Appl. Math. J. Chinese Univ. Ser. B
2007, 22(4): 405-412

PERTURBATION RESULTS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

Huang Zhibo Chen Zongxuan

Abstract. Two perturbation results on the linear differential function $f^{''}+\Pi(z)A(z)f=0$ are obtained, where $\Pi(z)$ and $A(z)$ are periodic entire functions with period $2\pi\mathrm{i}$ and $\sigma_e(\Pi)<\sigma_e(A)$.

§1 Introduction

The standard notation from Nevanlinna theory is used in this paper(see [1-3]). In addition, we use the notation $\sigma(f)$ and $\lambda(f)$ respectively to denote the order of growth and exponent of convergence of zeros for a meromorphic function f. The following results were proved in [4].

Theorem A. Let $A(z)=B(\mathrm{e}^z)$, where $B(\zeta)=g_1(1/\zeta)+g_2(\zeta), g_1$ and g_2 are entire functions with g_2 transcendental and $\sigma(g_2)$ not equal to a positive integer or infinity, and g_1 arbitrary.
(i) Suppose $\sigma(g_2)>1$.
(a) If f is a non-trivial solution of

$$f^{''}+A(z)f=0 \tag{1.1}$$

with $\lambda_e(f)<\sigma(g_2)$, then $f(z)$ and $f(z+2\pi\mathrm{i})$ are linearly dependent.
(b) If f_1 and f_2 are any two linearly independent solutions of (1.1), then $\lambda_e(f_1f_2)\geq\sigma(g_2)$.
(ii) Suppose $\sigma(g_2)<1$.
(a) If f is a non-trivial solution of (1.1) with $\lambda_e(f)<1$, then $f(z)$ and $f(z+2\pi\mathrm{i})$ are linearly dependent.
(b) If f_1 and f_2 are any two linearly independent solutions of (1.1), then $\lambda_e(f_1f_2)\geq 1$.

We remark that the conclusion of Theorem A is yet valid if we assume $\sigma(g_1)$ is not equal to an integer or infinity, with g_2 arbitrary and $B(\zeta)=g_1(1/\zeta)+g_2(\zeta)$. In this case when g_1 is transcendental with its order not equal to an integer or infinity and g_2 is arbitrary, we need only consider $B^*(\eta)=B(1/\eta)=g_1(\eta)+g_2(1/\eta)$, g_1 in $0<|\eta|<+\infty, \eta=1/\zeta$.

Theorem B. Let $g(\zeta)$ be a transcendental entire function and its order be not a positive integer

Received: 2007-01-15.
MR Subject Classification: 30D35, 34A20.
Keywords: complex oscillation, differential function, period, linearly dependent, linearly independent.
Digital Object Identifier(DOI): 10.1007/s11766-007-0404-1.
Supported by the College's Ph.D Foundation of China (20050574002) and the Natural Science Foundation of Guangdong Province(06025059).

and infinity. Let $A(z) = B(\mathrm{e}^z)$, where $B(\zeta) = g(1/\zeta) + \Sigma_{j=1}^{p} b_j \zeta^j$ and p is an odd positive integer, then $\lambda(f) = +\infty$ for each non-trivial solution f of (1.1). In fact, the stronger conclusion

$$\log^+ N(r, f) \neq o(r) \tag{1.2}$$

holds.

§2 Main results

Suppose (1.1) admits a non-trivial solution that has a finite e-type exponent of convergence of zeros. If $\Pi(z)$ is a periodic entire function with period $2\pi\mathrm{i}$ and $\sigma_e(\Pi) < \sigma_e(A)$, what can we say about the e-type exponent of convergence of zeros for any two linearly independent solutions of the equation

$$f'' + \Pi(z)A(z)f = 0? \tag{2.1}$$

A perturbation result for differential function

$$f'' + (A(z) + \Pi(z))f = 0 \tag{2.2}$$

where the coefficients of (2.2) are not necessarily periodic is given in [5], and a perturbation result of (2.2) where the coefficients are periodic is given in [4]. We answer the above problem based on the method used in [4,5] coupled with the special properties of periodic coefficients.

Theorem 1. Let $B(\zeta) = g_1(1/\zeta) + g_2(\zeta)$, $C(\zeta) = g_3$, where g_1, g_2 and g_3 are entire functions of finite order such that $\sigma(g_2)$ is a positive integer, and $\sigma(g_3) < \sigma(g_2)$. Suppose $A(z) = B(\mathrm{e}^z), \Pi(z) = C(\mathrm{e}^z)$ and furthermore that (1.1) admits a non-trivial solution f with $\lambda_e(f) < \sigma(g_2)$ and that $f(z)$ and $f(z + 2\pi\mathrm{i})$ are linearly independent. Then

(i) for any non-trivial solution h of (2.1) with $\lambda_e(h) < \sigma(g_2)$, $h(z)$ and $h(z+2\pi\mathrm{i})$ are linearly dependent.

(ii) for any two linearly independent solutions h_1 and h_2 of (2.1), we must have $\lambda_e(h_1 h_2) \geq \sigma(g_2)$.

We easily deduce the following result from Theorem 1.

Corollary. Under the assumption in Theorem 1, any two linearly independent solutions h_1 and h_2 of (2.1) must have $\lambda(h_1 h_2) = +\infty$.

Theorem B investigates the oscillation properties of any non-trivial solution of (1.1) with $B(\zeta) = g(1/\zeta) + \Sigma_{j=1}^{p} b_j \zeta^j$ while p is an odd positive integer and $\sigma(g)$ is not a positive integer and infinity. We now investigate perturbation problem of (2.1) precisely when $\sigma(g)$ is a positive integer.

Theorem 2. Let $g(\zeta)$ be a transcendental entire function of an integer order $\sigma(g)$, and $C(\zeta) \not\equiv 0$ be an entire function with $\sigma(C) < \sigma(g)$. Let $A(z) = B(\mathrm{e}^z)$, where $B(\zeta) = \Sigma_{j=1}^{p} b_{-j} \zeta^{-j} + g(\zeta)$, p is an odd positive integer and $\Pi(z) = C(\mathrm{e}^z)$. Suppose (1.1) admits a non-trivial solution f with $\lambda(f) < +\infty$, then any non-trivial solution h of (2.1) must have $\lambda(h) = +\infty$. In fact, the stronger conclusion (1.2) holds.

§3 Preliminaries for the proofs of main results

The main tools that we use in this paper are Nevanlinna theory in C_0[2,6] where $C_0 = C \setminus \{z : |z| \leq R_0\}$ and Valiron representation for functions analytic in C_0[7], and the fact that if f_1 and f_2

are two non-trivial, linearly independent solutions of (1.1), then the product $E(z) = f_1(z)f_2(z)$ satisfies the differential equation

$$-4A(z) = \frac{c^2}{E^2(z)} - \left(\frac{E'(z)}{E(z)}\right)^2 + 2\frac{E''(z)}{E(z)}, \tag{3.1}$$

where $c \neq 0$ is the Wronskian of f_1 and f_2 (see [1]), and

$$T(r,E) = N(r,1/E) + \tfrac{1}{2}T(r,A) + O(\log rT(r,E)) \qquad \text{n.e.} \tag{3.2}$$

In general, we use "n.e." to denote that an asymptotic relation holds except a set of finite measure. Our argument actually depends on an analogous formula of (3.2) on the Valiron representation of periodic functions.

In this paper, we use the following definitions[4,8].

Let $A(z)$ be an entire function. We define

$$\sigma_e(A) = \overline{\lim_{r\to+\infty}} \frac{\log T(r,A)}{r} \tag{3.3}$$

to be the e-type order of $A(z)$, and also define

$$\lambda_e(A) = \overline{\lim_{r\to+\infty}} \frac{\log^+ N(r,1/A)}{r} \tag{3.4}$$

to be the e-type exponent of convergence of zeros of $A(z)$. We shall have occasions to consider the zeros of $A(z)$ in the right-half plane only. In that case, we define the upper limit in (3.4) by $\lambda_{eR}(A)$ when we only count the zeros of $A(z)$ in the right-half plane, and similarly, we define $\lambda_{eL}(A)$ to be the upper limit in (3.4) when we only count the zeros of $A(z)$ in the left-half plane.

Let $B(\zeta)$ be analytic in $0 < |\zeta| < +\infty$, hence we have a representation $B(\zeta) = g_1(1/\zeta) + g_2(\zeta)$, where both $g_1(\zeta)$ and $g_2(\zeta)$ are entire functions. Let $A(z) = B(\mathrm{e}^z) = A_1(z) + A_2(z)$, where $A_1(z) = g_1(\mathrm{e}^{-z})$ and $A_2(z) = g_2(\mathrm{e}^z)$. Observe that the transformation $\zeta = \mathrm{e}^z$ is a one-to-one correspondence between the sets $\{z : -\log\rho \le \mathrm{Re}z \le \log\rho, -\pi < \mathrm{Im}z \le \pi\}$ and $\{\zeta : \rho^{-1} \le |\zeta| \le \rho\}$. By the periodicity of e^z, we have

$$\begin{aligned} \max_{\rho^{-1}\le|\zeta|\le\rho} |B(\zeta)| &= \max_{\substack{-\log\rho\le\mathrm{Re}z\le\log\rho,\\ -\pi<\mathrm{Im}z\le\pi}} |A(z)| \le \max_{|z|\le\log\rho+\pi} |A(z)| \\ &\le \max_{\substack{-(\log\rho+\pi)\le\mathrm{Re}z\le\log\rho+\pi,\\ -\pi<\mathrm{Im}z\le\pi}} |A(z)| = \max_{(\mathrm{e}^\pi\rho)^{-1}\le|\zeta|\le(\mathrm{e}^\pi\rho)} |B(\zeta)|, \end{aligned}$$

so deduce that

$$\sigma_e(A) = \overline{\lim_{\rho\to+\infty}} \frac{\log\log \max\limits_{\rho^{-1}\le|\zeta|\le\rho} |B(\zeta)|}{\log\rho}. \tag{3.5}$$

From $\max\limits_{\rho^{-1}\le|\zeta|\le\rho} |B(\zeta)| = \max\{\max\limits_{|\zeta|=\rho^{-1}} |B(\zeta)|, \max\limits_{|\zeta|=\rho} |B(\zeta)|\}$, this together with (3.5) yields

$$\sigma_e(A) = \overline{\lim_{\rho\to+\infty}} \frac{\log\log \max\limits_{\rho^{-1}\le|\zeta|\le\rho} |B(\zeta)|}{\log\rho} = \max\{\sigma(g_1), \sigma(g_2)\}. \tag{3.6}$$

In particular, $\sigma_e(A_1) = \sigma(g_1), \sigma_e(A_2) = \sigma(g_2)$.

Let us now turn to the discussion of zeros. Let $n(D, 1/F)$ be the number of zeros of $F(z)$ in the set D, then we deduce

$$\begin{aligned} n\left(\rho^{-1} \le |\zeta| \le \rho, 1/B(\zeta)\right) &= n\left(\begin{Bmatrix} -\log\rho \le \mathrm{Re}z \le \log\rho \\ -\pi < \mathrm{Im}z \le \pi \end{Bmatrix}, 1/A(z)\right) \\ &\le n\left(\{|z| \le \log\rho + \pi\}, 1/A(z)\right) \end{aligned}$$

$$\begin{aligned} &\leq \ 2\left(\frac{(\log\rho+\pi)-\pi}{2\pi}+1\right)n\left(\left\{\begin{matrix}-(\log\rho+\pi)\leq \mathrm{Re}z\leq \log\rho+\pi\\ -\pi<\mathrm{Im}z\leq\pi\end{matrix}\right\},1/A(z)\right)\\ &= \ \left(\frac{\log\rho}{\pi}+2\right)n\left(\{(\mathrm{e}^{\pi}\rho)^{-1}\leq|\zeta|\leq \mathrm{e}^{\pi}\rho\},1/B(\zeta)\right). \end{aligned}$$

Thus

$$\lambda_e(A)=\overline{\lim_{\rho\to+\infty}}\frac{\log n\big(\{\rho^{-1}\leq|\zeta|\leq\rho\},1/B(\zeta)\big)}{\log\rho}. \tag{3.7}$$

Similarly, we have

$$\begin{aligned} \lambda_{eR}(A)&=\overline{\lim_{\rho\to+\infty}}\frac{\log n(\{1<|\zeta|\leq\rho\},1/B(\zeta))}{\log\rho};\\ \lambda_{eL}(A)&=\overline{\lim_{\rho\to+\infty}}\frac{\log n(\{\rho^{-1}\leq|\zeta|\leq1\},1/B(\zeta))}{\log\rho}. \end{aligned} \tag{3.8}$$

Thus $\lambda_e(A)=\max\{\lambda_{eL}(A),\lambda_{eR}(A)\}$.

If f is analytic in C_0, then [7] implies that

$$f(z)=z^n\Theta(z)u(z), \tag{3.9}$$

where n is an integer, $\Theta(z)$ is analytic and non-vanishing on $C_0\cup\{\infty\}$, and $u(z)$ is an entire function with $u(z)=\pi(z)\mathrm{e}^{h(z)}$, the function $\pi(z)$ is a Weierstrass product formed from the zeros of f in C_0, and $h(z)$ is an entire function.

Letting $R_0=1$, we may regard $B(\zeta)$ to be analytic in C^*, where $C^*:=C\setminus\{z:|z|\leq1\}$. By (3.9) we have a similar representation

$$B(\zeta)=\zeta^nR(\zeta)b(\zeta), \tag{3.10}$$

where n is an integer, $R(\zeta)$ is analytic and non-vanishing on $C^*\cup\{\infty\}$, and $b(\zeta)$ is an entire function. From (3.10) we obtain

$$\sigma(g_2)=\overline{\lim_{\rho\to+\infty}}\frac{\log\log\max\limits_{|\zeta|=\rho}|B(\zeta)|}{\log\rho}=\overline{\lim_{\rho\to+\infty}}\frac{\log\log M(\rho,b(\zeta))}{\log\rho}=\sigma(b(\zeta)). \tag{3.11}$$

Since the zeros of $B(\zeta)$ and $b(\zeta)$ coincide in $1<|\zeta|<+\infty$, we deduce from (3.8) that

$$\lambda_{eR}(A)=\lambda(b(\zeta)). \tag{3.12}$$

Suppose $w(z)$ is meromorphic in $C_0:=\{z:R_0<|z|<+\infty\}$. By a similar argument as in [7], $w(z)$ has a representation

$$w(z)=z^n\Theta(z)f(z), \tag{3.13}$$

where n is an integer, $\Theta(z)$ is analytic and non-vanishing on $C_0\cup\{\infty\}$, f is a meromorphic function in C. Let $T_1(r,w)$ denote the Nevanlinna characteristic function [1,6] for $w(z)$ in C_0, which is defined by $T_1(r,w)=m_1(r,w)+N_1(r,w)$, where $m_1(r,w)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|w(r\mathrm{e}^{\mathrm{i}\varphi})|d\varphi$, and $N_1(r,w)$ is the counting function for the poles of w in C_0.

From (3.13) we deduce that

$$m_1(r,w)=m(r,f)+O(\log r), \tag{3.14}$$

and $N_1(r,w)=N(r,f)$. Thus

$$T_1(r,w)=T(r,f)+O(\log r). \tag{3.15}$$

Since $T(r,f)=T(r,1/f)+O(1)$, so

$$T_1(r,1/w)=T(r,1/f)+O(\log r)=T(r,f)+O(\log r)=T_1(r,w)+O(\log r), \tag{3.16}$$

that is $T_1(r,1/w)=T_1(r,w)+O(\log r)$. As in [1,6], we define the order of w in C_0 by

$\sigma_1(w)=\overline{\lim\limits_{r\to+\infty}}\frac{\log T_1(r,w)}{\log r}$.

§4 Proof of Theorem 1

Lemma 4.1[1]**.** Let $F(r)$ and $G(r)$ are non-decreasing functions on $(0,\infty)$. If

(i) $F(r)\le G(r)$ n.e.;

or

(ii) $F(r)\le G(r)$, where $r\notin H\cup(0,1]$, $H\subset(0,\infty)$ is a set that has finite logarithmic measure.

Then for any constant $\alpha(\alpha>1)$, there exists a constant $r_0(r_0>0)$ such that $F(r)\le G(\alpha r)$ when $r>r_0$.

Lemma 4.2[9]**.** Let $A_0(z),A_1(z),\cdots,A_{k-2}(z)$ be periodic entire functions with period 2πi and functions in e^z when $k\ge2$, and $A_j(z)$ satisfies $T(r,A_j)=o\{T(r,A_0)\}$ n.e., $j=1,2,\cdots,k-2$. Assume $f\not\equiv0$ is a solution of equation $w^{(k)}+A_{k-2}w^{k-2}+\cdots+A_0w=0$. If

(i) A_0 is rational in e^z and satisfies $\log^+N(r,1/f)=o(r)$, or

(ii) A_0 is transcendental in e^z and satisfies $\log^+N(r,1/f)=O(r)$,

then there exists positive number q and $1\le q\le k$ such that $f(z)$ and $f(z+q2\pi\mathrm{i})$ are linearly dependent.

Proof of Theorem 1. (i) Let f be a non-trivial solution of (1.1) with $\lambda_e(f)<\sigma(g_2)$, and $f(z)$ and $f(z+2\pi\mathrm{i})$ are linearly independent. Since $\lambda_e(f)<\sigma(g_2)<+\infty$, Lemma 4.2 implies that $f(z)$ and $f(z+4\pi\mathrm{i})$ must be linearly dependent. Let $E(z)=f(z)f(z+2\pi\mathrm{i})$, then

$$E(z+2\pi\mathrm{i})=f(z+2\pi\mathrm{i})f(z+4\pi\mathrm{i})=c_1f(z)f(z+2\pi\mathrm{i})=c_1E(z),$$

for some non-zero constant c_1. Clearly $E'(z)/E(z)$ and $E''(z)/E(z)$ are both periodic functions with period 2πi, while $A(z)$ is periodic by definition. Since (3.1) shows that $E^2(z)$ is also a periodic function with period 2πi, we can find an analytic function $\Phi(\zeta)$ in $0<|\zeta|<+\infty$ such that $E^2(z)=\Phi(\mathrm{e}^z)$. Substituting this representation into (3.1) yields

$$-4B(\zeta)=\tfrac{c^2}{\Phi}+\zeta\tfrac{\Phi'}{\Phi}-\tfrac{3}{4}\zeta^2\left(\tfrac{\Phi'}{\Phi}\right)^2+\zeta^2\tfrac{\Phi''}{\Phi}. \tag{4.1}$$

Since both $B(\zeta)$ and $\Phi(\zeta)$ are analytic in $C^*:=\{\zeta:1<|\zeta|<+\infty\}$, the Valiron theory gives their representation as

$$B(\zeta)=\zeta^nR(\zeta)b(\zeta),\quad \Phi(\zeta)=\zeta^{n_1}K_1(\zeta)\phi(\zeta), \tag{4.2}$$

where n and n_1 are some integers, $R(\zeta)$ and $K_1(\zeta)$ are functions being analytic and non-vanishing on $C^*\cup\{\infty\}$ and $b(\zeta)$ and $\phi(\zeta)$ are entire functions. By (3.14) and the lemma on logarithmic derivative in [10], we deduce from (4.1) that

$$m_1(\rho,1/\Phi)=m_1(\rho,B)+O\{\log(\rho\log T(\rho,\phi))\}\quad \text{n.e.} \tag{4.3}$$

Since $N_1(\rho,B)\equiv0$, (4.3) implies

$$T_1(\rho,1/\Phi)=N_1(\rho,1/\Phi)+T_1(\rho,B)+O\{\log(\rho\log T(\rho,\phi))\}. \tag{4.4}$$

Applying (3.15)-(3.16) to (4.4) and using the fact that $N_1(\rho,1/\Phi)=N(\rho,1/\phi)$, we deduce

$$T(\rho,1/\phi)=N(\rho,1/\phi)+T(\rho,b)+O\{\log(\rho\log T(\rho,\phi))\}. \tag{4.5}$$

It is easy to see that $\lambda_e(f)=\lambda_e(E)=\lambda_e(E^2)$. Since $\lambda_e(f)<\sigma(g_2)$, so $\lambda_{eR}(E^2)\le\lambda_e(E^2)=\lambda_e(f)<\sigma(g_2)$. As in (3.12), $\lambda(\phi)=\lambda_{eR}(\Phi)=\lambda_{eR}(E^2)$. But $\sigma(g_2)=\sigma(b(\zeta))$ by (3.11), hence

$\lambda(\phi) < \sigma(b(\zeta))$. Notice that (4.5) satisfied by ϕ is an analogous formula to (3.2) satisfied by E. It follows from (4.5) and Lemma 4.1 that $\sigma(\phi) = \sigma(b(\zeta))$, thus $\lambda(\phi) < \sigma(\phi)$. We know $\sigma(g_2)$ is a positive integer, so are $\sigma(b(\zeta))$ and $\sigma(\phi(\zeta))$. Thus we may rewrite $\phi(\zeta) = \pi_1(\zeta)e^{P(\zeta)}$, where $\pi_1(\zeta)$ is an entire function, $\sigma(\pi_1) < \sigma = \sigma(g_2)$, $P(\zeta) = \alpha\zeta^\sigma$, and α is a non-zero constant.

Let us now suppose (2.1) possesses a non-trivial solution $h(z)$ such that $\lambda_e(h) < \sigma(g_2)$ but $h(z)$ and $h(z+2\pi i)$ are linearly independent. Let $F(z) = h(z)h(z+2\pi i)$. By a similar argument that we have applied to $E(z)$ above, we conclude that there exists an analytic function $\Psi(\zeta)$ in $0 < |\zeta| < +\infty$ such that $F^2(z) = \Psi(e^z)$. Similarly, $\Psi(\zeta)$ has a Valiron representation

$$\Psi(\zeta) = \zeta^{n_2}K_2(\zeta)\psi(\zeta) \tag{4.6}$$

in C^*, where n_2 is an integer, $K_2(\zeta)$ is analytic and non-vanishing on $C^* \cup \{\infty\}$, and $\psi(\zeta)$ is an entire function in C.

We now substitute $F^2(z) = \Psi(\zeta)$, with $\zeta = e^z$, into (3.1) with $A(z)$ replaced by $\mathit{\Pi}(z)A(z)$. This yields

$$-4C(\zeta)B(\zeta) = \tfrac{c_2^2}{\Psi} + \zeta\tfrac{\Psi'}{\Psi} - \tfrac{3}{4}\zeta^2\left(\tfrac{\Psi'}{\Psi}\right)^2 + \zeta^2\tfrac{\Psi''}{\Psi}, \tag{4.7}$$

where c_2 is a the Wronskian of $h(z)$ and $h(z+2\pi i)$.

In a similar fashion of $\phi(\zeta)$,we have

$$\lambda(\psi) = \lambda_{eR}(F^2) \le \lambda_e(F^2) = \lambda_e(h) < \sigma(g_2) = \sigma(b(\zeta)).$$

We then apply a similar argument to (4.7) to obtain

$$T(\rho,\psi) = N(\rho, 1/\psi) + T(\rho, d) + O\{\log\rho\log T(\rho,\psi)\}. \tag{4.8}$$

As to (4.1) for (4.5), $d(\zeta)$ is an entire function appearing in Valiron representation of

$$C(\zeta)B(\zeta) = \zeta^{n_3}R_d(\zeta)d(\zeta),$$

where functions $R_d(\zeta)$ and $d(\zeta)$ play the same role of corresponding functions in (4.2), and it is easy to check that $\sigma(d) = \sigma(b(\zeta))$. It follows from (4.8) and Lemma 4.1 that $\sigma(\psi) = \sigma(b(\zeta))$, and thus $\lambda(\psi) < \sigma(\psi)$. Since $\sigma(g_2)$ is a positive integer, so are $\sigma(b(\zeta))$ and $\sigma(\psi)$. Thus we may rewrite $\psi(\zeta) = \pi_2(\zeta)e^{Q(\zeta)}$, where $\pi_2(\zeta)$ is an entire function, $Q(\zeta) = \beta\zeta^\sigma$, $\sigma(\pi_2) < \sigma = \sigma(g_2)$, and β is a non-zero constant.

Letting $\mathit{\Phi}(\zeta) = R_1(\zeta)e^{P(\zeta)}$ and $\Psi(\zeta) = R_2(\zeta)e^{Q(\zeta)}$, where $R_1(\zeta) = \zeta^{n_1}K_1(\zeta)\pi_1(\zeta)$ and $R_2(\zeta) = \zeta^{n_2}K_2(\zeta)\pi_2(\zeta)$, $\max\{\sigma_1(R_1), \sigma_1(R_2)\} < \sigma = \sigma(g_2)$, and substituting them into (4.1) and (4.7) respectively, we get

$$\begin{aligned} -4B(\zeta) \;&=\; \frac{c^2}{R_1e^P} + \zeta(\frac{R_1'}{R_1} + P') - \frac{3}{4}\{(\zeta\frac{R_1'}{R_1})^2 + 2\zeta^2\frac{R_1'}{R_1}P' + \zeta^2{P'}^2\} \\ &\quad +\zeta^2(\frac{R_1''}{R_1} + 2\frac{R_1'}{R_1}P' + {P'}^2 + P''), \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} -4C(\zeta)B(\zeta) \;&=\; \frac{c_2^2}{R_2e^Q} + \zeta(\frac{R_2'}{R_2} + Q') - \frac{3}{4}\{(\zeta\frac{R_2'}{R_2})^2 + 2\zeta^2\frac{R_2'}{R_2}Q' + \zeta^2{Q'}^2\} \\ &\quad +\zeta^2(\frac{R_2''}{R_2} + 2\frac{R_2'}{R_2}Q' + {Q'}^2 + Q''). \end{aligned} \tag{4.10}$$

To multiple (4.9) by $C(\zeta)$ yields

$$\begin{aligned}-4C(\zeta)B(\zeta) &= \frac{c^2}{R_1\mathrm{e}^P}C(\zeta)+\zeta(\frac{R_1'}{R_1}+P')C(\zeta)-\frac{3}{4}\left\{\left(\zeta\frac{R_1'}{R_1}\right)^2+2\zeta^2\frac{R_1'}{R_1}P'+\zeta^2P'^2\right\}C(\zeta)\\ &\quad+\zeta^2\left(\frac{R_1''}{R_1}+2\frac{R_1'}{R_1}P'+P'^2+P''\right)C(\zeta). \qquad (4.11)\end{aligned}$$

Subtracting (4.11) form (4.10) leads to

$$0\equiv\tfrac{c^2}{R_1\mathrm{e}^P}C(\zeta)-\tfrac{c_2^2}{R_2\mathrm{e}^Q}+H(\zeta), \qquad (4.12)$$

where $H(\zeta)$ is meromorphic function in C^*. In fact, $H(\zeta)$ is a differential polynomial in $\frac{R_1'}{R_1},\frac{R_2'}{R_2},C(\zeta),P',Q'$ and their derivatives. We can deduce from the definitions of $R_1,R_2,C(\zeta),P,Q$ that $\sigma_1(H)<\sigma=\sigma(g_2)$. Rewrite (4.12) as

$$\mathrm{e}^{-P}+H_1\mathrm{e}^{-Q}=H_2, \qquad (4.13)$$

where H_1 and H_2 are meromorphic functions in C^*, with

$$H_1=-\frac{c_2^2}{c^2}\frac{R_1}{R_2}\frac{1}{C(\zeta)},\quad H_2=-\frac{R_1}{c^2}\frac{H(\zeta)}{C(\zeta)},$$

and $\max\{\sigma_1(H_1),\sigma_1(H_2)\}<\sigma(g_2)$.
Differentiating (4.13) yields

$$\mathrm{e}^{-P}+\tfrac{H_1Q'-H_1'}{P'}\mathrm{e}^{-Q}=-\tfrac{H_2'}{P'}. \qquad (4.14)$$

Using (4.14) to eliminate e^{-P} from (4.13) yields

$$H_3\mathrm{e}^{-Q}=H_4, \qquad (4.15)$$

where $H_3=H_1'+(P'-Q')H_1$ and $H_4=H_2P'+H_2'$ are meromorphic functions in C^* with $\sigma_1(H_3),\sigma_1(H_4)<\sigma(g_2)$. Thus $H_4\equiv 0$, i.e. $H_2=c_3\mathrm{e}^{-P}$, where c_3 is a non-zero constant, from a simple order consideration in (4.15). We can obtain $\sigma_1(H_2)=\sigma=\sigma(g_2)$. This is a contradiction to $\sigma_1(H_2)<\sigma(g_2)$. Hence $h(z)$ and $h(z+2\pi\mathrm{i})$ must be linearly dependent.

(ii) Suppose (2.1) possesses two non-trivial solutions h_1 and h_2 that are linearly independent and $\lambda_e(h_1h_2)<\sigma(g_2)$, then $\lambda_e(h_j)<\sigma(g_2)$ for $j=1,2$. Part (i) implies that $h_j(z)$ and $h_j(z+2\pi\mathrm{i})$ are linearly dependent for $j=1,2$. Let $E(z)=f(z)f(z+2\pi\mathrm{i})$ and $F(z)=h_1(z)h_2(z)$, then $F(z+2\pi\mathrm{i})=c_4F(z)$ for some non-zero constant c_4. Applying a similar argument to $E(z)$ and $F(z)$ as in part (i) yields $\sigma_1(H_2)=\sigma(g_2)$. This is a contradiction. Hence $\lambda_e(h_1h_2)\geq\sigma(g_2)$. This completes the proof of Theorem 1.

§5 Proof of Theorem 2

Lemma 5.1 [11]**.** Let $A(z)=B(\mathrm{e}^{\alpha z})$ ba a periodic entire function with period $\omega=2\pi\mathrm{i}/\alpha$, and be transcendent in $\mathrm{e}^{\alpha z}$, i.e., $B(\zeta)$ is transcendent and analytic on $0<|\zeta|<+\infty$. If $B(\zeta)$ has a pole of odd order at $\zeta=\infty$ or $\zeta=0$ (including those which can be changed into this case by varying the period of $A(z)$), and (1.1) has a solution $f\not\equiv 0$ which satisfies condition

$$\log^+N(r,1/f)=o(r)\quad\text{as}\quad r\to+\infty, \qquad (5.1)$$

then $f(z)$ and $f(z+\omega)$ are linearly independent.

Proof of Theorem 2. Suppose (1.1) possesses a non-trivial solution f with $\lambda(f)<+\infty$, hence $\lambda_e(f)=0<\sigma(g)$ by definition (3.4). Thus, Lemma 5.1 implies that $f(z)$ and $f(z+2\pi i)$

are linearly independent solutions of (1.1), and $f(z)$ and $f(z+2\pi\mathrm{i})$ satisfy the hypotheses of Theorem 1. Suppose (2.1) admits a non-trivial solution $h(z)$ with $\lambda_e(h) < \sigma(g)$, then Lemma 5.1 again implies that $h(z)$ and $h(z+2\pi\mathrm{i})$ are linearly independent. Part (ii) of Theorem 1 shows that $\lambda_e(h(z)h(z+2\pi\mathrm{i})) \geq \sigma(g) > 0$. So $\lambda_e(h) = \lambda_e(h(z)h(z+2\pi\mathrm{i})) > 0$, and thus (1.2) holds. This completes the proof of Theorem 2.

References

1 Gao Shian, Chen Zongxuan, Chen Tewei.Oscillation Theory of Linear Differential Equations, Wuhan: Huazhong University of Science and Technology Press,1998(Chinese).

2 Laine I. Nevanlinna Theory and Complex Differential Equations, Berlin: Walter de Gruyter, 1993.

3 Hayman W K. Meromorphic Functions. Oxford: Clarendon Press, 1975.

4 Chiang Yikman, Gao Shian. On a problem in complex oscillation theory of periodic second order linear differential equations and some related perturbation results, Ann Acad Sci Fenn, 2002, 27:273-290.

5 Bank S B, Laine I, Langley J. Oscillation results for solutions of linear differential equations in the complex domain, Resulate Math, 1989, 16: 3-15.

6 Bank S B, Laine I. Representations of solutions of periodic second order linear differential equations, J Reine Angew Math, 1983, 344: 1-21.

7 Valiron G. Lectures on the General Theory of Integer Functions, New York: Chelsea, 1949.

8 Gao Shian. A result concerning the complex oscillation for periodic second order linear differential equations of transcendental type, Math Appl, 2002, 15(2): 85-88.

9 Bank S B, Langley J. Oscinallation theorems for higher order linear differential equations with entire periodic coefficients, Comment Math Univ St Paul, 1992, 41: 65-85.

10 Yang L. Value Distribution Theory and It's New Researches, Beijing: Science Press, 1982.

11 Gao Shian. A further result on the complex oscillation theory of periodic second order linear differential equations, Proc Edinburgh Math Soc, 1990, 33: 143-158.

School of Math. Sci., South China Normal Univ., Guangzhou 510631, China.