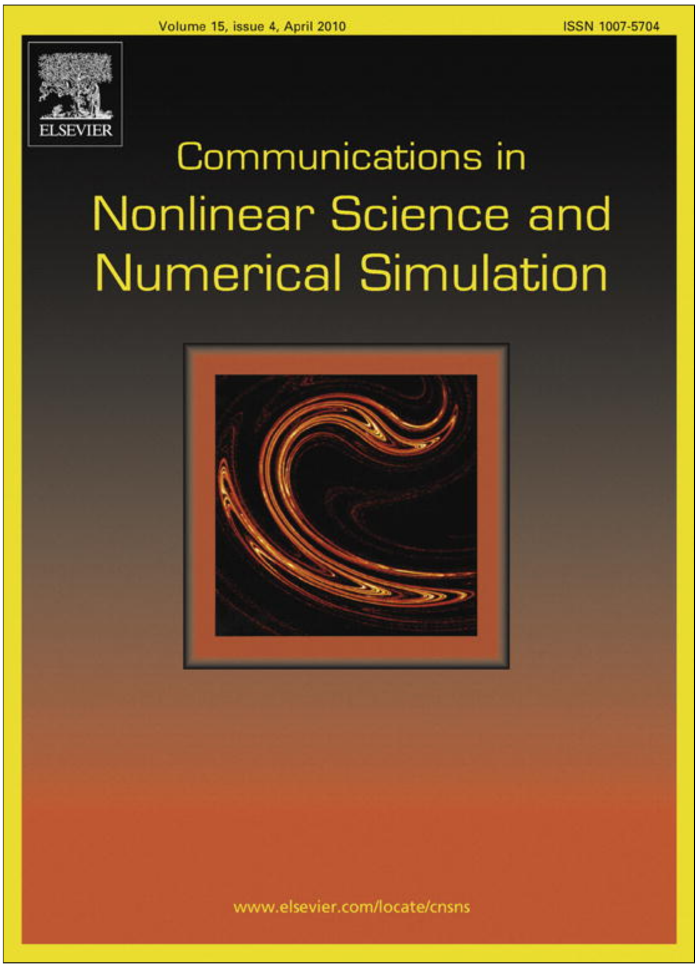
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Subnormal solutions of second order nonhomogeneous linear periodic differential equations

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ABSTRACT

We obtain the representations of the subnormal solutions of nonhomogeneous equations

$$f'' + [P_1(e^z) + Q_1(e^{-z})]f' + [P_2(e^z) + Q_2(e^{-z})]f = R_1(e^z) + R_1(e^{-z}),$$

where $P_1(z),\ P_2(z),\ Q_1(z),\ Q_2(z),\ R_1(z)$ and $R_2(z)$ are polynomials in z such that $P_1(z),\ P_2(z),\ Q_1(z)$ and $Q_2(z)$ are not all constants, $\deg P_1 \leqslant \deg P_2$. We resolve the question raised by Gundersen and Steinbart in 1994.

1. Introduction

We use the standard notation from Nevanlinna theory in this paper (see [4,8,11]).

The study of the properties of solutions of a linear differential equation with periodic coefficients is one of the difficult aspects in the complex oscillation theory of differential equations. However, it is also one of the important aspects since it relates to many special functions. Some important results were done by different authors, see, for instance, [1–3,5–7,9,10].

Now, we consider second order nonhomogeneous linear differential equation

$$f'' + [P_1(e^z) + Q_1(e^{-z})]f' + [P_2(e^z) + Q_2(e^{-z})]f = R_1(e^z) + R_1(e^{-z}), \tag{1.1}$$

where $P_1(z),\ P_2(z),\ Q_1(z),\ Q_2(z),\ R_1(z)$ and $R_2(z)$ are polynomials in z such that $P_1(z),\ P_2(z),\ Q_1(z)$ and $Q_2(z)$ are not all constants. It is well known that every solution $f(z)$ of (1.1) is an entire function.

Let $f(z)$ be an entire function. We define

$$\rho_e(f) = \overline{\lim_{r\to+\infty}} \frac{\log T(r,f)}{r} \tag{1.2}$$

to be the e-type order of $f(z)$.

If $f(z) \not\equiv 0$ is a solution of (1.1) and if $f(z)$ satisfies $\rho_e(f) = 0$, then we say that $f(z)$ is a subnormal solution of (1.1). For convenience, we also say that $f(z) \equiv 0$ is a subnormal solution of (1.1).

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In [5], Gundersen and Steinbat have raised the following open problem, i.e., what about the forms of the subnormal solutions of Eq. (1.1)?

In [6], we have obtained the all forms of subnormal solutions of homogeneous equation

$$f'' + [P_1(e^z) + Q_1(e^{-z})]f' + [P_2(e^z) + Q_2(e^{-z})]f = 0,$$

where $P_1(z),\ P_2(z),\ Q_1(z)$ and $Q_2(z)$ are polynomials in z such that $P_1(z),\ P_2(z),\ Q_1(z)$ and $Q_2(z)$ are not all constants (see Theorems 1.2–1.4 in [6]).

In [7], we have obtained the forms of subnormal solutions of nonhomogeneous equation (1.1) when $\deg P_1 > \deg P_2$, i.e.,

Theorem 1.A [7]. *Suppose that $f(z)$ is a subnormal solution of (1.1), where $P_1(z),\ P_2(z),\ Q_1(z)$ and $Q_2(z)$ are polynomials in z such that $P_1(z),\ P_2(z),\ Q_1(z)$ and $Q_2(z)$ are not all constants.*

(i) *If $\deg P_1 > \deg P_2$ and $\deg P_1 > \deg R_1$, then $f(z)$ must have the form*

$$f(z) = e^{\beta z}[g_1(e^z) + g_2(e^{-z})],$$

where β is a constant, $g_1(z)$ and $g_2(z)$ are polynomials in z.

(ii) *If $\deg P_1 > \deg P_2$ and $\deg P_1 \leqslant \deg R_1$, then $f(z)$ must have the form*

$$f(z) = e^{\beta z}[g_1(e^z) + g_2(e^{-z})] + c_1 z g_3(e^{-z}) + c_2 g_4(e^{-z}) + g_0(e^z),$$

where β is a constant, c_1 and c_2 are constants that may or may not be equal to zero, $g_0(z)$ may be equal to zero or may be a polynomial in z, $g_1(z),\ g_2(z),\ g_3(z)$ and $g_4(z)$ are polynomials in z with $\deg\{g_3\} \geqslant 1$.

In this paper, we will obtain the forms of subnormal solutions of nonhomogeneous equation (1.1) when $\deg P_1 \leqslant \deg P_2$, and resolve completely the open problem raised by Gundersen and Steinbat in 1994, i.e.,

Theorem 1.1. *Suppose that $f(z)$ is a subnormal solution of (1.1), where $P_1(z),\ P_2(z),\ Q_1(z),\ Q_2(z),\ R_1(z)$ and $R_2(z)$ are polynomials in z such that $P_1(z),\ P_2(z),\ Q_1(z)$ and $Q_2(z)$ are not all constants. If $\deg P_1 < \deg P_2$, then $f(z)$ must have the form*

$$f(z) = e^{\beta z}[g_1(e^z) + g_2(e^{-z})], \tag{1.3}$$

where β is a constant, $g_1(z)$ and $g_2(z)$ are polynomials in z.

Theorem 1.2. *Suppose that $f(z)$ is a subnormal solution of (1.1), where $P_1(z),\ P_2(z),\ Q_1(z),\ Q_2(z),\ R_1(z)$ and $R_2(z)$ are polynomials in z such that $P_1(z),\ P_2(z),\ Q_1(z)$ and $Q_2(z)$ are not all constants. If $\deg P_1 = \deg P_2 \geqslant 1$, then $f(z)$ must have one of the following two forms:*

$$f(z) = ce^{\beta_1 z}[g_1(e^z) + g_2(e^{-z})] + e^{\beta_2 z}[g_3(e^z) + g_4(e^{-z})], \tag{1.4}$$

where β_1 and β_2 are constants such that β_1 is not an integer, c is a constant that may or may not be equal to zero, and $g_1(z),\ g_2(z),\ g_3(z)$ and $g_4(z)$ are polynomials in z, or

$$f(z) = e^{nz}\{e^{\beta z}[g_1(e^z) + g_2(e^{-z})] + c_1 z g_3(e^{-z}) + c_2 g_4(e^{-z}) + g_0(e^z)\}, \tag{1.5}$$

where n is an integer and β is a constant, c_1 and c_2 are constants that may or may not be equal to zero, $g_0(z)$ may be equal to zero or may be a polynomial in z, and $g_1(z),\ g_2(z),\ g_3(z)$ and $g_4(z)$ are polynomials in z with $\deg\{g_3\} \geqslant 1$.

2. Proof of Theorem 1.1

We begin with some lemmas.

Lemma 2.1 [6, Theorem 1.3]. *Suppose that $f(z)$ is a subnormal solution of homogeneous equation*

$$f'' + [P_1(e^z) + Q_1(e^{-z})]f' + [P_2(e^z) + Q_2(e^{-z})]f = 0, \tag{2.1}$$

where $P_1(z),\ P_2(z),\ Q_1(z)$ and $Q_2(z)$ are polynomials in z such that $P_1(z),\ P_2(z),\ Q_1(z)$ and $Q_2(z)$ are not all constants. If $\deg P_1 < \deg P_2$, then the only subnormal solution $f(z)$ of (2.1) is $f(z) \equiv 0$.

Lemma 2.2 [6, Lemma 2.3]. *Suppose that $f(z)$ is an entire and subnormal solution of*

$$P_0(e^z, e^{-z})f^{(n)} + P_1(e^z, e^{-z})f^{(n-1)} + \cdots + P_n(e^z, e^{-z})f = P_{n+1}(e^z, e^{-z}), \tag{2.2}$$

where $P_j(e^z, e^{-z})(j = 0, 1, 2, \ldots, n+1)$ are polynomials in e^z and e^{-z} with $P_0(e^z, e^{-z}) \not\equiv 0$, and that $f(z)$ and $f(z + 2\pi i)$ are linearly dependent. Then $f(z)$ has the form

$$f(z) = e^{\beta z}[g_1(e^z) + g_2(e^{-z})]$$

where β is a constant, $g_1(z)$ and $g_2(z)$ are polynomials in z.

Proof of Theorem 1.1. Suppose that $f(z)$ is a subnormal solution of (1.1), so is $f(z+2\pi i)$. Thus,

$$f(z)-f(z+2\pi i)$$

is a subnormal solution of (2.1). Since $\deg P_1 < \deg P_2$, it follows from Lemma 2.1 that

$$f(z)-f(z+2\pi i)\equiv 0.$$

Hence, we have $f(z)$ has the form of (1.3) by Lemma 2.2. This completes the proof of Theorem 1.1.

Example 2.1. If n and q are any two integers, then $f(z)=e^{nz}+e^{-qz}$ is a solution of

$$f''+(e^z+q+e^{-z}-n)f'+(e^{2z}+qe^z-nq+qe^{-z})f=e^{(n+2)z}+(n+q)e^{(n+1)z}+(n+q)e^{(n-1)z}+e^{(2-q)z}.$$

This is an example of Theorem 1.1 when $\deg P_1 < \deg P_2$.

3. Proof of Theorem 1.2

We need the following lemmas.

Lemma 3.1 [6, Theorem 1.2]. *Suppose that $f(z)$ is a subnormal solution of (2.1), where $P_1(z)$, $P_2(z)$, $Q_1(z)$ and $Q_2(z)$ are polynomials in z and are not all constants.*

(i) *If $\deg P_1 > \deg P_2$ and $P_2+Q_2\equiv 0$, then any subnormal solution $f(z)$ of (2.1) must be a constant.*
(ii) *If $\deg P_1 > \deg P_2$ and $P_2+Q_2\not\equiv 0$, then $f(z)\not\equiv 0$ must have the form*

$$f(z)=g_2(e^{-z}),$$

where $g_2(z)$ is a polynomial in z with $\deg\{g_2\}\geqslant 1$.

Lemma 3.2 [9]. *Suppose that $f(z)$ is a subnormal solution of (2.1), where $P_1(z),\ P_2(z),\ Q_1(z)$ and $Q_2(z)$ are polynomials in z and are not all constants. Then $f(z)$ must have the form*

$$f(z)=e^{\beta z}[g_1(e^z)+g_2(e^{-z})],$$

where β is a constant, $g_1(z)$ and $g_2(z)$ are polynomials in z.

Proof of Theorem 1.2. Suppose that $f(z)$ is a subnormal solution of (1.1), so is $f(z+2\pi i)$. Thus $f(z)-f(z+2\pi i)$ is a subnormal solution of (2.1).

If $f(z)-f(z+2\pi i)\equiv 0$, it follows from Lemma 2.2 that $f(z)$ must have the form

$$f(z)=e^{\beta z}[g_1(e^z)+g_2(e^{-z})], \tag{3.1}$$

where β is a constant, $g_1(z)$ and $g_2(z)$ are polynomials in z. This is the form of (1.4). In this case the constant c is equal to zero.

If $f(z)-f(z+2\pi i)\not\equiv 0$. Since $f(z)-f(z+2\pi i)$ is a subnormal solution of (2.1), we have by Lemma 3.2,

$$f(z)-f(z+2\pi i)=e^{\beta_1 z}[g_1(e^z)+g_2(e^{-z})], \tag{3.2}$$

where β_1 is a constant, $g_1(z)$ and $g_2(z)$ are polynomials in z.

Now, we will discuss the following two cases.

Case 3.1. Suppose that the constant β_1 in (3.2) is not an integer. Set

$$g(z)=f(z)+\frac{1}{e^{2\pi i\beta_1}-1}e^{\beta_1 z}[g_1(e^z)+g_2(e^{-z})], \tag{3.3}$$

where the constant β_1 is not an integer, $g_1(z)$ and $g_2(z)$ are polynomials in z. Then from (3.2) and (3.3), $g(z)$ is a subnormal solution of (1.1). However, by (3.3),

$$g(z+2\pi i)=f(z+2\pi i)+\frac{1}{e^{2\pi i\beta_1}-1}e^{\beta_1 z}\cdot[g_1(e^z)+g_2(e^{-z})]\cdot e^{2\pi i\beta_1}. \tag{3.4}$$

Thus, by (3.2)–(3.4), we have

$$g(z)-g(z+2\pi i)\equiv 0.$$

Thus $g(z)$ is a subnormal solution of (1.1) and $g(z)\equiv g(z+2\pi i)$. By Lemma 2.2, we obtain that $g(z)$ has the form

$$g(z)=e^{\beta_2 z}[g_3(e^z)+g_4(e^{-z})], \tag{3.5}$$

where β_2 is a constant, $g_3(z)$ and $g_4(z)$ are polynomials in z. It follows from (3.3) and (3.5) that

$$f(z) = \frac{1}{1-e^{2\pi i\beta_1}} e^{\beta_1 z} \cdot [g_1(e^z) + g_2(e^{-z})] + e^{\beta_2 z}[g_3(e^z) + g_4(e^{-z})],$$

where β_1 and β_2 are constants such that β_1 is not an integer, $g_1(z),\ g_2(z),\ g_3(z)$ and $g_4(z)$ are polynomials in z. This is the form of (1.4). In this case the constant $c = \frac{1}{1-e^{2\pi i\beta_1}}$.

Case 3.2. Suppose that the constant β_1 in (3.2) is an integer. Let α be a constant such that

$$\deg\{P_2 - \alpha P_1\} < \deg P_1 = \deg P_2, \tag{3.6}$$

and set

$$h_1(z) = e^{\alpha z} f(z). \tag{3.7}$$

Since $f(z)$ is a subnormal solution of (1.1), we obtain that $h_1(z)$ is a subnormal solution of

$$h'' + [P_3(e^z) + Q_3(e^{-z})]h' + [P_4(e^z) + Q_4(e^{-z})]h = e^{\alpha z}[R_1(e^z) + R_2(e^{-z})], \tag{3.8}$$

where

$$P_3(e^z) = P_1(e^z) - 2\alpha, \quad Q_3(e^{-z}) = Q_1(e^{-z}),$$
$$P_4(e^z) = P_2(e^z) - \alpha P_1(e^z) + \alpha^2, \quad Q_4(e^{-z}) = Q_2(e^{-z}) - \alpha Q_1(e^{-z}).$$

Thus, $P_3(z),\ P_4(z),\ Q_3(z)$ and $Q_4(z)$ are polynomials in z with $\deg P_3 > \deg P_4$ by (3.6).

Since $f(z)$ is a subnormal solution of (1.1), so is $f(z + 2\pi i)$. Similar to the proof that $h_1(z)$ is a subnormal solution of (3.8), we obtain

$$h_2(z) = e^{\alpha z} f(z + 2\pi i), \tag{3.9}$$

is also a subnormal solution of (3.8). Thus

$$h(z) = h_1(z) - h_2(z) \tag{3.10}$$

is a subnormal solution of

$$h'' + [P_3(e^z) + Q_3(e^{-z})]h' + [P_4(e^z) + Q_4(e^{-z})]h = 0, \tag{3.11}$$

where $P_3(z),\ P_4(z),\ Q_3(z)$ and $Q_4(z)$ are polynomials in z with $\deg P_3 > \deg P_4$.

It follows from (3.2), (3.7), (3.9) and (3.10) that

$$h(z) = e^{(\alpha+\beta_1)z}[g_1(e^z) + g_2(e^{-z})], \tag{3.12}$$

where β_1 is an integer, $g_1(z)$ and $g_2(z)$ are polynomials in z.

Now, we will discuss the following two subcases.

Subcase 3.1. If $P_4 + Q_4 \equiv 0$, we obtain from Lemma 3.1 (i) that $h(z) = C$, where C is a constant. Thus, by (3.12),

$$e^{(\alpha+\beta_1)z}[g_1(e^z) + g_2(e^{-z})] = C, \tag{3.13}$$

where C is a constant. From (3.13), $\alpha + \beta_1$ must be an integer. Since we have supposed β_1 is an integer, we also obtain α is an integer. Hence (3.8) turns into

$$h'' + [P_3(e^z) + Q_3(e^{-z})]h' + [P_4(e^z) + Q_4(e^{-z})]h = S_5(e^z) + S_6(e^{-z}), \tag{3.14}$$

where $P_3(z),\ P_4(z),\ Q_3(z),\ Q_4(z),\ S_5(z)$ and $S_6(z)$ are polynomials in z and $\deg P_3 > \deg P_4$. Since $h_1(z)$ is a subnormal solution of (3.14), it follows from Theorem 1.A that

$$h_1(z) = e^{\beta z}[g_1(e^z) + g_2(e^{-z})] + c_1 z g_3(e^{-z}) + c_2 g_4(e^{-z}) + g_0(e^z), \tag{3.15}$$

where β is a constant and c_1 and c_2 are constants that may and may not be equal to zero, $g_0(z)$ may be equal to zero or may be a polynomial in z, and $g_1(z),\ g_2(z),\ g_3(z)$ and $g_4(z)$ are polynomials in z with $\deg\{g_3\} \geqslant 1$. By (3.7) and (3.15), we obtain

$$f(z) = e^{-\alpha z}\{e^{\beta z}[g_1(e^z) + g_2(e^{-z})] + c_1 z g_3(e^{-z}) + c_2 g_4(e^{-z}) + g_0(e^z)\}, \tag{3.16}$$

where α is an integer. This is the form of (1.5).

Subcase 3.2. If $P_4 + Q_4 \not\equiv 0$, we obtain from Lemma 3.1(ii) that

$$h(z) = g_3(e^{-z}), \tag{3.17}$$

where $g_3(z)$ is a polynomial in z with $\deg\{g_3\} \geqslant 1$. It follows from (3.12) and (3.17) that

$$e^{(\alpha+\beta_1)z}[g_1(e^z) + g_2(e^{-z})] = g_3(e^{-z}), \tag{3.18}$$

where β_1 is an integer, $g_1(z),\ g_2(z)$ and $g_3(z)$ are polynomials in z with $\deg\{g_3\} \geqslant 1$. From (3.18), $\alpha+\beta_1$ must be an integer. Since we have supposed β_1 is an integer, we also obtain α is an integer. Hence (3.8) turns into (3.14). Similar to the proof of Subcase 3.1 of Theorem 1.2, we can obtain that (3.15) and (3.16) hold. This is a form of (1.5). Subcase 3.2 of Theorem 1.2 is completed. Theorem 1.2 is completed completely.

Example 3.1. $f(z) = e^z + e^{-2z} = e^{-z}(e^{2z} + e^{-z})$ is a solution of

$$f'' + (e^z + e^{-z} + a + 3)f' + (2e^z + e^{-z} + 2a - 4)f = 3e^{2z} + 3ae^z - 6e^{-2z} - e^{-3z} + 2.$$

This is an example of Theorem 1.2 when $\deg P_1 = \deg P_2 \geqslant 1$. In this case $f(z) \equiv f(z+2\pi i)$.

Example 3.2. $f(z) = e^{(-1+i)z} + e^z$ is a solution of

$$f'' + (e^z + e^{-2z} + 1 - i)f' + (1-i)[e^z + e^{-2z}]f = (2-i)[e^{2z} + e^z + e^{-2z}].$$

This is an example of Theorem 1.2 when $\deg P_1 = \deg P_2 \geqslant 1$. In this case $f(z)f(z+2\pi i)$ and $\beta = i$ is not an integer.

Example 3.3. $f(z) = e^z[e^{2z} + e^z + z(1+e^{-z})]$ is a solution of

$$f'' + (e^z + e^{-2z} + 1)f' - (e^z + 1)f = 2e^{4z} + 12e^{3z} + 9e^{2z} + 6e^z + e^{-z} + 2.$$

This is an example of Theorem 1.2 when $\deg P_1 = \deg P_2 \geqslant 1$. In this case $f(z)f(z+2\pi i)$ and β is an integer.

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