

SEPARATION PROPERTIES FOR SELF-CONFORMAL SETS

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Abstract

For an one-to-one self-conformal contractive system $\{w_j\}_{j=1}^m$ on $\mathbb{R}^d$ with attractor K and conformality dimension α , Peres *et al* showed that the OSC and the SOSC are both equivalent to $0 < \mathcal{H}^\alpha(K) < \infty$. We will give a simple proof of this result as well as some further consideration of the properties related to the separation condition.

1. Introduction

Let $U_0 \subset \mathbb{R}^d$ be a bounded open set. Let $w_j : U_0 \rightarrow U_0$ ($j = 1, 2, \dots, m$) be contractive maps and there exists a nonempty compact subset $X \subseteq U_0$ such that for each $1 \leq j \leq m$, $w_j(X) \subseteq X$. Then there exists a compact subset $K \subseteq X$ such that $K = \bigcup_{j=1}^m w_j(K)$. We say that $\{w_j\}_{j=1}^m$ satisfies the *open set condition* (OSC) if there exists a nonempty bounded open set $U \subseteq U_0$ such that

$$w_j(U) \subseteq U \quad \text{and} \quad w_i(U) \cap w_j(U) = \emptyset \quad \text{for } i \neq j.$$

Such U is called a basic open set of $\{w_j\}_{j=1}^m$. If the above U satisfies further $U \cap K \neq \emptyset$, then $\{w_j\}_{j=1}^m$ is said to satisfy the *strong open set condition* (SOSC). In [S], Schief made use of an idea of Bandt [BG] and showed that for similitude, the two conditions are equivalent, and furthermore they are equivalent to $0 < \mathcal{H}^\alpha(K) < \infty$ where α is the similarity dimension of K .

Recently, Peres, Rams, Simon and Solomyak [P] extended Schief's theorem to self-conformal maps. A simple proof was also given by Lau, Rao and the author for the equivalence of the OSC and the SOSC [L]. In a private communication, Peres asked if the short proof can be used to prove the equivalence to $0 < \mathcal{H}^\alpha(K) < \infty$. In the note we answer his question positively. The main idea and some of the proofs are already in [L] and [FL], we will modify them to fit our purpose. In [LX] Lau and Xu considered the boundary dimension of self-similar set. Some of their results are extended to the self-conformal maps.

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For one-to-one contractive self-conformal IFS $\{w_j\}_{j=1}^m$, we define the *conformality dimension* of the IFS to be the (positive) number α such that the Ruelle operator $T_\alpha : C(K) \rightarrow C(K)$ defined by

$$T_\alpha f(x) = \sum_{j=1}^m |w'_j(x)|^\alpha f(w_j(x))$$

has spectral radius 1 [FL]. We let $\mathcal{H}^\alpha$ be the α -Hausdorff measure.

We prove following Theorem 1.1 by constructing a basic open set U satisfies both of the SOSC and $\dim_H(K \setminus U) < \alpha$. The key of the proof is Lemma 3.4. Furthermore we remark that in the previous consideration of the self-conformality, an additional connectedness assumption is added to the open set U in the OSC (see e.g. [MU], [P]), we will see that the assumption is redundant (Lemma 2.1 and the remark there.) Our basic results are

Theorem 1.1. *Let $\{w_j\}_{j=1}^m$ be one-to-one self-conformal contractive IFS with $\{|w'_j(x)|\}_{j=1}^m$ satisfies (2.1) and the Dini condition. Then the followings are equivalent:*

- (i) $\{w_j\}_{j=1}^m$ satisfies the OSC.
- (ii) $\{w_j\}_{j=1}^m$ satisfies the SOSC.
- (iii) $0 < \mathcal{H}^\alpha(K) < \infty$.

Theorem 1.2. *Let $\{w_j\}_{j=1}^m$ be as in Theorem 1.1 and satisfy the OSC. Then there exists a basic open set U such that*

$$\dim_H(K \setminus U) < \alpha.$$

2. Preliminaries

Let $\{w_j\}_{j=1}^m$ be self-conformal on U_0 (i.e. for each j and each $x \in U_0$, $w'_j(x)$ is a self-similar matrix and $w'_j(\cdot)$ is continuous). We assume that there exists a nonempty compact set X such that $X \subseteq U_0$, and for each $1 \leq j \leq m$, $w_j(X) \subseteq X$, w_j is one-to-one on U_0 and $w'(\cdot)$ is Dini continuous on U_0 with

$$0 < \inf_{x \in U_0} |w'_j(x)| \leq \sup_{x \in U_0} |w'_j(x)| < 1 \quad \text{for each } 1 \leq j \leq m \quad (2.1)$$

By enlarging X to $X_0 \subseteq U_0$ by a δ -neighborhood, we can show easily from the contractiveness of w_j 's that there exists k' such that

$$\bigcup_{|J|=k} w_J(X_0) \subseteq X_0 \quad \text{for any } k \geq k'$$

where $J = j_1 j_2 \cdots j_k$, $1 \leq j_i \leq m$, $w_J = w_{j_1} \circ w_{j_2} \circ \cdots \circ w_{j_k}$. Hence we may assume without loss of generality that $\overline{X^\circ} = X$ and $B(K, \delta) \subseteq X^\circ$ for some $\delta > 0$ ($B(K, \delta)$ denotes the open δ -neighborhood of K).

We will use the notation $\mathcal{J} = \{J = j_1 j_2 \cdots j_n : 1 \leq j_i \leq m, n \in \mathbb{N}\}$, and for any $J \in \mathcal{J}$ define

$$K_J = w_J(K), \quad r_J = \inf_{x \in X} |w'_J(x)|, \quad R_J = \sup_{x \in X} |w'_J(x)|.$$

Lemma 2.1. *Suppose X and $\{w_j\}_{j=1}^m$ are defined as above. Then*

(i) *there exists a $c_1 > 1$ such that*

$$R_J \leq c_1 r_J \quad \text{for any } J \in \mathcal{J} \quad (2.2)$$

and

$$c_1^{-1} r_I r_J \leq r_{IJ} \leq c_1 r_I r_J \quad \text{for any } I, J \in \mathcal{J}; \quad (2.3)$$

(ii) *there exist $c_2 \geq c_1$ and $\delta > 0$ such that for $x, y, z \in X$, $|x - y| \leq \delta$,*

$$c_2^{-1} |w'_J(z)| \leq \frac{|w_J(x) - w_J(y)|}{|x - y|} \leq c_2 |w'_J(z)| \quad \text{for any } J \in \mathcal{J}; \quad (2.4)$$

(iii) *there exist $c_3 \geq c_2$ and k_0 such that for any $x, y \in X$*

$$|w_J(x) - w_J(y)| \leq c_3 r_J |x - y| \quad \text{for any } J \in \mathcal{J} \text{ with } |J| > k_0. \quad (2.5)$$

Proof. The proof of (i) and (ii) is in [FL, Lemma 2.3]. We will include the proof of (ii) here for completeness. For any $x \in X$, there exists $\delta_x > 0$ such that the δ_x -open ball $B(x, \delta_x) \subseteq U_0$. Since X is compact, there exists $\delta > 0$ (the Lebesgue number) such that for any $x, y \in X$, if $|x - y| \leq \delta$, then $x, y \in B(x', \delta_{x'})$ for some $x' \in X$. Then we can apply the mean value theorem to show that for $|x - y| \leq \delta$

$$|w_J(x) - w_J(y)| = |w'_J(\xi)(x - y)|,$$

where ξ is in the line segment joining x and y . The self-similar property of $w_J(\cdot)$ yields

$$|w_J(x) - w_J(y)| = |w'_J(\xi)| |x - y|. \quad (2.6)$$

This together with (2.2) implies (ii).

(iii) follows directly from the choice of the above δ , (2.6) and (ii). $\blacksquare$

To make use of the local connectedness of X , we take $0 < \varepsilon < 2^{-1} c_3^{-1} \delta$, then $2c_3 \varepsilon \leq \delta$, and hence by the assumption on X , we have

$$B(K, c_3 \varepsilon) \subseteq X. \quad (2.7)$$

For $J \in \mathcal{J}$, let

$$G_J = w_J(B(K, \varepsilon)).$$

Consequently by (2.7) and (2.4), we have for any $x \in K$,

$$B(w_J(x), c_2^{-1}\varepsilon r_J) \subseteq w_J(B(x, \varepsilon)) \subseteq B(w_J(x), c_2\varepsilon r_J). \quad (2.8)$$

It follows that

$$\begin{aligned} B(K_J, c_2^{-1}\varepsilon r_J) &= \bigcup_{x \in K} B(w_J(x), c_2^{-1}\varepsilon r_J) \subseteq G_J = \bigcup_{x \in K} w_J(B(x, \varepsilon)) \\ &\subseteq \bigcup_{x \in K} B(w_J(x), c_2\varepsilon r_J) = B(K_J, c_2\varepsilon r_J). \end{aligned} \quad (2.9)$$

We remark that in [MU] and [P] the connectedness of X was used to apply the mean value theorem so as to deduce (2.8) and (2.9); The above argument says that the local connectedness of X is enough to deduce them. Hence we can study its separation properties in the absence of the connectedness assumption on U_0 so long as we regard the concerned sets to be union of subsets whose diameter are less than δ .

For $0 < b < 1$, we let

$$\Lambda_b = \{J = j_1 \cdots j_n : r_{j_1 \cdots j_n} < b \leq r_{j_1 \cdots j_{n-1}}\}.$$

As in [L], our most crucial difference from [S] and [P] is the following inductive way to define the index set $\Lambda(J)$, $J \in \mathcal{J}$: Let k_0 be defined as in Lemma 2.1(iii), for J with $|J| = k_0$, we define

$$\Lambda(J) = \{I \in \Lambda_{\dim G_J} : K_I \cap G_J \neq \emptyset\}.$$

Suppose $\Lambda(J)$ is defined, for any $1 \leq j \leq m$, we define

$$\Lambda(jJ) = \mathcal{A} \cup \mathcal{B}$$

where

$$\mathcal{A} = \{jI : I \in \Lambda(J)\}$$

and

$$\mathcal{B} = \{I \in \Lambda_{\dim G_{jJ}} : i_1 \neq j \text{ and } K_I \cap G_{jJ} \neq \emptyset\}.$$

(Note that in [S], the $\Lambda(J)$ is defined as $\{I \in \Lambda_{\dim G_J} : K_I \cap G_J \neq \emptyset\}$.) It is easy to see from the construction that each $I \in \Lambda(J)$ is of either type $\mathcal{A}$ or $\mathcal{B}$, $K_I \cap G_J \neq \emptyset$; also K_I and K_J are comparable in size by the following lemma.

Lemma 2.2. *Suppose $\{w_j\}_{j=1}^m$ is as in Lemma 2.1, then there exist k_1 and $c_4 > 0$ such that $c_4^{-1} \leq \frac{r_J}{r_I} \leq c_4$ for all $I \in \Lambda(J)$, $J \in \mathcal{J}$ with $|J| \geq k_1$.*

Proof. The idea is in [L, Lemma 3.1], we modify it to fit our purpose. Let $k_1 \geq k_0$ such that

$$\min\{|I| : I \in \Lambda(J) \text{ and } |J| \geq k_1\} > k_0.$$

For any $I \in \Lambda(J)$, $J \in \mathcal{J}$ with $|J| \geq k_1$. We consider the two cases

(i) If $i_1 \neq j_1$, by the construction of $\mathcal{B}$, we have $I \in \Lambda_{\text{diam} G_J}$. Then

$$r_I \leq \text{diam } G_J \leq r_{i_1 \dots i_{n-1}} \leq c_1 r^{-1} r_I \quad (2.10)$$

where $r = \min_{1 \leq j \leq m} \{r_j\}$. As $\varepsilon < 2^{-1} c_3^{-1} \delta < \delta$, it follows from (2.4) that

$$\text{diam } G_J \geq c_2^{-1} \varepsilon r_J.$$

Hence

$$c_2^{-1} \varepsilon r_J \leq \text{diam } G_J \leq c_1 r^{-1} r_I. \quad (2.11)$$

Also by (2.9), we have

$$\text{diam } G_J \leq 2c_2 \varepsilon r_J + |K_J|.$$

Then by (2.10) and (2.5), it follows that

$$r_I \leq \text{diam } G_J \leq c_3(2\varepsilon + |K|)r_J. \quad (2.12)$$

Hence (2.11) and (2.12) imply that there exists $a > 0$ such that

$$a^{-1} \leq \frac{r_J}{r_I} \leq a. \quad (2.13)$$

(ii) If $i_1 = j_1$, we write

$$J = j_1 \cdots j_l j_{l+1} \cdots j_n := j_1 \cdots j_l J', \quad I = j_1 \cdots j_l i_{l+1} \cdots i_m := j_1 \cdots j_l I'$$

where $j_{l+1} \neq i_{l+1}$. Then by the construction of $\mathcal{A}$, we see inductively that $I' \in \Lambda(J')$ and by (2.13), $a^{-1} \leq r_{J'}/r_{I'} \leq a$. This and Lemma 2.1(i) imply that

$$(ac_1^2)^{-1} \leq \frac{r_J}{r_I} \leq ac_1^2.$$

If we let $c_4 = ac_1^2$, then the lemma follows from the conclusion of the two cases. $\blacksquare$

We remark that for fixed $J_0 \in \mathcal{J}$, the construction of the set $\mathcal{A}$ implies trivially that

$$\Lambda(jJ_0) \supseteq \{jI : I \in \Lambda(J_0)\}, \quad j = 1, \dots, m.$$

The key to prove the SOSC is to find J_0 such that the equality holds (Lemma 3.4). In this case the set $\mathcal{B}$ is empty.

3. The proof of the main results

We need a few notations and lemmas. For any two subsets E, F in $\mathbb{R}^d$, we define

$$\begin{aligned} D(E, F) &= \inf\{|x - y| : x \in E, y \in F\}; \\ d(E, F) &= \inf\{\varepsilon : E \subseteq B(F, \varepsilon), F \subseteq B(E, \varepsilon)\}. \end{aligned}$$

Lemma 3.1. [FL, Lemma 2.8] *Let w be conformal and invertible, let D be a Borel subset in the domain of w , and $0 < \mathcal{H}^\alpha(D) < \infty$. Then we have the following formula of change of variable,*

$$\mathcal{H}^\alpha(w(D)) = \int_D |w'(x)|^\alpha d\mathcal{H}^\alpha(x).$$

Lemma 3.2. *Let $\{w_j\}_{j=1}^m$ be as in Theorem 1.1. Suppose $0 < \mathcal{H}^\alpha(K) < \infty$, then*

$$\mathcal{H}^\alpha(K_I \cap K_J) = 0 \quad \text{for any incomparable } I, J \in \mathcal{J}.$$

Proof. Since T_α has spectral radius 1, by [FL, Theorem 1.1], there exists $0 < h \in C(K)$ such that $h(x) = \sum_{j=1}^m |w'_j(x)|^\alpha h(w_j x)$. Then

$$\begin{aligned} \sum_{j=1}^m \int_{K_j} h(x) d\mathcal{H}^\alpha(x) &\geq \int_{\bigcup_{j=1}^m K_j} h(x) d\mathcal{H}^\alpha(x) = \int_K h(x) d\mathcal{H}^\alpha(x) \\ &= \int_K \sum_{j=1}^m |w'_j(x)|^\alpha h(w_j x) d\mathcal{H}^\alpha(x) = \sum_{j=1}^m \int_{K_j} h(x) d\mathcal{H}^\alpha(x). \end{aligned}$$

(The last equality is followed from Lemma 3.1.) This implies that

$$\mathcal{H}^\alpha(K_i \cap K_j) = 0 \quad \text{for any } i \neq j.$$

It follows immediately that

$$\mathcal{H}^\alpha(K_I \cap K_J) = 0 \quad \text{for any incomparable } I, J \in \mathcal{J}. \quad \blacksquare$$

Lemma 3.3. *Let $\{w_j\}_{j=1}^m$ be as in Lemma 3.2. Then there exists fixed $\delta_0 > 0$ such that for any $L \in \mathcal{J}$,*

$$\|w_I(\cdot) - w_J(\cdot)\|_{C(K)} \geq \delta_0 r_L \quad \text{for any } I, J \in \Lambda(L) \text{ with } I \neq J.$$

Proof. Since $0 < \mathcal{H}^\alpha(K) < \infty$, there exists an open set U such that $K \subseteq U \subset X$ and

$$0 < \mathcal{H}^\alpha(U) \leq \mathcal{H}^\alpha(K) + 1 < \infty.$$

Let c_1 and δ be defined as in Lemma 2.1 and let $0 < \eta < 2^{-1}c_1^{-\alpha}$, there exists an open covering $\{V_i\}_{i=1}^n$ of K such that

$$K \subseteq V := \bigcup_{i=1}^n V_i \subseteq U,$$

$$\delta' := D(K, V^c) < \delta \quad (3.1)$$

and

$$0 < \mathcal{H}^\alpha(K) \leq \sum_{i=1}^n |V_i|^\alpha < (1 + \eta)\mathcal{H}^\alpha(K). \quad (3.2)$$

For any pair of $I, J \in \Lambda(L)$, let us assume without loss of generality that $\mathcal{H}^\alpha(K_J) \geq \mathcal{H}^\alpha(K_I)$. Then for any given ε satisfying $c_1^\alpha \eta < \varepsilon < 1$, we have

$$\varepsilon \mathcal{H}^\alpha(K_I) < \mathcal{H}^\alpha(K_J). \quad (3.3)$$

We claim that $d(K_I, K_J) \geq \delta' r_I$. Otherwise if $d(K_I, K_J) < \delta' r_I$, by (3.1) and Lemma 2.1(ii), we have $D(K_I, w_I(V^c)) \geq \delta' r_I$, then $K_J \subseteq w_I(V)$. Hence by (3.3) and Lemma 3.2, we have

$$(1 + \varepsilon)\mathcal{H}^\alpha(K_I) < \mathcal{H}^\alpha(K_I) + \mathcal{H}^\alpha(K_J) = \mathcal{H}^\alpha(K_I \cup K_J) \leq \mathcal{H}^\alpha(w_I(V)).$$

This together with (3.2) imply that

$$\varepsilon r_I^\alpha \mathcal{H}^\alpha(K) \leq \varepsilon \mathcal{H}^\alpha(K_I) < \mathcal{H}^\alpha(w_I(V \setminus K)) \leq (c_1 r_I)^\alpha \mathcal{H}^\alpha(V \setminus K) < (c_1 r_I)^\alpha \eta \mathcal{H}^\alpha(K).$$

(The first and third inequalities are followed from Lemma 3.1 and Lemma 2.1(i).)

Then $\varepsilon < c_1^\alpha \eta$, it contradicts the choice of ε . The claim is proved, and the lemma follows. ■

Lemma 3.4. Let $\{w_j\}_{j=1}^m$ be as in Lemma 3.2. Then $\gamma := \sup_{|L| \geq k_1} \#\Lambda(L) < \infty$. If we let $J_0 \in \mathcal{J}$ such that $|J_0| \geq k_1$ and $\#\Lambda(J_0) = \gamma$, then

$$\Lambda(IJ_0) = \{IJ : J \in \Lambda(J_0)\} \quad \text{for all } I \in \mathcal{J}. \quad (3.4)$$

Proof. Let c_3, c_4 and δ_0 be constants given in Lemma 2.1, Lemma 2.2 and Lemma 3.3 respectively. Let $\delta' = (3c_3c_4)^{-1}\delta_0$. We can find a finite set $Z \subset K$ such that its δ' -neighborhood covers K . For any $L \in \mathcal{J}$ with $|L| \geq k_1$ and for all different $I, J \in \Lambda(L)$, by Lemma 3.3, there exists $x \in K$ such that $|w_I(x) - w_J(x)| \geq \delta_0 r_L$. For the above chosen x there exists $z \in Z$ such that $|x - z| < \delta'$, then by (2.5) and the choice of k_1 (see the proof of Lemma 2.2), we have

$$|w_I(x) - w_I(z)| \leq \frac{\delta_0}{3} r_L \quad \text{and} \quad |w_J(x) - w_J(z)| \leq \frac{\delta_0}{3} r_L.$$

It follows that for any different $I, J \in \Lambda(L)$, there exists some $z \in Z$ such that

$$|w_I(z) - w_J(z)| \geq \frac{\delta_0}{3} r_L. \quad (3.5)$$

For each $z \in Z$, set

$$P_z(L) = \{I \in \Lambda(L) : \exists J \in \Lambda(L) \ni (3.5) \text{ holds}\}.$$

Hence (3.5) implies that

$$\Lambda(L) = \bigcup_{z \in Z} P_z(L).$$

To prove $\sup_{|L| \geq k_1} \#\Lambda(L) < \infty$, we observe that for each $z \in Z$, the sets

$$\{B(w_I(z), \frac{\delta_0}{6} r_L) : I \in P_z(L)\}$$

are disjoint by (3.5) and are contained in $B(G_L, \text{diam } K_I + \frac{\delta_0}{6} r_L)$ by the definition of $\Lambda(L)$. By Lemma 2.1(iii) and Lemma 2.2, there exist $c > 0$ (independent of L) and $x \in K$ such that $B(G_L, \text{diam } K_I + \frac{\delta_0}{6} r_L) \subseteq B(x, cr_L)$. By a simple volume argument, we deduce that there exists an ℓ (independent of L) such that $\max_{z \in Z} \#P_z(L) \leq \ell$. Then

$$\#\Lambda(L) \leq (\#Z) \max_{z \in Z} \#P_z(L) \leq \ell(\#Z).$$

We conclude that $\gamma = \sup_{|L| \geq k_1} \#\Lambda(L) < \infty$. Hence there exists J_0 such that $|J_0| \geq k_1$ and $\#\Lambda(J_0) = \gamma$.

To prove (3.4), we have remarked after the definition of $\Lambda(J)$ that

$$\Lambda(jJ_0) \supseteq \{jI : I \in \Lambda(J_0)\}, \quad j = 1, \dots, m$$

is trivial. On the other hand, the choice of J_0 implies that $\#\{IJ : J \in \Lambda(J_0)\} = \gamma$. Thus the maximum of γ implies that $\#\Lambda(IJ_0) = \gamma$ also and (3.4) follows $\blacksquare$

Proof of Theorem 1.1.

It is obvious that (ii) implies (i). That (i) implies (iii) is in [MU] and [FL]. We are hence required to prove (iii) implies (ii). The proof needs only small modification of [S] and is the same as [L], we include it here for completeness. Let $J_0 \in \mathcal{J}$ be chosen as in Lemma 3.4. For any fixed $1 \leq l \leq m$ and $J = j_1 j_2 \cdots j_n \in \mathcal{J}$ with $j_1 \neq l$, we consider the family

$$\mathcal{K}_l = \{K_L : L \in \Lambda_{\text{diam } G_{JJ_0}} \text{ with } l_1 = l\}$$

where l_1 is the first index of the multiple indices of L . Then $\mathcal{K}_l$ is a cover of K_l . Since $j_1 \neq l_1$, (3.4) implies that $L \notin \Lambda(JJ_0)$. Then by the construction of $\mathcal{B}$, $K_L \cap G_{JJ_0} = \emptyset$. Hence by (2.9), we have $D(K_L, K_{JJ_0}) \geq c_2^{-1} \varepsilon r_{JJ_0}$, which implies

$$D(K_l, K_{JJ_0}) \geq c_2^{-1} \varepsilon r_{JJ_0} \quad \text{for } l \neq j_1. \quad (3.6)$$

Now we let $G_J^* = w_J(B(K, 2^{-1}c_2^{-2}\varepsilon))$ and let

$$U = \bigcup_{J \in \mathcal{J}} G_{JJ_0}^*.$$

We claim that the U satisfies the condition of the SOSC. Indeed U is a bounded open set, $U \cap K \neq \emptyset$ and

$$w_j(U) = \bigcup_{J \in \mathcal{J}} w_j(G_{JJ_0}^*) = \bigcup_{J \in \mathcal{J}} G_{jJJ_0}^* \subseteq U.$$

Now we prove that

$$w_i(U) \cap w_j(U) = \emptyset \quad \text{for } i \neq j.$$

For otherwise, there are I, J such that $G_{iIJ_0}^* \cap G_{jJJ_0}^* \neq \emptyset$. We assume $r_{iIJ_0} \geq r_{jJJ_0}$. Let y be in the intersection, then there exist $y_1 \in K_{iIJ_0}$ and $y_2 \in K_{jJJ_0}$ such that

$$d(y, y_1) < c_2 \cdot \frac{1}{2c_2^2} \varepsilon \cdot r_{iIJ_0} \leq \frac{c_2^{-1} \varepsilon}{2} r_{iIJ_0}$$

and

$$d(y, y_2) < c_2 \cdot \frac{1}{2c_2^2} \varepsilon \cdot r_{jJJ_0} \leq \frac{c_2^{-1} \varepsilon}{2} r_{iIJ_0}.$$

Then

$$d(y_1, y_2) < c_2^{-1} \varepsilon r_{iIJ_0}.$$

Hence

$$D(K_{iIJ_0}, K_j) < c_2^{-1} \varepsilon r_{iIJ_0}$$

which contradicts (3.6). This completes the proof. $\blacksquare$

Lemma 3.5. [FL, Theorem 2.9] *Let $\{w_j\}_{j=1}^m$ be as in Theorem 1.1 and satisfy the OSC. Let $\nu = \mathcal{H}^\alpha|_K$. Then ν is the invariant measure for T_α , i.e., $T_\alpha^* \nu = \nu$.*

Proof of Theorem 1.2.

By the assumption and Theorem 1.1, we have $0 < \mathcal{H}^\alpha(K) < \infty$. We will adopt the proof of Theorem 1.1 and let U be the one constructed there. To prove $\dim_H(K \setminus U) < \alpha$. Let $\mu = \mathcal{H}^\alpha(K)^{-1} \mathcal{H}^\alpha$, then by Lemma 3.5, μ is a probability invariant measure of T_α , i.e.,

$$\mu = \sum_{j=1}^m (|w_j'(x)|^\alpha \mu) \circ w_j^{-1}. \quad (3.7)$$

Let $k := |J_0|$. Then by Lemma 2.1 and Lemma 3.1, we have

$$\mu(K_{J_0}) \geq c_1^{-\alpha} r_{J_0}^\alpha; \quad (3.8)$$

for any $L \in \mathcal{J}$,

$$\mu(K_{LJ_0}) = \mathcal{H}^\alpha(K)^{-1} \int_K |w'_L(w_{J_0}x)|^\alpha |w'_{J_0}(x)|^\alpha d\mathcal{H}^\alpha(x) \geq r_L^\alpha r_{J_0}^\alpha \geq c_1^{-\alpha} r_{J_0}^\alpha \mu(K_L). \quad (3.9)$$

For any integer n , let

$$U_n = \bigcup_{\ell=0}^{n-1} \bigcup_{|J|=k\ell} G_{JJ_0}^*.$$

Then $U_n \subseteq U$. Let

$$\mathcal{J}(n) = \{(j_1 j_2 \cdots j_{kn}) : 1 \leq j_i \leq m\}$$

and

$$\mathcal{L}_n = \{L = (l_1 l_2 \cdots l_{kn}) \in \mathcal{J}(n) : (l_{k\ell+1} l_{k\ell+2} \cdots l_{k\ell+k}) \neq J_0 \quad \forall \quad 0 \leq \ell < n\}.$$

Since for any J with $0 \leq |J| = k\ell < kn$, we deduce from $K = \bigcup_{j=1}^m w_j(K)$ that

$$K_{JJ_0} = w_{JJ_0}(K) = \bigcup_{|J'|=k(n-1)-|J|} K_{JJ_0 J'}. \quad (3.10)$$

Then

$$\begin{aligned} K \setminus U &\subseteq K \setminus U_n = K \setminus \bigcup_{\ell=0}^{n-1} \bigcup_{|J|=k\ell} G_{JJ_0}^* \subseteq K \setminus \bigcup_{\ell=0}^{n-1} \bigcup_{|J|=k\ell} K_{JJ_0} \\ &= \bigcap_{\ell=0}^{n-1} \bigcap_{|J|=k\ell} \bigcap_{|J'|=k(n-1)-|J|} K_{JJ_0 J'}^c \subseteq \bigcup_{L \in \mathcal{L}_n} K_L. \end{aligned} \quad (3.11)$$

We need to estimate the value of $\mathcal{H}^\alpha(\bigcup_{L \in \mathcal{L}_n} K_L)$. For this we will prove inductively that

$$\sum_{L \in \mathcal{L}_n} \mu(K_L) \leq (1 - c_1^{-\alpha} r_{J_0}^\alpha)^n \quad \text{for any } n. \quad (3.12)$$

Indeed by Lemma 3.2, we have

$$\mu(K_I \cap K_J) = 0 \quad \text{for any } I, J \in \mathcal{L}_n \text{ with } I \neq J.$$

This together with (3.8) imply that

$$\sum_{L \in \mathcal{L}_1} \mu(K_L) = \mu\left(\bigcup_{\substack{L \neq J_0 \\ |L|=k}} K_L\right) = 1 - \mu(K_{J_0}) \leq 1 - c_1^{-\alpha} r_{J_0}^\alpha.$$

Assume that

$$\sum_{L \in \mathcal{L}_n} \mu(K_L) \leq (1 - c_1^{-\alpha} r_{J_0}^\alpha)^n.$$

Since

$$\sum_{|J|=k} \mu(K_{LJ}) = \mu(K_L),$$

then

$$\begin{aligned}
\sum_{L \in \mathcal{L}_{n+1}} \mu(K_L) &= \sum_{L \in \mathcal{L}_n} \sum_{|J|=k} \mu(K_{LJ}) - \sum_{L \in \mathcal{L}_n} \mu(K_{LJ_0}) \\
&= \sum_{L \in \mathcal{L}_n} \mu(K_L) - \sum_{L \in \mathcal{L}_n} \mu(K_{LJ_0}) \leq \sum_{L \in \mathcal{L}_n} \mu(K_L) - c_1^{-\alpha} r_{J_0}^\alpha \sum_{L \in \mathcal{L}_n} \mu(K_L) \quad \text{by (3.9)} \\
&= (1 - c_1^{-\alpha} r_{J_0}^\alpha) \sum_{L \in \mathcal{L}_n} \mu(K_L) \leq (1 - c_1^{-\alpha} r_{J_0}^\alpha) (1 - c_1^{-\alpha} r_{J_0}^\alpha)^n = (1 - c_1^{-\alpha} r_{J_0}^\alpha)^{n+1}.
\end{aligned}$$

Let $\delta_n := \max\{\text{diam } K_L : L \in \mathcal{L}_n\}$ and $r = \min_{1 \leq j \leq m} \{r_j\}$. If we take

$$\beta := \alpha - \frac{\log(1 - c_1^{-\alpha} r_{J_0}^\alpha)}{k \log r}.$$

Then $\beta < \alpha$. Denote $c_4 = (c_3 \text{diam } K)^\beta$. Then for large n , we have

$$\begin{aligned}
\mathcal{H}_{\delta_n}^\beta(K \setminus U) &\leq \mathcal{H}_{\delta_n}^\beta\left(\bigcup_{L \in \mathcal{L}_n} K_L\right) \leq \sum_{L \in \mathcal{L}_n} (\text{diam } K_L)^\beta \quad \text{by (3.11)} \\
&\leq \sum_{L \in \mathcal{L}_n} (c_3 r_L \text{diam } K)^\beta = c_4 \sum_{L \in \mathcal{L}_n} r_L^\beta = c_4 \sum_{L \in \mathcal{L}_n} r_L^{\beta-\alpha} r_L^\alpha \quad \text{by (2.5)} \\
&\leq c_4 \left(r^{nk(\beta-\alpha)} \sum_{L \in \mathcal{L}_n} \mu(K_L) \right) \leq c_4 (r^{k(\beta-\alpha)} (1 - c_1^{-\alpha} r_{J_0}^\alpha))^n \leq c_4. \quad \text{by (3.12)}
\end{aligned}$$

Since $\lim_{n \rightarrow \infty} \delta_n = 0$, then $\mathcal{H}^\beta(K \setminus U) \leq c_4 < \infty$, hence $\dim_H(K \setminus U) \leq \beta$. ■

Corollary 3.6. *Let $\{w_j\}_{j=1}^m$ be as in Theorem 1.2. Then*

$$\dim_H(w_i(K) \cap w_j(K)) < \alpha \quad \text{for } i \neq j.$$

Proof. Let J_0 and β be as the proof of Theorem 1.2. Using (3.12), we can show similarly to [LX, Theorem 1.6] that $\dim_H(w_i(K) \cap w_j(K)) \leq \beta < \alpha$. ■

Theorem 3.7. *Let $\{w_j\}_{j=1}^m$ be as in Theorem 1.2. If the $\{w_j\}_{j=1}^m$ has a basic open set U such that $U \setminus \bigcup_{j=1}^m w_j(\overline{U}) \neq \emptyset$, then $\dim_H K < d$.*

Proof. Suppose that $\dim_H K = d$, since $\{w_j\}_{j=1}^m$ satisfies the OSC, we know from [FL, Theorem 2.7] that T_d has spectral radius 1, i.e., $\alpha = d$. Since $U \setminus \bigcup_{j=1}^m w_j(\overline{U})$ is an open subset of $\mathbb{R}^d$, the proof will be finished if we can show that $\mathcal{H}^d(U \setminus \bigcup_{j=1}^m w_j(\overline{U})) = 0$. For this, let

$$V = U \setminus \bigcup_{j=1}^m w_j(\overline{U}). \quad (3.13)$$

We claim that

$$w_I(V) \cap w_J(V) = \emptyset \quad \forall \quad I, J \in \mathcal{J}, \quad I \neq J. \quad (3.14)$$

In fact for I, J comparable, namely $J = II_0$. Since U is a basic open set, we have

$$w_i(U) \subset U \quad \text{and} \quad w_i(U) \cap w_j(U) = \emptyset, \quad \forall i \neq j, \quad (3.15)$$

then $w_{I_0}(V) \subset \bigcup_{j=1}^m w_j(U)$, and then $w_{I_0}(V) \cap V = \emptyset$. Hence

$$w_I(V) \cap w_J(V) \subseteq w_I(V \cap w_{I_0}(V)) = \emptyset.$$

If I, J are incomparable, let $I = i_1 i_2 \cdots i_p$, $J = j_1 j_2 \cdots j_q$ and $r = \min\{k : i_k \neq j_k\}$. Denote $I_0 = i_1 i_2 \cdots i_{r-1}$. By (3.13) and (3.15), we have

$$w_I(V) \cap w_J(V) \subseteq w_{I_0}(w_{i_r}(U) \cap w_{j_r}(U)) = \emptyset.$$

This completes the proof of the claim.

By (3.14) and Lemma 3.1, we have

$$\sum_{n=1}^{\infty} \int_V \sum_{|J|=n} |w'_J(x)|^d d\mathcal{H}^d(x) = \sum_{n=1}^{\infty} \sum_{|J|=n} \mathcal{H}^d(w_J(V)) = \mathcal{H}^d\left(\bigcup_{J \in \mathcal{J}} w_J(V)\right) \leq \mathcal{H}^d(U) < \infty. \quad (3.16)$$

On the other hand, for any fixed $y_0 \in K$ and any $x \in X$, by Lemma 2.1(i)

$$c_1^{-d} |w'_J(y_0)|^d \leq |w'_J(x)|^d.$$

Hence

$$c_1^{-d} \sum_{|J|=n} |w'_J(y_0)|^d \leq \sum_{|J|=n} |w'_J(x)|^d, \quad x \in X.$$

Since $\alpha = d$, it follows from [FL, Theorem 1.1] that

$$\lim_n \sum_{|J|=n} |w'_J(\cdot)|^d = h(\cdot) \quad \text{uniformly on } K$$

where $0 < h \in C(K)$ is the 1-eigenfunction of the Ruelle operator T_d . Then

$$c_1^{-d} h(y_0) \mathcal{H}^d(V) = c_1^{-d} \cdot \lim_n \int_V \sum_{|J|=n} |w'_J(y_0)|^d d\mathcal{H}^d(x) \leq \liminf_n \int_V \sum_{|J|=n} |w'_J(x)|^d d\mathcal{H}^d(x).$$

By (3.16), the right side is 0, hence

$$\mathcal{H}^d(V) = \mathcal{H}^d\left(U \setminus \bigcup_{j=1}^m w_j(\overline{U})\right) = 0. \quad \blacksquare$$

Corollary 3.8. *Let $\{w_j\}_{j=1}^m$ be as in Theorem 1.1. If $\alpha = d$ and $\mathcal{H}^d(K) > 0$. Then $K^\circ \neq \emptyset$ and $\dim_H \partial K < d$.*

Proof. Let U be the basic open set constructed in the proof of Theorem 1.1. By the assumption and Theorem 3.7, we have $\mathcal{H}^d(U \setminus \bigcup_{j=1}^m w_j(\overline{U})) = 0$. Then

$$\overline{U} = \bigcup_{j=1}^m w_j(\overline{U}).$$

By the uniqueness of the invariant set K , we have $K = \overline{U}$, and then $K^\circ \supseteq U \neq \emptyset$. In view of the proof of Theorem 1.2, we have

$$\dim_H \partial K \leq \dim_H (K \setminus U) < d. \quad \blacksquare$$

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