f_n$, where $f_1, f_2, \cdots, f_n$ are n linearly independent solutions of the equation $(*)$. This paper extends previous results.

Keywords: Zeros Distribution; Linear differential equation; Borel Direction.

1. Introduction

We shall assume that the readers are familiar with the standard notations of Nevanlinna theory and complex differential equations (see [1, 3]).

Up to now, there are many papers about the zeros distribution of the solutions of a linear differential equations since it is one of the difficult aspects in the complex oscillation theory of differential equations (see $[5 - 15]$).

*The project was supported by the National Natural Science Foundation of China (No.10 871076), and also partly supposed by the School of Mathematical Sciences Foundation of SCNU, China.

In order to state our results, we give some definitions.

Let $g(z)$ be an entire function in the plane and let $\arg z = \theta \in [0, 2\pi)$ be a ray. We denote angular domain and sectorial domain , for any $\alpha < \beta$, respectively,

$$\Omega(\alpha,\beta) = \{z|\alpha \leq \arg z \leq \beta, |z| > 0\};$$
$$\Omega((\alpha,\beta),r) = \{z|z \in \Omega(\alpha,\beta), |z| < r\}.$$

Let $n(\Omega((\alpha,\beta),r), g = a)$ be the number of a-points, i.e. roots of the equation $g(z) = a$ in the sectorial domain $\Omega((\alpha,\beta),r)$.

The hyper order exponent of convergence of zero sequence of $g(z) - a$ in angular domain $\Omega(\alpha,\beta)$ is defined by

$$\lambda_2(\Omega(\alpha,\beta), g = a) = \overline{\lim_{r\to\infty}} \frac{\log\log n(\Omega((\alpha,\beta),r), g = a)}{\log r}.$$

We also denote, for each $\epsilon > 0$, the hyper order exponent of convergence of zero sequence of $g(z)$ in the angular domain $\Omega(\theta - \epsilon, \theta + \epsilon)$ by $\lambda_{2,\theta,\epsilon}(g)$, i.e.

$$\lambda_{2,\theta,\epsilon}(g) = \overline{\lim_{r\to\infty}} \frac{\log\log n(\Omega((\theta-\epsilon,\theta+\epsilon),r), g = a)}{\log r},$$

and by $\lambda_\theta(g) = \lim_{\epsilon\to 0} \lambda_{\theta,\epsilon}(g)$ and $\lambda_{2,\theta}(g) = \lim_{\epsilon\to 0} \lambda_{2,\theta,\epsilon}(g)$ respectively.

Our proofs also require the Nevanlinna characteristic function for an angular domain (see [2, 9]). If $0 < \beta - \alpha \leq 2\pi$ and $k = \frac{\pi}{\beta-\alpha}$ and $g(z)$ is meromorphic on the angular domain $\Omega(\alpha,\beta)$, we denote

$$A_{\alpha,\beta}(r,g) = \frac{k}{\pi}\int_1^r \left(\frac{1}{t^k} - \frac{t^k}{r^{2k}}\right)\left\{\log^+|g(te^{i\alpha})| + \log^+|g(te^{i\beta})|\right\}\frac{dt}{t};$$
$$B_{\alpha,\beta}(r,g) = \frac{2k}{\pi r^k}\int_\alpha^\beta \log^+|g(re^{i\theta})|\sin k(\theta-\alpha)d\theta;$$
$$C_{\alpha,\beta}(r,g) = 2\sum_{1<|b_v|<r}\left(\frac{1}{|b_v|^k} - \frac{|b_v|^k}{r^{2k}}\right)\sin k(\beta_v - \alpha);$$
$$D_{\alpha,\beta}(r,g) = A_{\alpha,\beta}(r,g) + B_{\alpha,\beta}(r,g);$$
$$S_{\alpha,\beta}(r,g) = A_{\alpha,\beta}(r,g) + B_{\alpha,\beta}(r,g) + C_{\alpha,\beta}(r,g),$$

where $b_v = |b_v|e^{i\beta_v} (v = 1, 2, \cdots)$ are the poles of $g(z)$ in angular domain $\Omega(\alpha,\beta)$, counting multiplicities. $S_{\alpha,\beta}(r,g)$ and $C_{\alpha,\beta}(r,g)$ are called the Nevanlinna's angular characteristic function and the angular counting function respectively. If we only consider the distinct poles of $g(z)$, we denote the corresponding angular counting function by $\overline{C}_{\alpha,\beta}(r,g)$. The sectorial hyper order $\rho_2(\Omega(\alpha,\beta),g)$ of $g(z)$ in an angular domain $\Omega(\alpha,\beta)$ will be defined by

$$\rho_2(\Omega(\alpha,\beta),g) = \overline{\lim_{r\to\infty}} \frac{\log\log S_{\alpha,\beta}(r,g)}{\log r}.$$

A ray $L : \arg z = \theta$ is called a Borel direction of hyper order $\rho(0 < \rho \leq +\infty)$ of $g(z)$ which has the hyper order ρ, if for any sufficiently small $\epsilon > 0$, we have

$$\overline{\lim_{r\to+\infty}} \frac{\log\log n\,(\Omega((\theta-\epsilon,\theta+\epsilon),r), g=a)}{\log r} = \rho,$$

with at most two exceptional values $a \in \mathbb{C}_\infty$.

In [5], we considered the equation

$$f'' + A(z)f = 0, \tag{1.1}$$

where $A(z)$ is an entire function, and obtained

Theorem 1.A. *Let $A(z)$ be an entire function with order $\sigma(A) = +\infty$ and hyper order $\sigma_2(A) = 0$ and let f_1 and f_2 be two linearly independent solutions of (1.1). Set $E = f_1 f_2$. Suppose that the hyper order exponent of convergence of zero sequence of E is $+\infty$. Then a ray $\arg z = \theta$ from the origin is a Borel direction of E with hyper order $+\infty$ and $\rho_2(\Omega(\theta-\epsilon,\theta+\epsilon), E) = +\infty$, if and only if $\lambda_{2,\theta}(E) = +\infty$.*

In [6], we extended Theorem 1.A for higher order differential equations

$$f^{(n)} + A_{n-2}(z)f^{(n-2)} + \cdots + A_1(z)f' + A_0(z)f = 0, (n \geq 2), \tag{1.2}$$

where $A_j(z)(j = 0, 1, \cdots, n-2)$ are entire functions. We obtained

Theorem 1.B. *Let $A_j(z)(j = 0, 1, \cdots, n-2)$ be entire functions with order $\sigma(A_j) = +\infty$ and hyper order $\sigma_2(A_j) = 0(j = 0, 1, 2, \cdots, n-2)$, and let $f_1, f_2, \cdots, f_n$ be n linearly independent solutions of (1.2). Set $E = f_1 f_2 \cdots f_n$. Suppose that the hyper order exponent of convergence of zeros sequence of E is $+\infty$. Then a ray $\arg z = \theta$ from the origin is a Borel direction of E with hyper order $+\infty$ and $\rho_2(\Omega(\theta-\epsilon,\theta+\epsilon), E) = +\infty$, if and only if $\lambda_{2,\theta}(E) = +\infty$.*

In [13], Zh.J. Wu and D.C.Sun considered equations (1.1) with $A(z)$ of finite order and obtained the following Theorem.

Theorem 1.C. *Let $A(z)$ be a transcendental meromorphic function of order σ Let f_1, f_2 be two linearly independent solutions of (1.1) and $E = f_1 f_2$. Suppose that $\sigma_2(E) > 0$. Then there exists a ray $L : \arg z = \theta$ such that $\lambda_{2,\theta}(E) = \sigma_2(E)$. where*

$$\lambda_{2,\theta}(E) = \lim_{\epsilon\to 0} \overline{\lim_{r\to\infty}} \frac{\log\log n(\Omega((\theta-\epsilon,\theta+\epsilon),r), E=0)}{\log r}.$$

We can find it is easy if equation (1.1) with coefficient $A(z)$ having finite order. We also find that Theorem 1.A and Theorem 1.B both have the condition $\rho_2(\Omega(\theta-\epsilon,\theta+\epsilon),E)=+\infty$ for all sufficient small $\epsilon>0$. Here, we omit this condition and obtain the following Theorem 1.1 by using the methods which is different from the methods used in [5, 6], but similar to the methods used in [11, 13].

Theorem 1.1. *Let $A_j(z)(j=0,1,\cdots,n-2)$ be entire functions with order $\sigma(A_j)=+\infty$ and hyper order $\sigma_2(A_j)=0(j=0,1,2,\cdots,n-2)$, and let $f_1,f_2,\cdots,f_n$ be n linearly independent solutions of (1.2). Set $E=f_1f_2\cdots f_n$. Suppose that the hyper order exponent of convergence of zeros sequence of E is $+\infty$. Then a ray $\arg z=\theta$ from the origin is a Borel direction of E with hyper order $+\infty$ if and only if $\lambda_{2,\theta}(E)=+\infty$.*

2. Lemmas for the Proof

In order to prove our result, we need the followings.

Now, suppose that $g(z)$ is analytic, then $g(z)$ has the power series representation

$$g(z)=\sum_{n=0}^{\infty}a_nz^n,\quad (0\le |z|<\infty).$$

Denote maximum item and center index of $g(z)$ by $\mu(r)$ and $\nu(r)$ respectively, i.e.

$$\mu(r)=\max_{n\ge 0}\{|a_n|r^n\},$$

and

$$\nu(r)=\max\{m:\mu(r)=|a_m|r^m\}.$$

Set $a=\max\limits_{n\ge 0}\{|a_n|\}$, we have

$$|a_n|r^n\le \mu(r)\le ar^{\nu(r)}.$$

Lemma 2.1. ([4, $P18$]) *Suppose that $g(z)$ is analytic, then for $r<R$ and $\mu(r)>1$,*

$$M(r,g)\le \mu(r)\{1+\log M(R,g)\}\frac{2R}{R-r}.$$

On the other hand, under the hypotheses of Lemma 2.1, we

$$T(r,g)\le \log M(r,g)\le \frac{R+r}{R-r}T(R,g).$$

Together with Lemma 2.1 in which we set $R = 2r$, we obtain

$$\begin{aligned} T(r,g) &\leq \log\mu(r) + \log\log M(2r,g) + O(1) \\ &\leq \nu(r)\log r + \log T(4r,g) + O(1). \end{aligned} \tag{2.1}$$

Lemma 2.2. ([7]) *Let $f_1, f_2, \cdots, f_n$ be n linearly independent meromorphic solutions of*

$$f^{(n)} + A_{n-1}(z)f^{(n-1)} + \cdots + A_1(z)f^{'} + A_0(z)f = 0, (n \geq 2),$$

with meromorphic coefficients. Then the Wronskian determinant

$$W = W(f_1, f_2, \cdots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1^{'} & f_2^{'} & \cdots & f_n^{'} \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix},$$

satisfying the differential equation $W^{'} + A_{n-1}(z)W = 0$. Specially, if $A_{n-1}(z)$ is an entire function, then for some $c \in \mathbf{C}, W(f_1, f_2, \cdots, f_n) = c\exp(-\varphi)$, where φ is a primitive function of $A_{n-1}(z)$.

Lemma 2.3. ([2]) *Suppose that $g(z)(\not\equiv$ constant$)$ is meromorphic in the plane and that $\Omega(\alpha, \beta)$ is an angular domain, where $0 < \beta - \alpha \leq 2\pi$. Then*

(i) *for any complex number $a \neq \infty$,*

$$S_{\alpha,\beta}\left(r, \frac{1}{g-a}\right) = S_{\alpha,\beta}\left(r, g\right) + O(1);$$

(ii) *for any $r < R$,*

$$A_{\alpha,\beta}\left(r, \frac{g^{'}}{g}\right) \leq K\left\{\left(\frac{R}{r}\right)^k \int_1^R \frac{\log T(t,g)}{t^{1+k}}dt + \log\frac{r}{R-r} + \log\frac{R}{r} + 1\right\},$$

and

$$B_{\alpha,\beta}\left(r, \frac{g^{'}}{g}\right) \leq \frac{4k}{r^k} m\left(r, \frac{g^{'}}{g}\right),$$

where $k = \dfrac{\pi}{\beta - \alpha}$ and K is a positive constant not depending on r and R.

3. The Proof of Theorem 1.1

Proof. The proof of Theorem 1.1 will be completed by the following three steps.

Step 1. We prove, for any sufficiently small $\epsilon > 0$, on $\Omega((\theta-\epsilon,\theta+\epsilon),r)$,

$$S_{\theta-\epsilon,\theta+\epsilon}(r,E) = O\left\{n\left(\Omega((\theta-\epsilon,\theta+\epsilon),r),\frac{1}{E}\right)+\exp((2r)^{\epsilon})\right\}.$$

Suppose that $f(z)$ is a non-trivial solution of (2.2). Then

$$\frac{f^{(n)}}{f}+A_{n-2}(z)\frac{f^{(n-2)}}{f}+\cdots+A_1(z)\frac{f'}{f}+A_0(z)=0. \tag{3.1}$$

We apply Wiman-Valiron theory to (3.1). Hence there exists a set $D_1 \subset [0,+\infty)$ of finite logarithmic measure such that if $r \notin D_1$ and z is a point on $|z|=r$ at which $|f(z)| = M(r,f)$, then

$$\left|\frac{f^{(j)}}{f}\right| = \left(\frac{\nu(r)}{z}\right)^j(1+o(1)), j=1,2,\cdots,n, \tag{3.2}$$

where $\nu(r)$ denotes the central index of f.

It follows from (3.1) and (3.2) that

$$\begin{aligned}&\nu(r)^n(1+o(1))+\nu(r)^{n-2}z^2A_{n-2}(z)(1+o(1))+\cdots\\ &\quad+\nu(r)z^{n-1}A_1(z)(1+o(1))+z^nA_0(z)=0.\end{aligned} \tag{3.3}$$

Set $\sigma_2 = \max\limits_{0\le j\le n-2}\{\sigma_2(A_j)\}$. For all arbitrary $\epsilon > 0$, there exists a set $D_2 \subset (1,+\infty)$ of finite logarithmic measure such that

$$|A_j(z)| \le \exp\{\exp\left(r^{\sigma_2+\epsilon}\right)\}, j=0,1,2,\cdots,n-2, \tag{3.4}$$

when $z \notin [0,1]\cup D_2$ and $r \to +\infty$.

It follows from (3.3) and (3.4) that

$$\nu(r) \le nr^n\exp\{\exp\left(r^{\sigma_2+\epsilon}\right)\} \le \exp\{\exp\left(r^{\sigma_2+2\epsilon}\right)\}. \tag{3.5}$$

Since $f(z)$ is analytic, $f(z)$ satisfies the condition of Lemma 2.1. Thus, (2.1) and (3.5) implies

$$\overline{\lim_{r\to+\infty}}\frac{\log\log\log T(r,f)}{\log r} \le \sigma_2. \tag{3.6}$$

Now we suppose that $f_1, f_2, \cdots, f_n$ be n linearly independent solutions of (1.2). Set $E = f_1f_2\cdots f_n$, and Wronskian determinant

$$W = W(f_1,f_2,\cdots,f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n\\ f_1' & f_2' & \cdots & f_n'\\ \vdots & \vdots & \ddots & \vdots\\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)}\end{vmatrix}. \tag{3.7}$$

It follows from Lemma 2.2, without loss of generality, we can set

$$W(f_1, f_2, \cdots, f_n) = 1.$$

From (3.6), we have

$$\overline{\lim_{r\to+\infty}} \frac{\log\log\log T(r, f_j)}{\log r} \le \sigma_2, (j = 1, 2, \cdots, n). \tag{3.8}$$

Hence

$$\overline{\lim_{r\to+\infty}} \frac{\log\log\log T(r, E)}{\log r} \le \sigma_2. \tag{3.9}$$

Now dividing (3.7) by E, we have

$$\begin{aligned}
\frac{1}{E} = \frac{W}{E} &= \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \frac{f_1'}{f_1} & \frac{f_2'}{f_2} & \cdots & \frac{f_n'}{f_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{f_1^{(n-1)}}{f_1} & \frac{f_2^{(n-1)}}{f_2} & \cdots & \frac{f_n^{(n-1)}}{f_n} \end{vmatrix} \\
&= \sum_{1\le j_s\ne j_1\le n} (-1)^{\tau(j_1, j_2, \cdots, j_n)} \cdot 1_{j_1} \cdot \frac{f_{j_2}'}{f_{j_2}} \cdot \frac{f_{j_3}''}{f_{j_3}} \cdots \frac{f_{j_s}^{(s-1)}}{f_{j_s}} \cdots \frac{f_{j_n}^{(n-1)}}{f_{j_n}} \\
&= \sum_{1\le j_s\ne j_1\le n} (-1)^{\tau(j_1, j_2, \cdots, j_n)} \prod_{s=2}^{n} \frac{f_{j_s}^{(s-1)}}{f_{j_s}},
\end{aligned} \tag{3.10}$$

where 1_{j_1} denotes the number 1 in row 1 and in column j_1 and $\tau(j_1, j_2, \cdots, j_n)$ denotes the inverse order number of $j_1, j_2, \cdots, j_n$, and $j_1, j_2, \cdots, j_n$ is an arrangement of $1, 2, \cdots, n$. We deduce from (3.8) and Lemma 2.3 (ii) in which we set $R = 2r$ that, for $j = 1, 2, \cdots n$,

$$\begin{aligned}
A_{\theta-\epsilon,\theta+\epsilon}\left(r, \frac{f_j'}{f_j}\right) &\le K \int_1^{2r} \frac{\log T(r, f_j)}{t^{1+k}} dt + O(1) \\
&\le K \int_1^{2r} \frac{\exp(t^{\sigma_2+\epsilon})}{t^{1+\frac{\pi}{2\epsilon}}} dt + O(1) \le K \exp((2r)^{\sigma_2+\epsilon}).
\end{aligned}$$

for all sufficiently small $\epsilon > 0$, where K is a sufficiently large positive constant and the following K is the same but can be different.

Since, for $j = 1, 2, \cdots, n$, and for all sufficiently small $\epsilon > 0$,

$$m\left(r, \frac{f_j'}{f_j}\right) = O\left(\log T(2r, f_j) + \log r\right) \le K \exp((2r)^{\sigma_2+\epsilon}),$$

we deduce from Lemma 2.3 (ii) that, for $j = 1, 2, \cdots, n$, and for all sufficiently small $\epsilon > 0$,

$$B_{\theta-\epsilon,\theta+\epsilon}\left(r, \frac{f_j'}{f_j}\right) \le K \exp((2r)^{\sigma_2+\epsilon}),$$

therefore we have

$$D_{\theta-\epsilon,\theta+\epsilon}\left(r,\frac{f_j'}{f_j}\right)\leq K\exp((2r)^{\sigma_2+\epsilon}), j=1,2,\cdots,n. \tag{3.11}$$

for all sufficiently small $\epsilon>0$.

Similarly, we have, for $j=1,2,\cdots,n$, and for all sufficiently small $\epsilon>0$,

$$D_{\theta-\epsilon,\theta+\epsilon}\left(r,\frac{f_j^{(s)}}{f_j}\right)\leq\sum_{l=1}^{s}D_{\theta-\epsilon,\theta+\epsilon}\left(r,\frac{f_j^{(l)}}{f_j^{(l-1)}}\right)\leq K\exp((2r)^{\sigma_2+\epsilon}). \tag{3.12}$$

It follows from (3.10) and (3.12) that

$$\begin{aligned}D_{\theta-\epsilon,\theta+\epsilon}\left(r,\frac{1}{E}\right) &= D_{\theta-\epsilon,\theta+\epsilon}\left(r,\sum_{1\leq j_s\neq j_1\leq n}(-1)^{\tau(j_1,j_2,\cdots,j_n)}\prod_{s=2}^{n}\frac{f_{j_s}^{(s-1)}}{f_{j_s}}\right)\\ &\leq K\exp((2r)^{\sigma_2+\epsilon}),\end{aligned}$$

for all sufficiently small $\epsilon>0$.

Since, by Lemma 2.3 (i),

$$S_{\theta-\epsilon,\theta+\epsilon}(r,E)=S_{\theta-\epsilon,\theta+\epsilon}(r,\frac{1}{E})+O(1)=D_{\theta-\epsilon,\theta+\epsilon}(r,\frac{1}{E})+C_{\theta-\epsilon,\theta+\epsilon}(r,\frac{1}{E})+O(1),$$

we have, for all sufficiently small $\epsilon>0$,

$$S_{\theta-\epsilon,\theta+\epsilon}(r,E)\leq K\left\{C_{\theta-\epsilon,\theta+\epsilon}(r,\frac{1}{E})+\exp\left((2r)^{\sigma_2+\epsilon}\right)\right\}. \tag{3.13}$$

Let $a_\nu=|a_\nu|e^{i\alpha_\nu}(\nu=1,2,\cdots)$ be the zeros of E in the angular domain $\Omega(\theta-\epsilon,\theta+\epsilon)$. Then

$$\begin{aligned}C_{\theta-\epsilon,\theta+\epsilon}\left(r,\frac{1}{E}\right) &= 2\sum_{1<|a_\nu|<r}\left(\frac{1}{|a_\nu|^k}-\frac{|a_\nu|^k}{r^{2k}}\right)\sin k(\alpha_\nu-\theta+\epsilon)\\ &\leq 2\sum_{1<|a_\nu|<r}\frac{1}{|a_\nu|^k}=2\int_1^r\frac{1}{t^k}dn(t)\\ &\leq 2n\left(\Omega\left((\theta-\epsilon,\theta+\epsilon),r\right),\frac{1}{E}\right)+O(1)\end{aligned} \tag{3.14}$$

It follows from (3.13) and (3.14) that, for all sufficiently small $\epsilon>0$ and $\sigma_2=\max_{0\leq j\leq n}\{\sigma_2(A_j)\}=0$,

$$S_{\theta-\epsilon,\theta+\epsilon}(r,E)=O\left\{n\left(\Omega((\theta-\epsilon,\theta+\epsilon),r),\frac{1}{E}\right)+\exp((2r)^{\sigma_2+\epsilon})\right\}. \tag{3.15}$$

Step 2. We prove, for any sufficiently small $\epsilon > 0$ and $k = \dfrac{\pi}{2\epsilon}$, on $\Omega((\theta-\epsilon,\theta+\epsilon),r)$,

$$S_{\theta-\epsilon,\theta+\epsilon}(r,E) \geq \left(1-\frac{1}{r^{2k}}\right)\frac{n\left(\Omega\left(\left(\theta-\frac{2\epsilon}{3},\theta+\frac{2\epsilon}{3}\right),r\right),\frac{1}{E}\right)}{r^k}.$$

Suppose that $a_\nu = |a_\nu|e^{i\alpha_\nu}(\nu = 1,2,\cdots)$ are the roots of $E = 0$, counting multiplicities, in angular domain $\Omega(\theta-\epsilon,\theta+\epsilon)$. We first observe that $\theta - \dfrac{2\epsilon}{3} < \alpha_v < \theta + \dfrac{2\epsilon}{3}$ implies for $k = \dfrac{\pi}{2\epsilon}$ the inequalities

$$k\cdot\frac{\epsilon}{3} < k(\alpha_v-\theta+\epsilon) < \pi - k\cdot\frac{\epsilon}{3}.$$

Hence

$$\sin k(\alpha_v-\theta_0+\epsilon) \geq \sin(k\cdot\frac{\epsilon}{3}) = \sin\frac{\pi}{6} = \frac{1}{2}. \tag{3.16}$$

Moreover, we write a sum below as a $Stieltjes - integral$,

$$\begin{aligned}\sum\left(\frac{1}{|a_\nu|^k}-\frac{|a_\nu|^k}{r^{2k}}\right) &= \sum\left(\frac{1}{|a_\nu|^k}\right)-\sum\left(\frac{|a_\nu|^k}{r^{2k}}\right)\\ &= \int_1^r\frac{dn(t)}{t^k}-\frac{1}{r^{2k}}\int_1^r t^k dn(t),\end{aligned}$$

where a short hand notation $n(t) = n\left(\Omega((\theta-\dfrac{2\epsilon}{3},\theta+\dfrac{2\epsilon}{3}),t),\dfrac{1}{E}\right)$ will be used.

Application of Lemma 2.3(i), (3.16) and the partial integration of the above $Stieltjes - integrals$ and the definition of $S_{\alpha,\beta}(r,E)$ now results in

$$\begin{aligned}S_{\theta-\epsilon,\theta+\epsilon}(r,E) &= S_{\theta-\epsilon,\theta+\epsilon}(r,\frac{1}{E})+O(1) \geq C_{\theta-\epsilon,\theta+\epsilon}(r,\frac{1}{E})+O(1)\\ &= 2\sum_{1<|a_\nu|<r}\left(\frac{1}{|a_\nu|^k}-\frac{|a_\nu|^k}{r^{2k}}\right)\sin k(\alpha_\nu-\theta+\epsilon)+O(1)\\ &\geq 2\sum_{\substack{1<|a_\nu|<r\\ \theta-\frac{2\epsilon}{3}<\alpha_\nu<\theta+\frac{2\epsilon}{3}}}\left(\frac{1}{|a_\nu|^k}-\frac{|a_\nu|^k}{r)^{2k}}\right)\sin(k\cdot\frac{\epsilon}{3})+O(1)\\ &= 2\left\{\int_1^r\frac{dn(t)}{t^k}-\frac{1}{r^{2k}}\int_1^r t^k dn(t)\right\}\sin\frac{\pi}{6}+O(1)\\ &= \frac{n(r)}{r^k}+k\int_1^r\frac{n(t)}{t^{1+k}}dt-\frac{r^k n(r)}{r^{2k}}+\frac{k}{r^{2k}}\int_1^r t^{k-1}n(t)dt+O(1)\\ &\geq \left(1-\frac{1}{r^{2k}}\right)\frac{n(r)}{r^k}+O(1),\end{aligned} \tag{3.17}$$

where $n(r)$ is the numbers of the roots of the equation $E(z) = 0$, counting multiplicities, on the sector $\Omega((\theta-\dfrac{2\epsilon}{3},\theta+\dfrac{2\epsilon}{3}),r)$.

Step 3. We prove that $\lambda_{2,\theta}(E) = +\infty$ if and only if for each sufficiently small $\epsilon > 0$,

$$\overline{\lim_{r\to+\infty}} \frac{\log\log S_{\theta-\epsilon,\theta+\epsilon}(r,E)}{\log r} = +\infty.$$

The prove is trivial from (3.15) and (3.17).

This concludes the proof of Theorem 1.1. ■

Acknowledgement. The authors are very grateful to the referee for his(her) many valuable comments and suggestions which greatly improved the presentation of this paper.

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