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Almost controllable graphs and beyond

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ABSTRACT

An eigenvalue λ of a graph G of order n is a main eigenvalue if its eigenspace is not orthogonal to the all-ones vector j_n. In 1978, Cvetković proved that G has exactly one main eigenvalue if and only if G is regular, and posed the following long-standing problem: characterize the graphs with exactly k $(2 \le k \le n)$ main eigenvalues. Graphs of order n with $n, n-1$ main eigenvalues are called controllable, almost controllable, respectively. Cographs, threshold graphs are frequently studied in structural graph theory and computer science. In this paper, all almost controllable cographs, all almost controllable threshold graphs and all almost controllable graphs with second largest eigenvalue less than or equal to $\frac{\sqrt{5}-1}{2}$ are characterized. Furthermore, we give some results about cographs with exactly $n-2$ main eigenvalues, and propose some additional problems for further study.

1. Introduction

Throughout this paper, G is a simple graph with vertex set $V(G) = \{v_1, \cdots, v_n\}$ and edge set $E(G)$, where $|V(G)| = n$ is the order of G and $|E(G)|$ is the number of edges in G. If the vertices v_i and v_j are adjacent, we write $v_i \sim v_j$, then $e = v_i v_j$ is an edge that belongs to $E(G)$ and we say v_i (v_j) is *incident* to e. Let $N_G(u)$ be the neighbourhood set of u in G and $d_G(u) = |N_G(u)|$ be the *degree* of the vertex u in G. Two vertices u, v of G are called *duplicate* vertices if $u \nsim v$ and $N_G(u) = N_G(v)$, *co-duplicate* vertices if $u \sim v$ and $N_G(u) \backslash \{v\} = N_G(v) \backslash \{u\}$. The *complement* of a graph G is denoted by $\overline{G}$. Let $K_n, K_{a,b}, P_n$ be the complete graph, complete bipartite graph, path of order n where $a + b = n$, respectively. The *union* of two disjoint graphs G and H is denoted by $G \cup H$. The *join* $G \triangledown H$ of two disjoint graphs G and H is the graph obtained from $G \cup H$ by joining each vertex of G to each vertex of H. Suppose $V' \subseteq V(G)$, the *induced subgraph* of G with respect to V' is a graph with vertex set V' and edge set E', where $v_i v_j \in E'$ if $v_i v_j \in E(G)$ for any $v_i, v_j \in V'$, $G - V'$ is the graph obtained from G after deleting each vertex $v \in V'$ and all edges that are incident to v. An *elementary* graph is a graph in which each component is K_2 or a cycle.

Let $A(G) = [a_{ij}]$ be the $n \times n$ *adjacency matrix* of G for which $a_{ij} = 1$ if $v_i \sim v_j$ and $a_{ij} = 0$ if $v_i \nsim v_j$. The *eigenvalues* of G are the eigenvalues of its adjacency matrix $A(G)$. The *spectrum* of G is the multiset of all eigenvalues of G, and we denote it by Spec(G). An eigenvalue λ of G is said to be a *main eigenvalue* if its eigenspace is not orthogonal to the all-ones vector $j_n = [1, 1, \ldots, 1]^T$. By [6], all main eigenvalues of G are distinct. Let MainSpec(G) denote the set of all main eigenvalues of G.

Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ $(1 \le m \le n)$ be the distinct eigenvalues of G, and $\lambda_1, \lambda_2, \ldots, \lambda_k$ $(1 \le k \le m)$ be the main eigenvalues of G. Then $1 \le k \le n$. Let G be a connected graph. Then $A(G)$ is an irreducible matrix with non-negative entries, and thus the largest eigenvalue of G is always main by the famous Perron-Frobenius Theorem. In 1978, Cvetković proved that G has

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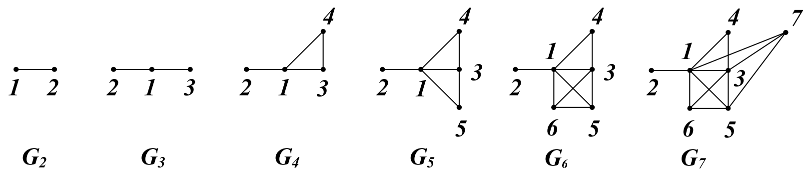

Fig. 1. The graphs G_i for $i \in \{2, 3, 4, 5, 6, 7\}$.

exactly one main eigenvalue if and only if G is regular. Besides, he posed the following long-standing problem: characterize the graphs with exactly k $(2 \le k \le n)$ main eigenvalues [6].

There are a series of papers characterizing the graphs with exactly $2, n-1, n$ main eigenvalues. All trees, unicyclic, bicyclic and tricyclic graphs with exactly 2 main eigenvalues are characterized in [21–23]. For the other relevant results, one can refer to Feng et al. [15], Hagos [19], Hayat et al. [20], Lepović [26], etc. For graphs with all eigenvalues main, Cvetković et al. defined them as *controllable graphs* through their correlation with control theory [10], and for the relevant results, we refer the readers to Cvetković et al. [10,11], Farrugia [14] and Stanić [31]. For graphs of order n with $n-1$ main eigenvalues, Wang et al. defined them as *almost controllable graphs* [32], and for the recent research on almost controllable graphs, one can refer to [12,27,28,32].

A graph G is called *reconstructible* if it can be determined from the knowledge only of all one-vertex-deleted subgraphs. In [17], the authors proved that a graph G of order n is reconstructible if all but at most one of the eigenvalues of $A(G)$ are simple, with the corresponding eigenvectors not being orthogonal to j_n. Thus characterizing the graphs with exactly k main eigenvalues (especially $k = n-1, n$) is of great importance as such graphs are reconstructible.

In this paper, we focus on almost controllable graphs and the paper is organized as follows. In Section 2, all almost controllable cographs and threshold graphs are characterized. In Section 3, almost controllable graphs with the second largest eigenvalue less than or equal to $\frac{\sqrt{5}-1}{2}$ are determined. In Section 4, we present some results about cographs with exactly $n-2$ main eigenvalues. Furthermore, some problems are proposed for further research.

2. Almost controllable cographs and threshold graphs

In this section, almost controllable cographs and threshold graphs are characterized.

First we define a graph G_n of order n $(n \ge 1)$ recursively: (1) $G_1 \cong K_1$ where $V(G_1) = \{v_1\}$; (2) $G_2 \cong P_2$ where $V(G_2) = \{v_1, v_2\}$; (3) For $n \ge 3$, G_n is obtained from G_{n-1} by adding a new vertex v_n adjacent to each vertex $v \in N_{G_{n-1}}(v_{n-1})$, and $v_n \sim v_{n-1}$ if $v_{n-1} \nsim v_{n-2}$, $v_n \nsim v_{n-1}$ if $v_{n-1} \sim v_{n-2}$, where $V(G_{n-1}) = \{v_1, v_2, \cdots, v_{n-1}\}$. It is not hard to find that v_n, v_{n-1} is a pair of duplicate vertices in G_n for odd n (≥ 3), and a pair of co-duplicate vertices in G_n for even n (≥ 2). The graphs G_i are shown in Fig. 1 for $i \in \{2, 3, 4, 5, 6, 7\}$.

By the symmetry of v_{n-2}, v_{n-1} in G_{n-1} and the definition of G_n $(n \ge 3)$, we can conclude the following proposition.

Proposition 2.1. *The graph G_n $(n \ge 2)$ is unique up to isomorphism.*

In [12], the authors raised the following problem.

Problem 2.2. ([12]) Given any integer t, how can one construct graphs (of order n) with t as the unique non-main eigenvalue for sufficiently large n?

Next we give an example for the cases $t = 0, -1$ of Problem 2.2, and we will give some lemmas first.

Lemma 2.3. *Let $n \ge 1$. Then $\overline{G_{n+1}} \cong G_n \cup K_1$.*

Proof. By the definition of G_n, it is not hard to find that $\overline{G_1} \cong K_1$, $\overline{G_2} \cong 2K_1$, and for $n \ge 3$, $\overline{G_n}$ can be obtained from $\overline{G_{n-1}}$ after adding a new vertex v_n adjacent to each vertex $v \in N_{\overline{G_{n-1}}}(v_{n-1})$, and $v_n \sim v_{n-1}$ if $v_{n-1} \nsim v_{n-2}$ in $\overline{G_{n-1}}$, $v_n \nsim v_{n-1}$ if $v_{n-1} \sim v_{n-2}$ in $\overline{G_{n-1}}$. Hence $\{\overline{G_n}\}_{n\ge 3}$ has the same recurrence relation as $\{G_n\}_{n \ge 3}$ but the initial conditions are different.

It is easy to check that $d_{G_3}(v_1) = 2$, $d_{G_4}(v_1) = 3$, $\cdots$, $d_{G_{n+1}}(v_1) = n$ by the definition of G_{n+1}, then $d_{\overline{G_{n+1}}}(v_1) = 0$ for $n \ge 2$. Hence $\{\overline{G_{n+1}} - v_1\}_{n \ge 3}$ and $\{G_n\}_{n \ge 3}$ have the same recurrence relation by $d_{\overline{G_{n+1}}}(v_1) = 0$.

We note that $\overline{G_2} - v_1 \cong G_1$, $\overline{G_3} - v_1 \cong G_2$, therefore $\overline{G_{n+1}} - v_1 \cong G_n$ for $n \ge 1$ since $\{\overline{G_{n+1}} - v_1\}_{n \ge 3}$ and $\{G_n\}_{n \ge 3}$ have the same recurrence relation, and this implies $\overline{G_{n+1}} \cong G_n \cup K_1$ by $d_{\overline{G_{n+1}}}(v_1) = 0$. $\square$

Lemma 2.4. ([29]) *A graph G and its complement $\overline{G}$ have the same number of main eigenvalues.*

Lemma 2.5. *Let G be a graph of order n with main eigenvalues $\lambda_1, \cdots, \lambda_k$ $(k \le n)$ where $\lambda_i \ne 0$ for $1 \le i \le k$. Then* $\text{MainSpec}(G \cup K_1) = \{\lambda_1, \cdots, \lambda_k, 0\}$.

Proof. Let $A(G)x_i = \lambda_i x_i$ for $i \in \{1, 2, \cdots, n\}$. Then $j_n^T x_i \neq 0$ for $i \in \{1, 2, \cdots, k\}$ and $j_n^T x_i = 0$ for $i \in \{k+1, k+2, \cdots, n\}$.

Let $y_i = (x_i^T, 0)^T$ for $i \in \{1, 2, \cdots, n\}$ and $y_{n+1} = (0, \cdots, 0, 1)^T$. It is easy to check that $A(G \cup K_1) = \begin{bmatrix} A(G) & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix}$, $A(G \cup K_1) y_i = \lambda_i y_i$ for $i \in \{1, 2, \cdots, n\}$, and $A(G \cup K_1) y_{n+1} = 0 \cdot y_{n+1}$. Clearly, $j_{n+1}^T y_i \neq 0$ for $i \in \{1, \cdots, k, n+1\}$ and $j_{n+1}^T y_i = 0$ for $i \in \{k+1, \cdots, n\}$, then we complete the proof. □

By direct calculation, we have the following results.

Proposition 2.6. *Let G be a graph of order n. If G has a pair of duplicate (or co-duplicate) vertices v_i, v_j, then 0 (or -1) is a non-main eigenvalue of G with the corresponding eigenvector $(0, \cdots, 0, 1, 0, \cdots, 0, -1, 0, \cdots, 0)^T$, where $1, -1$ is the i-th, j-th entry, respectively.*

By Proposition 2.6 and the definition of G_n $(n \geq 2)$, it is easy to find that -1 is a non-main eigenvalue of G_n if n is even, and 0 is a non-main eigenvalue of G_n if n is odd. Next we show that -1 (or 0) is the unique non-main eigenvalue of G_n.

Lemma 2.7 *(Sachs' Coefficient Theorem [9]). Let G be a graph on n vertices with characteristic polynomial $P_G(x) = x^n + c_1 x^{n-1} + \cdots + c_{n-1} x + c_n$, $\mathcal{H}_i$ be the set of all elementary subgraphs of G with i vertices for $1 \leq i \leq n$. For each H in $\mathcal{H}_i$, let $p(H)$ denote the number of components of H and $c(H)$ denote the number of cycles in H. Then*

$$c_i = \sum_{H \in \mathcal{H}_i} (-1)^{p(H)} 2^{c(H)}, \qquad \text{for all } i = 1, \ldots, n.$$

Lemma 2.8 *(Interlacing Theorem [9]). Let G be a graph with n vertices and eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, H be an induced subgraph of G with m vertices and eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$. Then $\lambda_i \geq \mu_i \geq \lambda_{n-m+i}$ for $i \in \{1, 2, \ldots, m\}$.*

Lemma 2.9. *Let $n \geq 2$. Then $0 \notin \mathrm{Spec}(G_n)$ for n is even, and 0 is a simple non-main eigenvalue of G_n for n is odd.*

Proof. We prove this by the following two cases.

Case 1. n is even.

We will prove $0 \notin \mathrm{Spec}(G_n)$ by showing that the constant c_n of $P_{G_n}(x)$ is non-zero. By the definition of G_n, it is not hard to find that $d_{G_n}(v_1) = n-1$, $d_{G_n}(v_2) = 1$, $d_{G_n}(v_3) = n-2$, $d_{G_n}(v_4) = 2$, $d_{G_n}(v_5) = n-3$, $d_{G_n}(v_6) = 3$, $\cdots$. In general, $N_{G_n}(v_i) = \{v_1, v_3, v_5, \cdots, v_{i-1}\}$ if i $(2 \leq i \leq n)$ is even, then $d_{G_n}(v_{n-3}) = \frac{n}{2} + 1$, $d_{G_n}(v_{n-2}) = \frac{n}{2} - 1$, $d_{G_n}(v_{n-1}) = \frac{n}{2}$, $d_{G_n}(v_n) = \frac{n}{2}$.

Let $\mathcal{H}_n$ be the set of all elementary subgraphs of G_n with n vertices. Now we consider the possible elementary subgraph $H \in \mathcal{H}_n$ which contributes to c_n by Lemma 2.7. It is not hard to find that there is only one H belongs to $\mathcal{H}_n$ where $E(H) = \{v_1v_2, v_3v_4, \cdots, v_{n-1}v_n\}$ since $N_{G_n}(v_i) = \{v_1, v_3, v_5, \cdots, v_{i-1}\}$ if i $(2 \leq i \leq n)$ is even. Therefore, H can only be isomorphic to $\frac{n}{2}K_2$ which implies $c_n = (-1)^{\frac{n}{2}} 2^0 \neq 0$. Thus $0 \notin \mathrm{Spec}(G_n)$ if n is even.

Case 2. n is odd.

Since G_n has a pair of duplicate vertices, 0 is a non-main eigenvalue of G_n by Proposition 2.6.

If 0 is an eigenvalue of G_n with multiplicity at least 2, then there exists some i $(1 \leq i \leq n-1)$ such that $\lambda_i = 0 \geq \mu_i \geq \lambda_{i+1} = 0$ by Lemma 2.8 and G_{n-1} is an induced subgraph of G_n, where $\lambda_i, \lambda_{i+1} \in \mathrm{Spec}(G_n)$ and $\mu_i \in \mathrm{Spec}(G_{n-1})$. That is to say $0 \in \mathrm{Spec}(G_{n-1})$, and it is impossible since $n-1$ is even and $0 \notin \mathrm{Spec}(G_{n-1})$ by Case 1. Thus 0 is a simple non-main eigenvalue of G_n for n is odd. □

Now we prove that the graph G_n $(n \geq 2)$ has 0 or -1 as its unique non-main eigenvalue.

Theorem 2.10. *Let $n \geq 2$. Then the graph G_n is almost controllable. In fact, -1 is the unique non-main eigenvalue of G_n if n is even, and 0 is the unique non-main eigenvalue of G_n if n is odd.*

Proof. Firstly, we show $|\mathrm{MainSpec}(G_n)| = n-1$ by induction on n.

It is easy to check that $\mathrm{Spec}(G_2) = \{1, -1\}$ where -1 is the unique non-main eigenvalue, and $\mathrm{Spec}(G_3) = \{\sqrt{2}, 0, -\sqrt{2}\}$ where 0 is the unique non-main eigenvalue. Therefore, $|\mathrm{MainSpec}(G_2)| = 1$ and $|\mathrm{MainSpec}(G_3)| = 2$.

By Lemmas 2.3 and 2.4, we have $|\mathrm{MainSpec}(G_4)| = |\mathrm{MainSpec}(\overline{G_4})| = |\mathrm{MainSpec}(G_3 \cup K_1)|$. Since 0 is a simple non-main eigenvalue of G_3, we have $|\mathrm{MainSpec}(G_3 \cup K_1)| = |\mathrm{MainSpec}(G_3)| + 1 = 3$ by Lemma 2.5 and $|\mathrm{MainSpec}(G_3)| = 2$, and thus $|\mathrm{MainSpec}(G_4)| = 3$. Similarly, $|\mathrm{MainSpec}(G_5)| = |\mathrm{MainSpec}(\overline{G_5})| = |\mathrm{MainSpec}(G_4 \cup K_1)|$. By Lemma 2.9, we have $0 \notin \mathrm{Spec}(G_4)$. Then $|\mathrm{MainSpec}(G_4 \cup K_1)| = |\mathrm{MainSpec}(G_4)| + 1$ by Lemma 2.5. Thus $|\mathrm{MainSpec}(G_5)| = 4$ by $|\mathrm{MainSpec}(G_4)| = 3$.

Suppose $|\mathrm{MainSpec}(G_{i-1})| = i-2$ for $i \geq 6$. Now we show $|\mathrm{MainSpec}(G_i)| = i-1$.

By Lemmas 2.3 and 2.4, we have $|\mathrm{MainSpec}(G_i)| = |\mathrm{MainSpec}(\overline{G_i})| = |\mathrm{MainSpec}\ (G_{i-1} \cup K_1)|$.

If i is even, then 0 is a simple non-main eigenvalue of G_{i-1} by Lemma 2.9, and thus we have $|\mathrm{MainSpec}(G_{i-1} \cup K_1)| = |\mathrm{MainSpec}(G_{i-1})| + 1 = i - 1$ by Lemma 2.5 and induction hypothesis.

If i is odd, then $0 \notin \mathrm{Spec}(G_{i-1})$ by Lemma 2.9, and thus we have $|\mathrm{MainSpec}(G_{i-1} \cup K_1)| = |\mathrm{MainSpec}(G_{i-1})| + 1 = i - 1$ by Lemma 2.5 and induction hypothesis.

By Proposition 2.6, the rest part of the theorem is obvious. □

By Theorem 2.10, the graph G_n is an example for the cases $t = 0, -1$ of Problem 2.2, that is, -1 is the unique non-main eigenvalue of G_n if n is even, and 0 is the unique non-main eigenvalue of G_n if n is odd.

A graph G is called *complement reducible* (a *cograph* for short) if for any induced subgraph H of G with at least two vertices, either H or $\overline{H}$ is disconnected. Cographs have a characterization in terms of forbidden induced subgraphs: they are graphs containing no P_4 as an induced subgraph [5]. Cographs are frequently studied in structural graph theory and have been rediscovered numerous times, see [2,16,30].

The following two lemmas are presented in [5] and improved by [4].

Lemma 2.11. ([4]) *The class of cographs can be defined recursively as follows:*
(i) *A single vertex is a cograph.*
(ii) *If H_1, H_2 are two disjoint cographs, then so is their union $H_1 \cup H_2$.*
(iii) *If H_1, H_2 are two disjoint cographs, then so is their join $H_1 \triangledown H_2$.*

Lemma 2.12. ([4]) *If G is a cograph, then every non-trivial induced subgraph H of G has two vertices which are (co-)duplicate in H.*

By the properties of the cographs, we have the following proposition.

Proposition 2.13. *Let G be a cograph. Then any induced subgraph of G is a cograph.*

Lemma 2.14. *Let G be a cograph, then $\overline{G}$ is a cograph.*

Proof. Suppose to the contrary, then $\overline{G}$ contains P_4 as an induced subgraph. However, $\overline{P_4} \cong P_4$ which implies G has P_4 as an induced subgraph, and this contradicts with G is a cograph. □

Lemma 2.15. *Let $n \geq 1$. Then the graph G_n ($\overline{G_n}$) is a cograph.*

Proof. By Lemmas 2.3, 2.11 and 2.14, $G_1 \cong K_1$ is a cograph, $\overline{G_2} \cong G_1 \cup K_1$ is a cograph and so G_2 is. Similarly, $\overline{G_i} \cong G_{i-1} \cup K_1$ is a cograph and G_i is also a cograph for $i \geq 3$. □

Next we characterize almost controllable cographs.

An *automorphism* of a graph G is a permutation σ of the vertex set $V(G)$ such that the pair of vertices $v_i \sim v_j$ if and only if $\sigma(v_i) \sim \sigma(v_j)$. The set of automorphisms of G under the composition operation, form a group, called the *automorphism group* of G and denoted by $\mathrm{Aut}(G)$. It is well-known that a graph and its complement share the same automorphism group.

Lemma 2.16. ([10]) *Controllable graphs have only trivial automorphism group.*

By Lemmas 2.12 and 2.16, it is clear that there is no controllable cograph [11]. However, there exist almost controllable cographs. In fact, G_n (or $\overline{G_n}$) is an almost controllable cograph by Theorem 2.10 and Lemma 2.15.

Lemma 2.17. ([12]) *Let G be a graph of order n with $n-1$ main eigenvalues, then its automorphism group $\mathrm{Aut}(G)$ is either trivial or generated by a transposition $\sigma = (v_i, v_j)$ for some $v_i, v_j \in V(G)$, where σ fixes all vertices $w \in V(G) \backslash \{v_i, v_j\}$.*

By the definition of the automorphism of a graph G and the relationship between G and $\mathrm{Aut}(G)$, we have the following result immediately.

Proposition 2.18. *Let G be a graph of order n. Then G has a pair of vertices v_i, v_j such that $N_{G-v_j}(v_i) = N_{G-v_i}(v_j)$ if and only if $(v_i, v_j) \in \mathrm{Aut}(G)$ for some $i, j \in \{1, 2, \cdots, n\}$. In fact, such v_i, v_j is a pair of duplicate vertices in G if $v_i \nsim v_j$, and a pair of co-duplicate vertices in G if $v_i \sim v_j$.*

By Lemma 2.17, if G is an almost controllable graph, then $|\mathrm{Aut}(G)| \in \{1, 2\}$. Besides, if $|\mathrm{Aut}(G)| = 2$, then G has a unique pair of vertices v_i, v_j such that $N_{G-v_j}(v_i) = N_{G-v_i}(v_j)$ for some $i, j \in \{1, 2, \cdots, n\}$ by Proposition 2.18.

Lemma 2.19. *Let G be an almost controllable cograph of order n (≥ 2). Then $G \cong H \cup K_1$ or $G \cong H \triangledown K_1$, where H is a cograph of order $n-1$. Especially, $G \cong H \triangledown K_1$ if H is disconnected.*

Proof. By Lemma 2.11, we have $G \cong H_1 \cup H_2$ or $G \cong H_1 \triangledown H_2$ where H_1, H_2 are two disjoint cographs. If $|V(H_i)| \geq 2$ for $i \in \{1,2\}$, then each pair of (co-)duplicate vertices in H_i is also a pair of (co-)duplicate vertices in G, which implies $|\mathrm{Aut}(G)| > 2$, and this contradicts G being almost controllable by Lemma 2.17. Then there is at most one H_i satisfies $|V(H_i)| \geq 2$ for $i \in \{1,2\}$, and thus $G \cong H \cup K_1$ or $G \cong H \triangledown K_1$, where H is a cograph of order $n-1$.

Now we show if H is disconnected with $|V(H)| \geq 2$, then $G \cong H \triangledown K_1$. Suppose to the contrary, we have $G \cong H \cup K_1$, where H is disconnected with $|V(H)| \geq 2$. By Lemma 2.12 and $G \cong H \cup K_1$, we know that every non-trivial connected component of H has two vertices which are (co-)duplicate in H and G. If there are at least two components in H with at least two vertices, then this will lead to a contradiction with G has a unique pair of vertices x, y such that $N_{G-y}(x) = N_{G-x}(y)$. If there is one component in H with at least two vertices, then we additionally have $2K_1$ in G, a contradiction. Obviously, $H \cong 2K_1$ can also lead to a contradiction. Thus $G \cong H \triangledown K_1$ if H is disconnected. □

Theorem 2.20. *Let G be a cograph of order $n \geq 2$. If G has a unique pair of vertices x, y such that $N_{G-y}(x) = N_{G-x}(y)$, then $G \in \{G_n, \overline{G_n}\}$.*

Proof. It is obvious that $(x, y) \in \mathrm{Aut}(G)$, where x, y is a pair of duplicate vertices if $x \nsim y$, and a pair of co-duplicate vertices if $x \sim y$ by Proposition 2.18.

For $n = 2$, then $G \cong P_2 \cong G_2$ or $G \cong 2K_1 \cong \overline{G_2}$. Thus $G \in \{G_2, \overline{G_2}\}$.

For $n \geq 3$, by Lemma 2.12, Proposition 2.18 and G is a cograph, we can suppose $(s,t) \in \mathrm{Aut}(H)$ where $H = G - x$ and $|V(H)| \geq 2$.

Claim 1. $y \in \{s,t\}$.

Suppose to the contrary, $y \neq s$ and $y \neq t$. Then $V(H) = \{s,t\} \cup V_1 \cup V_2$ where $V_1 = \{v | v \sim s, v \sim t\}$, $V_2 = \{v | v \nsim s, v \nsim t\}$. If $y \in V_1$, then $x \sim s, x \sim t$ in G since $(x, y) \in \mathrm{Aut}(G)$. However, in this case we have $(s,t) \in \mathrm{Aut}(G)$, which contradicts the uniqueness of x, y. Similarly, if $y \in V_2$, then $x \nsim s, x \nsim t$ in G, and this implies $(s,t) \in \mathrm{Aut}(G)$, a contradiction.

By Claim 1, we can suppose $t = y$, and thus $(s, y) \in \mathrm{Aut}(H)$.

Claim 2. If $x \sim y$, then $s \nsim y$; if $x \nsim y$, then $s \sim y$.

Suppose to the contrary, there are $x \sim y$ and $s \sim y$ in G. Since $(x, y) \in \mathrm{Aut}(G)$ and $(s, y) \in \mathrm{Aut}(H)$, we have $x \sim s$ and $N_G(x) = V_1 \cup \{y, s\}$ where V_1 is defined in Claim 1. Then both x, s and y, s are pairs of co-duplicate vertices in G, which contradicts the uniqueness of x, y. By similar way, if $x \nsim y$ and $s \nsim y$, then both x, s and y, s are pairs of duplicate vertices in G, a contradiction.

Claim 3. s, y is the unique pair of vertices of H such that $N_{H-y}(s) = N_{H-s}(y)$.

Suppose to the contrary, there is an automorphism $(s', y) \in \mathrm{Aut}(H)$ where $s' \neq s$.

If $x \sim y$, then $s \nsim y$ and $s' \nsim y$ by Claim 2, thus $x \nsim s$ and $x \nsim s'$ by $(x, y) \in \mathrm{Aut}(G)$ and $s' \nsim s$ by $(s, y) \in \mathrm{Aut}(H)$. Therefore, we have $N_H(s) = N_H(y) = N_H(s')$ by $(s, y) \in \mathrm{Aut}(H)$ and $(s', y) \in \mathrm{Aut}(H)$. Combining the above arguments, we can conclude that $N_G(s) = N_G(s')$, and thus s, s' is a pair of duplicate vertices in G, which contradicts the uniqueness of x, y.

If $x \nsim y$, then $s \sim y$ and $s' \sim y$ by Claim 2. By similar arguments, we have $x \sim s, x \sim s'$, $s \sim s'$ and $N_G(s) \backslash \{s'\} = N_G(s') \backslash \{s\}$. Thus s, s' is a pair of co-duplicate vertices in G, which contradicts the uniqueness of x, y.

Claim 4. Let $|V(H)| \geq 2$. Then the graphs G and H have the same number of components. Moreover, if G is disconnected, then $G \cong Q_1 \cup K_1$ and $H \cong (Q_1 - x) \cup K_1$ where $Q_1, Q_1 - x$ are connected.

It is obvious that G and H have the same number of components by $(x, y) \in \mathrm{Aut}(G)$.

Now we prove the rest part of Claim 4. Let $G \cong Q_1 \cup Q_2 \cup \cdots \cup Q_l$ where Q_i is connected $(1 \leq i \leq l, l \geq 2)$. By Lemma 2.12, Q_i has at least one pair of (co-)duplicate vertices if $|V(Q_i)| \geq 2$ for $i \in \{1, \cdots, l\}$. Suppose $(u_i, v_i) \in \mathrm{Aut}(Q_i)$ for some $i \in \{1, 2, \cdots, l\}$ where $(u_i, v_i) \neq (x, y)$. Then $N_{Q_i - v_i}(u_i) = N_{Q_i - u_i}(v_i) = N_{G - v_i}(u_i) = N_{G - u_i}(v_i)$, which implies $(u_i, v_i) \in \mathrm{Aut}(G)$, a contradiction. Thus there is at most one Q_i has $|V(Q_i)| \geq 2$ and we can suppose $G \cong Q_1 \cup (l-1)K_1$ where $l \geq 2$. However, if $l \geq 3$ and $|V(Q_1)| = 1$, then $G \cong 3K_1$ has 3 pairs of duplicate vertices, a contradiction; if $l \geq 3$ and $|V(Q_1)| \geq 2$, then G has at least 2 pairs of (co-)duplicate vertices, a contradiction. Thus $G \cong Q_1 \cup K_1$.

We note that $d_G(x) = d_G(y) \geq 1$ by $(x, y) \in \mathrm{Aut}(G)$. Then from this and $G \cong Q_1 \cup K_1$, we have $H = G - x \cong (Q_1 - x) \cup K_1$ where $Q_1 - x$ is connected. Then Claim 4 holds.

Now we show $G \in \{G_n, \overline{G_n}\}$.

From the above discussion, H is a cograph with a unique pair of vertices s, y such that $N_{H-y}(s) = N_{H-s}(y)$, then we can conclude that there are Claim 1'- Claim 4' between H and $H - y$ $(\cong H - s)$ which are similar to Claims 1-4.

Let $V(G) = \{v_1, v_2, \cdots, v_n\}$, where v_n, v_{n-1} is the unique pair of vertices of G such that $N_{G-v_{n-1}}(v_n) = N_{G-v_n}(v_{n-1})$. Also let $F_n = G$ and $F_i = G - \{v_n, v_{n-1}, \cdots, v_{i+1}\}$ for $2 \leq i \leq n-1$. It is obvious that F_i $(2 \leq i \leq n)$ are cographs. Then by the above arguments, each pair of graphs F_i, F_{i-1} have 4 Claims, where v_i, v_{i-1} is the unique pair of vertices of F_i $(3 \leq i \leq n)$ such that $N_{F_i - v_{i-1}}(v_i) = N_{F_i - v_i}(v_{i-1})$.

If G is connected, then F_2 is a connected graph of order 2, which implies $F_2 \cong P_2 \cong G_2$. By the relationships between F_3 and F_2, it is easy to find that $F_3 \cong P_3 \cong G_3$. By the definition of G_n and the relationships between F_i and F_{i-1} $(3 \le i \le n)$, we have $G \cong G_n$.

If G is disconnected, then $G \cong Q_1 \cup K_1$ where Q_1 is a connected graph by Claim 4. Let $d_G(v_1) = 0$. Then $d_{F_i}(v_1) = 0$ for $i \in \{2, 3, 4, \cdots, n-1\}$. Therefore, we have F_2 is a disconnected graph of order 2, which implies $F_2 \cong 2K_1 \cong \overline{G_2}$. By the relationships between F_3 and F_2, it is easy to find that $F_3 \cong P_2 \cup K_1 \cong \overline{G_3}$. By the definition of G_n and the relationships between F_i and F_{i-1} $(3 \le i \le n)$, we have $G \cong \overline{G_n}$. □

Now we determine all almost controllable cographs.

Theorem 2.21. *Let $n \ge 2$. Then G is an almost controllable cograph of order n if and only if $G \cong G_n$ or $G \cong \overline{G_n}$.*

Proof. If $G \cong G_n$ or $G \cong \overline{G_n}$, then G is an almost controllable cograph of order n by Theorem 2.10 and Lemma 2.15.

Let G be an almost controllable cograph of order n. Then by Lemma 2.17, $\mathrm{Aut}(G)$ is either trivial or generated by a transposition (x, y).

If $\mathrm{Aut}(G)$ is trivial, then G is not a cograph by Lemma 2.12.

If $\mathrm{Aut}(G)$ is generated by a transposition (x, y), then G has a unique pair of vertices x, y such that $N_{G-y}(x) = N_{G-x}(y)$ by Proposition 2.18. By Theorem 2.20, we have $G \in \{G_n, \overline{G_n}\}$. Thus $G \cong G_n$ or $G \cong \overline{G_n}$. □

A *threshold graph* can be obtained from K_1 by repeatedly performing one of the following two operations: (a) adding a new vertex adjacent to none of the former vertices; (b) adding a new vertex adjacent to all of the former vertices.

Let G be a threshold graph of order n and $V(G) = \{v_1, v_2, \cdots, v_n\}$ where v_i is the added vertex in the i-th step of the operations. We can use a $\{0, 1\}$-sequence $b = (b_1, b_2, \cdots, b_n)$ to represent G, where $b_i = 0$ if v_i is a vertex not adjacent to any of the vertices in $\{v_1, v_2, \ldots, v_{i-1}\}$ and $b_i = 1$ if v_i is adjacent to v_j for $1 \le j \le i-1$. There is a considerable body of knowledge on the spectral properties of threshold graphs [1,24,25].

We note that the complement of a threshold graph is also a threshold graph since the complement of $(b_1, b_2, \cdots, b_n)$ is $(1-b_1, 1-b_2, \cdots, 1-b_n)$. For example, $(0, 1) \cong P_2$, $(1, 0) \cong 2K_1$, $(1, 1, 0) \cong P_2 \cup K_1$, $(0, 0, 1) \cong P_3$.

It is well known that threshold graphs are characterized in terms of forbidden induced subgraphs: they are graphs without P_4, C_4 and $2K_2$ as induced subgraphs [18]. By the properties of threshold graphs and cographs, the following proposition is self-evident.

Proposition 2.22. *A threshold graph is also a cograph.*

Now we determine all almost controllable threshold graphs.

Theorem 2.23. *Let $n \ge 2$. Then G is an almost controllable threshold graph of order n if and only if $G \cong G_n$ or $G \cong \overline{G_n}$.*

Proof. It is clear that $G_2 \cong P_2$ is a threshold graph. Then by Lemma 2.3 and the definition of threshold graphs, we have $\overline{G_3} \cong G_2 \cup K_1$, $G_4 \cong \overline{G_3 \cup K_1} \cong \overline{G_3} \triangledown K_1$, $\overline{G_5} \cong G_4 \cup K_1$, $\cdots$, $\overline{G_{n-1}} \cong G_{n-2} \cup K_1$ (or $G_{n-1} \cong \overline{G_{n-2}} \triangledown K_1$), $G_n \cong \overline{G_{n-1}} \triangledown K_1$ (or $\overline{G_n} \cong G_{n-1} \cup K_1$) are threshold graphs. Therefore, by the fact that the complement of a threshold graph is also a threshold graph, we have G_n and $\overline{G_n}$ are threshold graphs for all $n \ge 2$. By Theorem 2.10, we have G_n and $\overline{G_n}$ are almost controllable threshold graphs.

On the other hand, if G is an almost controllable threshold graph, then G is an almost controllable cograph by Proposition 2.22, and thus $G \cong G_n$ or $G \cong \overline{G_n}$ by Theorem 2.21. □

3. Almost controllable $\frac{\sqrt{5}-1}{2}$-graphs

It is well known that connected graphs except for complete multipartite (including complete) graphs have the second largest eigenvalue greater than 0. The graph G with $\lambda_2(G) \le \frac{1}{3}$ ([3]), $\lambda_2(G) \le \frac{1}{2}$ ([33]) and $\lambda_2(G) \le \frac{\sqrt{5}-1}{2}$ ([8]) are characterized respectively.

A graph G with $\lambda_2(G) \le \frac{\sqrt{5}-1}{2}$ will be called $\frac{\sqrt{5}-1}{2}$-*graphs*. In [11], the authors proved that there are no controllable $\frac{\sqrt{5}-1}{2}$-graphs. In this section, we determine all almost controllable $\frac{\sqrt{5}-1}{2}$-graphs.

Lemma 3.1. ([8]) *If G is a $\frac{\sqrt{5}-1}{2}$-graph and contains P_4 as an induced subgraph, then any vertex v outside P_4 is one of the four types a, b, c, d shown in Fig. 2.*

Let a *pendant* vertex be a vertex of degree 1, and a *next-to-pendant* vertex be a vertex adjacent to a pendant vertex.

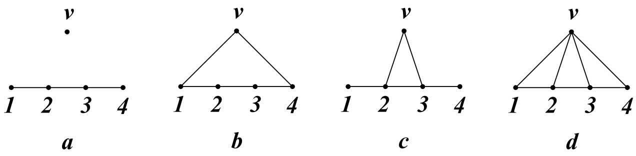

Fig. 2. The types a, b, c, d in Lemma 3.1.

Lemma 3.2. ([13]) *Let G be a graph which contains an induced subgraph $H \cong lP_4$ for $l \geq 1$. Then G is not almost controllable if any vertex $v \in V(G) \backslash V(H)$ satisfies one of the following three cases:*
(i) *v is either adjacent to every vertex or no vertex of some P_4;*
(ii) *v is adjacent to an even number of pendant vertices of H;*
(iii) *v is adjacent to an even number of next-to-pendant vertices of H.*

Theorem 3.3. *Let G be an almost controllable graph of order n (≥ 2) with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then $\lambda_2(G) \leq \frac{\sqrt{5}-1}{2}$ if and only if $2 \leq n \leq 8$, and $G \in \{G_2, G_3, G_4, G_5, G_6, G_7, \overline{G_2}, \overline{G_3}, \overline{G_4}, \overline{G_5}, \overline{G_6}, \overline{G_7}, \overline{G_8}\}$.*

Proof. Let G be an almost controllable graph of order n (≥ 2) with $\lambda_2(G) \leq \frac{\sqrt{5}-1}{2}$. Then we show $2 \leq n \leq 8$ and $G \in \{G_2, G_3, G_4, G_5, G_6, G_7, \overline{G_2}, \overline{G_3}, \overline{G_4}, \overline{G_5}, \overline{G_6}, \overline{G_7}, \overline{G_8}\}$ by the following two cases.

Case 1. G contains P_4 as an induced subgraph.

Then we have $\lambda_2(G) \geq \lambda_2(P_4) = \frac{\sqrt{5}-1}{2}$ by Lemma 2.8, and thus $\lambda_2(G) = \frac{\sqrt{5}-1}{2}$. By Lemmas 3.1 and 3.2, G is not almost controllable. Thus there is no such G.

Case 2. G does not contain P_4 as an induced subgraph.

Then G is a cograph, and we have $G \in \{G_n, \overline{G_n}\}$ by Theorem 2.21. By direct calculation, we have $\lambda_2(G_7) \approx 0.537$ and $\lambda_2(G_8) \approx 0.697$, combining with Lemma 2.8 and the definition of G_n, we have $\lambda_2(G_2) \leq \cdots \leq \lambda_2(G_7) < \frac{\sqrt{5}-1}{2} < \lambda_2(G_8) \leq \lambda_2(G_9) \leq \cdots$. By Lemmas 2.3 and 2.8, we have $\lambda_2(\overline{G_2}) \leq \cdots \leq \lambda_2(\overline{G_8} \cong G_7 \cup K_1) < \frac{\sqrt{5}-1}{2} < \lambda_2(\overline{G_9} \cong G_8 \cup K_1) \leq \lambda_2(\overline{G_{10}}) \leq \cdots$.

Combining the above arguments, we have $G \in \{G_2, G_3, G_4, G_5, G_6, G_7\}$ if G is connected with $\lambda_2(G) \leq \frac{\sqrt{5}-1}{2}$, and $G \in \{\overline{G_2}, \overline{G_3}, \overline{G_4}, \overline{G_5}, \overline{G_6}, \overline{G_7}, \overline{G_8}\}$ if G is disconnected with $\lambda_2(G) \leq \frac{\sqrt{5}-1}{2}$.

On the other hand, suppose G is an almost controllable graph with $\lambda_2(G) \leq \frac{\sqrt{5}-1}{2}$. By the calculation in Case 2, we have $G_2, G_3, G_4, G_5, G_6, G_7, \overline{G_2}, \overline{G_3}, \overline{G_4}, \overline{G_5}, \overline{G_6}, \overline{G_7}, \overline{G_8}$ are almost controllable graphs with the second largest eigenvalue less than $\frac{\sqrt{5}-1}{2}$. □

By Theorem 3.3, we know that there is no almost controllable graph G with $\lambda_2(G) = \frac{\sqrt{5}-1}{2}$. Naturally, we propose Problem 3.5 based on Lemma 3.4.

Lemma 3.4. ([13]) *Let G be a graph which contains an induced subgraph $H \cong lP_5$ for $l \geq 1$. Then G is not almost controllable if any vertex $v \in V(G) \backslash V(H)$ satisfies one of the following three cases:*
(i) *v is either adjacent to every vertex or no vertex of some P_5;*
(ii) *v is adjacent to an even number of pendant vertices of H;*
(iii) *v is adjacent to an even number of next-to-pendant vertices of H.*

Problem 3.5. Characterize all almost controllable graphs G with $\frac{\sqrt{5}-1}{2} < \lambda_2(G) \leq 1 = \lambda_2(P_5)$.

4. Cographs with $n-2$ main eigenvalues

By the discussion in Section 2, cographs of order n with n (controllable), $n-1$ (almost controllable) main eigenvalues are characterized. In this section, we study the cographs of order n with $n-2$ main eigenvalues. Two vertices $v_i, v_j \in V(G)$ belong to the same *orbit* if there is an automorphism $\sigma \in \mathrm{Aut}(G)$ such that $\sigma(v_i) = v_j$.

Lemma 4.1. ([7]) *The number of main eigenvalues of a graph G does not exceed the number of orbits into which $V(G)$ is partitioned by the automorphism group $\mathrm{Aut}(G)$.*

Theorem 4.2. *Let G be a graph of order n (≥ 3) with $n-2$ main eigenvalues. Then $\mathrm{Aut}(G)$ is one of the following cases where e is the identity transformation.*
(i) $\mathrm{Aut}(G) = \{e\}$;
(ii) $\mathrm{Aut}(G) = \{e, (u, v)\}$;

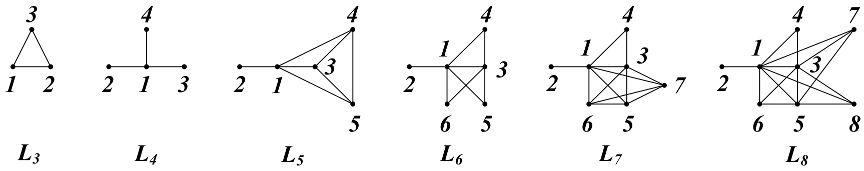

Fig. 3. The graphs L_i for $i \in \{3,4,5,6,7,8\}$.

(iii) $\mathrm{Aut}(G) = \{e, (u, p, q), (u, q, p)\}$;
(iv) $\mathrm{Aut}(G) = \{e, (u, p), (v, q), (u, p)(v, q)\}$;
(v) $\mathrm{Aut}(G) = \{e, (u, p)(v, q)\}$.

Proof. If G is a graph of order n with $n-2$ main eigenvalues, then the number of orbits of $V(G)$ is greater than or equal to $n-2$ by Lemma 4.1.

If $V(G)$ has n (or $n-1$) orbits, then $\mathrm{Aut}(G) = \{e\}$ (or $\mathrm{Aut}(G) = \{e, (u, v)\}$) is obvious by Lemma 2.17.

If $V(G)$ has $n-2$ orbits, then $n-2$ vertices have been used since each orbit has at least one vertex, and thus the remaining 2 vertices, say p, q, must belong to some of the $n-2$ orbits. If p, q are in the same orbit, then $(u, p, q) \in \mathrm{Aut}(G)$ for some $u \in V(G)$ which implies $\mathrm{Aut}(G) = \{e, (u, p, q), (u, q, p)\}$. If p, q are not in the same orbit, then there exist $u, v \in V(G)$ such that $(u, p), (v, q) \in \mathrm{Aut}(G)$ or $(u, p)(v, q) \in \mathrm{Aut}(G)$, and this implies $\mathrm{Aut}(G) = \{e, (u, p), (v, q), (u, p)(v, q)\}$ or $\mathrm{Aut}(G) = \{e, (u, p)(v, q)\}$. $\square$

Now we introduce a graph L_n of order n ($n \geq 3$), and show that L_n and $\overline{L_n}$ are cographs of order n with $n-2$ main eigenvalues.

As in Section 2, let $n \geq 3$ and $V(G_{n-1}) = \{v_1, \cdots, v_{n-1}\}$. We define L_n ($n \geq 3$) as the graph of order n that is obtained from G_{n-1} after adding a new vertex v_n such that $v_n \sim v$ for any $v \in N_{G_{n-1}}(v_{n-1})$, and $v_n \sim v_{n-1}$ if and only if $v_{n-1} \sim v_{n-2}$. The graphs L_i are shown in Fig. 3 for $i \in \{3,4,5,6,7,8\}$.

From the definitions of L_n and G_{n-1}, we know that any two of $\{v_{n-2}, v_{n-1}, v_n\}$ is a pair of (co-)duplicate vertices in L_n, then $\mathrm{Aut}(L_n) = \{e, (v_{n-2}, v_{n-1}, v_n), (v_{n-2}, v_n, v_{n-1})\}$. Therefore, L_n ($\overline{L_n}$) is a cograph by G_{n-1} ($\overline{G_{n-1}}$) is a cograph. Moreover, L_n ($\overline{L_n}$) is a threshold graph by the definition of threshold graphs and $(v_{n-2}, v_{n-1}, v_n) \in \mathrm{Aut}(L_n)$. It should be noted that L_n and its complement are cographs and threshold graphs, which can also be derived from Lemma 4.3 that will be proved below.

Next we show L_n ($\overline{L_n}$) has $n-2$ main eigenvalues for $n \geq 3$.

Lemma 4.3. *Let $n \geq 3$. Then $\overline{L_{n+1}} \cong L_n \cup K_1$.*

Proof. By the definition of L_n, we have $\overline{L_n}$ is obtained from $\overline{G_{n-1}}$ after adding a new vertex v_n such that $v_n \sim v$ for any $v \in N_{\overline{G_{n-1}}}(v_{n-1})$, and $v_n \sim v_{n-1}$ if and only if $v_{n-1} \sim v_{n-2}$ in $\overline{G_{n-1}}$.

Similarly, we have $\overline{L_{n+1}}$ is obtained from $\overline{G_n}$ after adding a new vertex v_{n+1} such that $v_{n+1} \sim v$ for any $v \in N_{\overline{G_n}}(v_n)$, and $v_{n+1} \sim v_n$ if and only if $v_n \sim v_{n-1}$ in $\overline{G_n}$. By Lemma 2.3 and the definition of $\overline{G_n}$, we have $\overline{G_n} \cong G_{n-1} \cup K_1$ and $d_{\overline{G_n}}(v_1) = 0$, then $d_{\overline{L_{n+1}}}(v_1) = 0$ is obvious. Thus $\overline{L_{n+1}} - v_1$ is isomorphic to the graph that is obtained from G_{n-1} after adding a new vertex v' such that $v' \sim v$ for any $v \in N_{G_{n-1}}(v_{n-1})$, and $v' \sim v_{n-1}$ if and only if $v_{n-1} \sim v_{n-2}$ in G_{n-1}. By the definition of L_n, we have $\overline{L_{n+1}} - v_1 \cong L_n$, and thus $\overline{L_{n+1}} \cong L_n \cup K_1$ by $d_{\overline{L_{n+1}}}(v_1) = 0$. $\square$

Lemma 4.4. *Let $n \geq 3$. If n is even, then 0 is a non-main eigenvalue of L_n with multiplicity 2. If n is odd, then $0 \notin \mathrm{Spec}(L_n)$ and -1 is a non-main eigenvalue of L_n with multiplicity 2.*

Proof. Let $\mathrm{Spec}(L_n) = \{\lambda_1, \ldots, \lambda_n\}$, $\mathrm{Spec}(G_{n-1}) = \{\mu_1, \ldots, \mu_{n-1}\}$, $x_1 = (0, \ldots, 0, -1, 1, 0)^T$, $x_2 = (0, \ldots, 0, -1, 0, 1)^T$ are n-dimensional vectors. Then we prove the results by the following two cases.

Case 1. n is even.

By the definitions of L_n and G_{n-1}, we know that any two of $\{v_{n-2}, v_{n-1}, v_n\}$ is a pair of duplicate vertices in L_n. Then it is easy to find that $A(L_n)x_i = 0 \cdot x_i$ for $i \in \{1, 2\}$. Thus 0 is an eigenvalue of L_n with multiplicity greater than or equal to 2.

Next we show 0 is an eigenvalue of L_n with multiplicity 2. Suppose the multiplicity of 0 is greater than 2, then there exists some i ($2 \leq i \leq n-1$) such that $\lambda_{i-1} = \lambda_i = \lambda_{i+1} = 0$. Then we have $0 = \lambda_{i-1} \geq \mu_{i-1} \geq \lambda_i = 0 \geq \mu_i \geq \lambda_{i+1} = 0$ by Lemma 2.8. That is to say, 0 is an eigenvalue of G_{n-1} with multiplicity greater than or equal to 2. This is a contradiction since 0 is a simple eigenvalue of G_{n-1} by Lemma 2.9. Therefore, 0 is a non-main eigenvalue of L_n with multiplicity 2 by $j_n^T x_1 = j_n^T x_2 = 0$.

Case 2. n is odd.

We will prove $0 \notin \mathrm{Spec}(L_n)$ by showing that the constant c'_n of $P_{L_n}(x)$ is non-zero. By the definitions of L_n and G_{n-1}, it is not hard to find that $N_{L_n}(v_i) = \{v_1, v_3, v_5, \cdots, v_{i-1}\}$ if i $(2 \le i \le n-3)$ is even, and any two of $\{v_{n-2}, v_{n-1}, v_n\}$ is a pair of co-duplicate vertices.

Let $\mathcal{H}'_n$ be the set of all elementary subgraphs of L_n with n vertices. Now we consider the possible elementary subgraph $H \in \mathcal{H}'_n$ which contributes to c'_n by Lemma 2.7. It is not hard to find that there is only one H belongs to $\mathcal{H}'_n$, where $E(H) = \{v_1v_2, v_3v_4, \cdots, v_{n-4}v_{n-3}, v_{n-2}v_{n-1}, v_{n-2}v_n, v_{n-1}v_n\}$ since $N_{L_n}(v_i) = \{v_1, v_3, v_5, \cdots, v_{i-1}\}$ if i $(2 \le i \le n-3)$ is even. Therefore, H can only be isomorphic to $\frac{n-3}{2}K_2 \cup C_3$ which implies $c'_n = (-1)^{\frac{n-1}{2}} 2^1 \neq 0$. Thus $0 \notin \mathrm{Spec}(L_n)$ if n is odd.

By the definitions of L_n and G_{n-1}, we know that any two of $\{v_{n-2}, v_{n-1}, v_n\}$ is a pair of co-duplicate vertices in L_n. Then it is easy to find that $A(L_n)x_i = -1 \cdot x_i$ for $i \in \{1, 2\}$. Thus -1 is an eigenvalue of L_n with multiplicity greater than or equal to 2.

Next we show -1 is a non-main eigenvalue of L_n with multiplicity 2. Suppose the multiplicity of -1 is great than 2, then there exists some i $(2 \le i \le n-1)$ such that $\lambda_{i-1} = \lambda_i = \lambda_{i+1} = -1$, and thus we have $-1 = \lambda_{i-1} \ge \mu_{i-1} \ge \lambda_i = -1 \ge \mu_i \ge \lambda_{i+1} = -1$ by Lemma 2.8. That is to say, -1 is an eigenvalue of G_{n-1} with multiplicity greater than or equal to 2. This is a contradiction since -1 is a simple non-main eigenvalue of G_{n-1} by Theorem 2.10. Therefore, -1 is a non-main eigenvalue of L_n with multiplicity 2 by $j_n^T x_1 = j_n^T x_2 = 0$. □

Theorem 4.5. *Let $n \ge 3$. Then both L_n and $\overline{L_n}$ are cographs with $n-2$ main eigenvalues. In fact, -1 is the non-main eigenvalue of L_n with multiplicity 2 if n is odd, and 0 is the non-main eigenvalue of L_n with multiplicity 2 if n is even.*

Proof. By direct calculation, $L_3 \cong C_3$ and $\mathrm{Spec}(C_3) = \{2, -1, -1\}$, where 2 is the unique main eigenvalue. Similarly, $L_4 \cong K_{1,3}$ and $\mathrm{Spec}(K_{1,3}) = \{\sqrt{3}, -\sqrt{3}, 0, 0\}$, and only $\sqrt{3}, -\sqrt{3}$ are main eigenvalues.

Similar to the proof of Theorem 2.10, we have L_n and $\overline{L_n}$ are cographs with $n-2$ main eigenvalues by Lemmas 2.4, 2.5, 4.3 and 4.4.

The rest part of the theorem is obvious by Lemma 4.4. □

Now we study the cographs of order n with exactly $n-2$ main eigenvalues.

Theorem 4.6. *Let G be a cograph of order n (≥ 3) with exactly $n-2$ main eigenvalues. Then G is one of the following four cases:*
(i) $G \cong L_n$ *or* $G \cong \overline{L_n}$;
(ii) $G \cong H \cup K_1$ *or* $G \cong H \triangledown K_1$, *where H is obtained from G_{n-3} (or $\overline{G_{n-3}}$) after adding 2 vertices v_{n-2}, v_{n-1} and some edges such that* $\mathrm{Aut}(H) =< (v_{n-4}, v_{n-3}), (v_{n-2}, v_{n-1}) >$;
(iii) $G \cong G_i \cup G_{n-i}$ *or* $G \cong G_i \cup \overline{G_{n-i}}$ *for some* i $(2 \le i \le n-2)$;
(iv) $G \cong G_i \triangledown \overline{G_{n-i}}$ *or* $G \cong \overline{G_i} \triangledown \overline{G_{n-i}}$ *for some* i $(2 \le i \le n-2)$.

Proof. Case 1. $\mathrm{Aut}(G) = \{e\}$ or $\mathrm{Aut}(G) = \{e, (u, v)\}$.

Then there is no such G by Lemmas 2.12, 2.16 and Theorems 2.10, 2.20.

Case 2. $\mathrm{Aut}(G) = \{e, (u, p)(v, q)\}$.

Then there is no such G by Lemma 2.12.

Case 3. $\mathrm{Aut}(G) = \{e, (u, p, q), (u, q, p)\}$.

Then any two of $\{u, p, q\}$ is a pair of (co-)duplicate vertices in G. It is easy to find that $G - u$ $(\cong G - p \cong G - q)$ has a unique pair of vertices p, q such that $N_{G-\{u,q\}}(p) = N_{G-\{u,p\}}(q)$. Combining with $G - u$ is a cograph, we have $G - u \in \{G_{n-1}, \overline{G_{n-1}}\}$ by Theorem 2.20. The same conclusion applies to $G - p, G - q$. Then (i) holds by the definition of L_n.

Case 4. $\mathrm{Aut}(G) = \{e, (u, p), (v, q), (u, p)(v, q)\}$.

By Lemma 2.11, we have $G \cong H_1 \cup H_2$ or $G \cong H_1 \triangledown H_2$, where H_1, H_2 are cographs.

Subcase 4.1. $(u, p), (v, q) \in \mathrm{Aut}(H_1)$.

Then $|V(H_2)| = 1$, otherwise H_2 has (co-)duplicate vertices by Lemma 2.12 and this contradicts with $\mathrm{Aut}(G) = \{e, (u, p), (v, q), (u, p)(v, q)\}$. Thus $G \cong H_1 \cup K_1$ or $G \cong H_1 \triangledown K_1$, where H_1, K_1 are cographs. It is easy to find that $H_1 - \{u, p\}$ (or $H_1 - \{v, q\}$) is a cograph with a unique pair of vertices v, q such that $N_{H_1-\{u,p,q\}}(v) = N_{H_1-\{u,p,v\}}(q)$ (or u, p such that $N_{H_1-\{v,q,p\}}(u) = N_{H_1-\{v,q,p\}}(p)$). Thus $H_1 - \{u, p\} \in \{G_{n-3}, \overline{G_{n-3}}\}$ (or $H_1 - \{v, q\} \in \{G_{n-3}, \overline{G_{n-3}}\}$). Then (ii) holds.

Subcase 4.2. $(u, p) \in \mathrm{Aut}(H_1)$ and $(v, q) \in \mathrm{Aut}(H_2)$.

Then H_1 (or H_2) is a cograph with a unique pair of vertices u, p (or v, q) such that $N_{H_1-p}(u) = N_{H_1-u}(p)$ (or $N_{H_2-q}(v) = N_{H_2-v}(q)$). Thus $H_1 \in \{G_i, \overline{G_i}\}$ and $H_2 \in \{G_{n-i}, \overline{G_{n-i}}\}$ by Theorem 2.20 for $2 \le i \le n-2$.

If $G \cong H_1 \cup H_2$, then G has $n-2$ main eigenvalues if and only if $\mathrm{MainSpec}(H_1) \cap \mathrm{MainSpec}(H_2) = \emptyset$, and thus $H_1 \cong \overline{G_i}, H_2 \cong \overline{G_{n-i}}$ cannot be both true since $0 \in \mathrm{MainSpec}(\overline{G_i}) \cap \mathrm{MainSpec}(\overline{G_{n-i}})$ for $2 \le i \le n-2$. Then (iii) holds.

If $G \cong H_1 \triangledown H_2$, then $|\mathrm{MainSpec}(G)| = |\mathrm{MainSpec}(H_1 \triangledown H_2)| = |\mathrm{MainSpec}(\overline{H_1} \cup \overline{H_2})| = n-2$ if and only if $\mathrm{MainSpec}(\overline{H_1}) \cap \mathrm{MainSpec}(\overline{H_2}) = \emptyset$, and thus $\overline{H_1} \cong \overline{G_i}, \overline{H_2} \cong \overline{G_{n-i}}$ cannot be both true since $0 \in \mathrm{MainSpec}(\overline{G_i}) \cap \mathrm{MainSpec}(\overline{G_{n-i}})$ for $2 \le i \le n-2$. Then (iv) holds. □

Example 4.7. Let $n \ge 5$, $W_n \cong H \cup K_1$ be the graph of order n, $H \cong G_{n-3} \cup 2K_1$. Suppose $d_H(v_{n-2}) = d_H(v_{n-1}) = 0$. It is easy to check that $\mathrm{Aut}(H) =< (v_{n-4}, v_{n-3}), (v_{n-2}, v_{n-1}) >$. Thus $G \cong W_n$ is a cograph belonging to (ii) of Theorem 4.6. By

Theorem 2.10, $W_n \cong G_{n-3} \cup 3K_1$ has $n-3$ main eigenvalues, where $\{-1, 0, 0\}$ are the non-main eigenvalues if n is odd, and $\{0, 0, 0\}$ are the non-main eigenvalues if n is even.

Example 4.8. Let H be a graph of order 4 that is obtained from G_2 after adding vertices p, q and edges such that $N(p) = N(q) = V(G_2)$, in fact, $H \cong P_3 \triangledown K_1$. Then $G \cong H \triangledown K_1$ is a cograph of order 5 which belongs to (ii) of Theorem 4.6 by $\mathrm{Aut}(H) =< (v_1, v_2), (p, q) >$. However, by calculation, $\mathrm{Spec}(G) = \{1+\sqrt{7}, 1-\sqrt{7}, -1, -1, 0\}$, and only $1+\sqrt{7}, 1-\sqrt{7}$ are main eigenvalues.

Examples 4.7 and 4.8 demonstrate that (ii) of Theorem 4.6 does not provide a characterization of cographs with $n-2$ main eigenvalues. On the other hand, we did not find the graph G belongs to case (iii) (or (iv)) of Theorem 4.6 such that $|\mathrm{MainSpec}(G)| < n-2$. This suggests that not all the cographs satisfying (ii) of Theorem 4.6 are the cographs of order n with $n-2$ main eigenvalues, but the cographs satisfying (iii) (or (iv)) of Theorem 4.6 are likely to be the all cographs of order n with $n-2$ main eigenvalues. We therefore propose the following problem.

Problem 4.9. Characterize the cographs of order n (≥ 3) with exactly $n-2$ main eigenvalues.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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