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Further results on almost controllable graphs

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An eigenvalue λ of a graph G of order n is a main eigenvalue if its eigenspace is not orthogonal to the all-ones vector j. In 1978, Cvetković proved that G has exactly one main eigenvalue if and only if G is regular, and posed the following long-standing problem: characterize the graphs with exactly k $(2 \leq k \leq n)$ main eigenvalues. Graphs of order n with n, $n-1$ main eigenvalues are called controllable, almost controllable, respectively. In this paper, we study the properties of almost controllable graphs. For almost controllable trees, unicyclic and bicyclic graphs, we show that the diameters of their complements are less than or equal to 3, determine all complements with diameter 3, and obtain the results about the controllable graphs. Moreover, all integral almost controllable graphs are determined, and some further problems are proposed.

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1. Introduction

Throughout this paper, G is a simple graph with vertex set $V(G) = \{v_1, \cdots, v_n\}$ and edge set $E(G)$. If the vertices v_i and v_j are adjacent, we write $v_i \sim v_j$, then $e = v_iv_j$ is an edge belonging to $E(G)$ and we say v_i (v_j) is *incident* to e. We say that two edges of G are adjacent if they are incident to a common vertex. Let $|V(G)|$ be the order of G and $|E(G)|$ be the number of edges in G. We say that G is *empty* if $|V(G)| = 0$, and *null* if $|E(G)| = 0$. Let $N_G(u)$ ($N(u)$ for short) be the neighbourhood set of u in G. Two vertices u, v are called *duplicate* vertices if $u \nsim v$ and $N(u) = N(v)$, *co-duplicate* vertices if $u \sim v$ and $N(u)\backslash\{v\} = N(v)\backslash\{u\}$. The *distance* between vertices u and v in G, denoted by $d_G(u, v)$, is the length of the shortest $u - v$ path in G. The *diameter* $\mathrm{diam}(G)$ of a connected graph G is the maximum distance between two vertices of G. The *complement* of a graph G is denoted by $\overline{G}$. Suppose $V' \subseteq V(G)$, the *induced subgraph* of G with respect to V' is a graph with vertex set V' and edge set E', where $v_iv_j \in E'$ if $v_iv_j \in E(G)$ for any $v_i, v_j \in V'$, $G - V'$ is the graph obtained from G after deleting each vertex $v \in V'$ and all edges that are incident to v. Let K_n, $K_{1,n-1}$, C_n, P_n be the complete graph, star, cycle, path of order n, respectively.

Let $A(G) = [a_{ij}]$ be the $n \times n$ *adjacency matrix* of G where $a_{ij} = 1$ if $v_i \sim v_j$ and $a_{ij} = 0$ otherwise. The *eigenvalues* of G are the eigenvalues of its adjacency matrix $A(G)$. The *spectrum* of G is the multiset of all eigenvalues of G, and we denote by $\mathrm{Spec}(G)$. An eigenvalue λ of G of order n is said to be a *main eigenvalue* if its eigenspace is not orthogonal to the all-ones vector $j = [1, 1, \ldots, 1]^T$ of length n, and an eigenvector x of G is a *main eigenvector* if $j^T x \neq 0$. By [4], all main eigenvalues of G are distinct.

For a connected graph G, since its adjacency matrix $A(G)$ is irreducible, the famous Perron-Frobenius Theorem ensures that the largest eigenvalue of G is always main. In 1978, Cvetković proved that G has exactly one main eigenvalue if and only if G is regular. Besides, he posed the following long-standing problem: characterize the graphs of order n with exactly k ($2 \leq k \leq n$) main eigenvalues [4].

There are a series of papers characterizing the graphs with exactly $2, n-1, n$ main eigenvalues. All trees, unicyclic, bicyclic and tricyclic graphs with exactly two main eigenvalues are characterized in [12–14], and for the other relevant results, one can refer to Feng et al. [9], Hagos [15], Hayat et al. [16], Lepović [19], etc. For graphs with all eigenvalues main, Cvetković et al. defined them as *controllable graphs* through their correlation with control theory [5], and for the relevant results, we refer the readers to Cvetković et al. [5,6], Farrugia [8] and Stanić [22]. For graphs of order n with $n-1$ main eigenvalues, Wang, Liu and Wang defined them as *almost controllable graphs* [23], and for the recent research on almost controllable graphs, one can refer to [7,18,20,23].

A graph G is called reconstructible if it can be determined from the knowledge only of all one-vertex-deleted subgraphs. In [11], the authors proved that a graph G of order n is reconstructible if all but at most one of the eigenvalues of $A(G)$ are simple, with the corresponding eigenvectors not orthogonal to j_n. Thus characterizing the graphs with exactly k main eigenvalues (especially $k = n-1, n$) is of great importance as such graphs are reconstructible.

In this paper, we focus on almost controllable graphs and the paper is organized as follows. In Section 2, we study the properties of almost controllable graphs. In Sections 3 and 4, we show that the diameters of the complements of (almost) controllable trees, unicyclic and bicyclic graphs are less than or equal to 3 and determine all complements with diameter 3. In Section 5, we propose two conjectures about the (almost) controllable graphs for further research. In Section 6, we determine the integral almost controllable graphs.

2. Preliminaries

An *automorphism* of a graph G is a permutation σ of the vertex set $V(G)$ such that the pair of vertices $v_i \sim v_j$ if and only if $\sigma(v_i) \sim \sigma(v_j)$. The set of automorphisms of G under the composition operation, form a group, called the *automorphism group* of G and denoted by $\mathrm{Aut}(G)$. Let $|\mathrm{Aut}(G)|$ be the order of $\mathrm{Aut}(G)$. It is well-known that a graph and its complement share the same automorphism group.

Lemma 2.1 *([7]). Let G be a graph of order n with $n-1$ main eigenvalues, then its automorphism group $\mathrm{Aut}(G)$ is either trivial or generated by a transposition $\sigma = (v_i, v_j)$ fixing all vertices $w \notin \{v_i, v_j\}$. Furthermore, if $v_i \nsim v_j$ then 0 is an eigenvalue of G, and if $v_i \sim v_j$ then -1 is an eigenvalue of G.*

By Lemma 2.1, we may conclude that for an almost controllable graph G, $|\mathrm{Aut}(G)| \in \{1, 2\}$. Moreover, if $\mathrm{Aut}(G) =<(u, v)>$, then u, v is the unique pair of vertices in G such that $N_{G-v}(u) = N_{G-u}(v)$ by the definition of the automorphism of G. In fact, such u, v is a pair of duplicate vertices in G if $u \nsim v$, and a pair of co-duplicate vertices in G if $u \sim v$.

It is proved that controllable graphs have only the trivial automorphism group [5]. By Lemma 2.1, we know that the graph with a trivial automorphism group may not be controllable. Similarly, we want to know whether a graph G with $|\mathrm{Aut}(G)| \in \{1, 2\}$ can only be controllable or almost controllable. The answer is no. An extreme example is the Frucht graph F on 12 vertices with $\mathrm{Aut}(F) = 1$ (see Fig. 1), which is neither controllable nor almost controllable [10]. It is easy to check that F is regular of degree 3, thus F has exactly 1 main eigenvalue 3. Moreover, there exist other regular asymmetric graphs with trivial automorphism group [25], and Fig. 2 lists the graphs with the least number of vertices.

Before considering the properties of almost controllable graphs, we give two lemmas about controllable graphs first.

Let a *pendant* vertex be a vertex of degree 1, and a *next-to-pendant* vertex be a vertex adjacent to a pendant vertex.

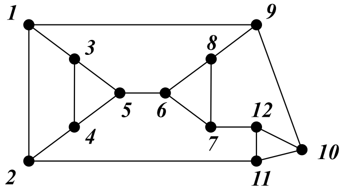

Fig. 1. The Frucht graph F of order 12.

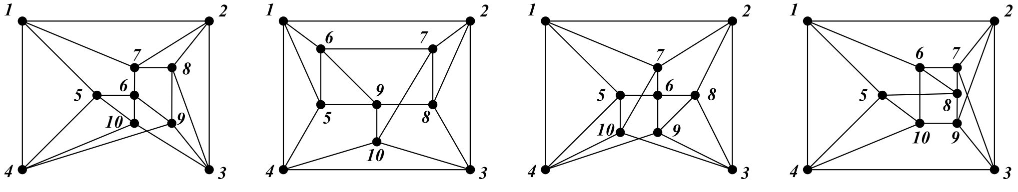

Fig. 2. The 4-regular asymmetric graphs of order 10.

Lemma 2.2 *([8]). Every controllable graph G on at least six vertices has P_4 as an induced subgraph.*

Lemma 2.3 *([6]). Let G be a controllable graph with P_4 as an induced subgraph. If v is any vertex in G, then none of the following hold:* (i) *v is adjacent to all vertices of P_4;* (ii) *v is adjacent to both pendant vertices of P_4;* (iii) *v is adjacent to both next-to-pendant vertices of P_4;* (iv) *v is non-adjacent to any vertex of P_4.*

By Lemmas 2.2 and 2.3, we know that controllable graphs have P_4 as an induced subgraph but the graphs with P_4 as an induced subgraph may not be controllable. Next we show a graph G with lP_4 $(l \geq 1)$ as an induced subgraph may not be almost controllable.

Proposition 2.4. *Let G be a graph which contains an induced subgraph H $(\cong lP_4)$ for $l \geq 1$. Then G is not almost controllable if any vertex $v \in V(G)\backslash V(H)$ satisfies one of the following three cases:*
(i) *v is either adjacent to every vertex or no vertex of some P_4;*
(ii) *v is adjacent to an even number of pendant vertices of H;*
(iii) *v is adjacent to an even number of next-to-pendant vertices of H.*

Proof. For $l = 1$, it is clear that G has an automorphism $(u_1, u_4)(u_2, u_3)$, where $P_4 = u_1u_2u_3u_4$. Then the result holds by Lemma 2.1.

For $l \geq 2$, by the structure of G, it is easy to find two non-main eigenvectors x_1 and x_2 of G with corresponding eigenvalues $\frac{-1-\sqrt{5}}{2}$, $\frac{-1+\sqrt{5}}{2}$, respectively, where

$$x_1 = (\underbrace{\xi_1, \cdots, \xi_1}_{l}, 0, \cdots, 0)^T, x_2 = (\underbrace{\xi_2, \cdots, \xi_2}_{l}, 0, \cdots, 0)^T,$$

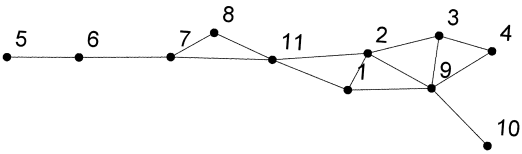

Fig. 3. The graph G in Example 2.5.

$\xi_1 = (-1, \frac{1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}, 1)^T$, $\xi_2 = (-1, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}, 1)^T$ and the first $4l$ entries of x_1, x_2 correspond to the vertices of H. □

Example 2.5. Let G be the graph of order 11 shown in Fig. 3. It is clear that G has an induced subgraph $H \cong 2P_4$ with $V(H) = \{1, 2, 3, 4, 5, 6, 7, 8\}$, and the connection ways between $\{9, 10, 11\}$ and $V(H)$ satisfy the cases of Proposition 2.4. By computer calculation, the automorphism group of G is trivial, and the eigenvalues of G are the roots of irreducible polynomials $x^6 - x^5 - 9x^4 + 12x^2 + 2x - 2$, $x^3 - 4x - 2$ and $\frac{-1\pm\sqrt{5}}{2}$. Moreover, $\frac{-1\pm\sqrt{5}}{2}$ are the only two non-main eigenvalues and then G is not almost controllable.

Furthermore, we show a graph G with lP_5 ($l \geq 1$) as an induced subgraph may not be almost controllable by the following result.

Proposition 2.6. *Let G be a graph which contains an induced subgraph H ($\cong lP_5$) for $l \geq 1$. Then G is not almost controllable if any vertex $v \in V(G)\backslash V(H)$ satisfies one of the following three cases:*
(i) *v is either adjacent to every vertex or no vertex of some P_5;*
(ii) *v is adjacent to an even number of pendant vertices of H;*
(iii) *v is adjacent to an even number of next-to-pendant vertices of H.*

Proof. Similar to the proof of Proposition 2.4, it is easy to find two non-main eigenvectors x_3 and x_4 of G with corresponding eigenvalues -1, 1, respectively, where

$$x_3 = (\underbrace{\xi_3, \cdots, \xi_3}_{l}, 0, \cdots, 0)^T, x_4 = (\underbrace{\xi_4, \cdots, \xi_4}_{l}, 0, \cdots, 0)^T,$$

$\xi_3 = (-1, 1, 0, -1, 1)^T$, $\xi_4 = (-1, -1, 0, 1, 1)^T$ and the first $5l$ entries of x_3, x_4 correspond to the vertices of H. □

Example 2.7. Let G be the graph of order 15 shown in Fig. 4. It is clear that G has an induced subgraph $H \cong 2P_5$ with $V(H) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and the connection ways between $\{11, 12, 13, 14, 15\}$ and $V(H)$ satisfy the cases of Proposition 2.6. By computer calculation, $|\text{Aut}(G)| = 2$, the eigenvalues of G are the roots of irreducible polynomial $x^{10} - 15x^8 - 4x^7 + 71x^6 + 28x^5 - 119x^4 - 44x^3 + 54x^2 + 4x - 4$ and $\pm\sqrt{2}, 0, \pm 1$. Moreover, $-1, 0, 1$ are the only three non-main eigenvalues of G, thus G is not almost controllable.

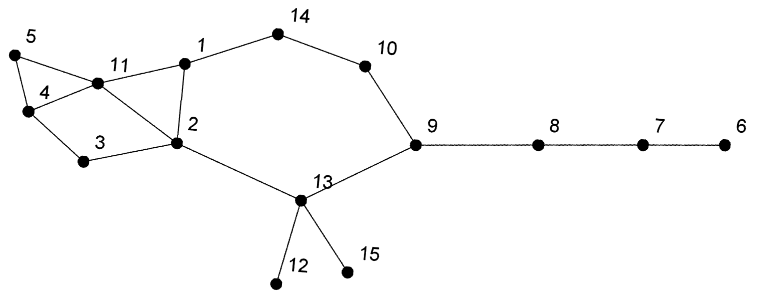

Fig. 4. The graph G in Example 2.7.

3. Almost controllable trees and unicyclic graphs

In this section, we study the diameters of the complements of almost controllable trees and unicyclic graphs, show $\mathrm{diam}(\overline{G}) \in \{2, 3\}$ if G is an almost controllable tree or an almost controllable unicyclic graph, and determine the graphs with $\mathrm{diam}(\overline{G}) = 3$. Furthermore, as a by-product, we show $\mathrm{diam}(\overline{G}) = 2$ if G is a controllable tree or a controllable unicyclic graph.

The following results are useful and interesting.

Lemma 3.1 *([17]). If* $\mathrm{diam}(G) \geq 3$, *then* $\mathrm{diam}(\overline{G}) \leq 3$.

Lemma 3.2 *([21]). A graph G and its complement $\overline{G}$ have the same number of main eigenvalues.*

Proposition 3.3. *Let G be a graph with the complement $\overline{G}$. Then* $\mathrm{diam}(\overline{G}) = 2$ *if and only if for any edge $uv \in E(G)$, there is a vertex $w \in V(G)$ satisfies $w \nsim u$ and $w \nsim v$ in G. In other words,* $\mathrm{diam}(\overline{G}) \neq 2$ *if and only if there is an edge $uv \in E(G)$ such that $N(u) \cup N(v) = V(G)$.*

Proof. If for any edge $uv \in E(G)$, there is a vertex w satisfies $w \nsim u$ and $w \nsim v$ in G, then there is a uwv path in $\overline{G}$ which implies $d_{\overline{G}}(u, v) = 2$. On the other hand, any two non-adjacent vertices of G will be adjacent in $\overline{G}$. Thus we have $\mathrm{diam}(\overline{G}) = 2$. The necessity is obvious and we complete the proof. □

Lemma 3.4. *Let T be a tree of order n. If* $|\mathrm{Aut}(T)| = 1$, *then* $\mathrm{diam}(\overline{T}) = 2$.

Proof. We prove this by indicating that any two adjacent vertices of T have at least one common non-adjacent vertex in T by Proposition 3.3.

Suppose to the contrary, there are two vertices $u \sim v$ in T such that $N(u) \cup N(v) = V(T)$. Firstly, for any vertices $w \in V(T) \backslash \{u, v\}$, $w \sim u$ and $w \sim v$ can not be both true, otherwise there will be a C_3 and this contradicts with T being a tree. Secondly, the subgraph T' induced by the vertex set $V(T) \backslash \{u, v\}$ is null, otherwise each pair of adjacent

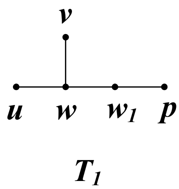

Fig. 5. The graph T_1.

vertices of T' will form a C_3 with u (v) or a C_4 with u, v in T, and this contradicts with T being a tree. Therefore, T is a star $K_{1,n}$ if $N(u) = \{v\}$ (or $N(v) = \{u\}$) or a double-star otherwise.

For $n = 4$, T must be isomorphic to P_4 or $K_{1,3}$. However, $|\mathrm{Aut}(P_4)| > 1$, $|\mathrm{Aut}(K_{1,3})| > 1$ and this contradicts with $|\mathrm{Aut}(T)| = 1$.

For $n \geq 5$, there will be at least one pair of duplicate vertices of degree 1 that are both adjacent to u or v in T, which contradicts with $|\mathrm{Aut}(T)| = 1$.

Combining the above arguments, we have $\mathrm{diam}(\overline{T}) = 2$. □

Since the automorphism group of controllable graphs is trivial [5], we have the following corollary by Lemma 3.2 and Lemma 3.4.

Corollary 3.5 *([22]). Let T be a controllable tree distinct from K_1. Then $\overline{T}$ is a controllable graph whose diameter is equal to 2.*

Theorem 3.6. *Let T be an almost controllable tree of order n (≥ 4). Then* $\mathrm{diam}(\overline{T}) \in \{2, 3\}$*. Moreover,* $\mathrm{diam}(\overline{T}) = 3$ *if and only if* $T \cong T_1$ (see Figure 5).

Proof. By Lemma 2.1 and T is almost controllable, we have $|\mathrm{Aut}(T)| \in \{1, 2\}$.

Case 1: $|\mathrm{Aut}(T)| = 1$.

We can directly get $\mathrm{diam}(\overline{T}) = 2$ by Lemma 3.4.

Case 2: $|\mathrm{Aut}(T)| = 2$.

Since the star $K_{1,n-1}$ is the only tree with diameter 2, we have $\mathrm{diam}(T) \geq 3$ since $|\mathrm{Aut}(K_{1,n-1})| = 2$ if and only if $n = 3$. Then $\mathrm{diam}(\overline{T}) \in \{2, 3\}$ by Lemma 3.1.

From the above two cases, we have $\mathrm{diam}(\overline{T}) \in \{2, 3\}$. Now we show the rest part of the result. Clearly, it is easy to check that T_1 is almost controllable with $\mathrm{diam}(\overline{T_1}) = 3$.

If T is almost controllable with $\mathrm{diam}(\overline{T}) = 3$, we will show that $T \cong T_1$. By Lemma 2.1, we can suppose $\mathrm{Aut}(T) =< (u, v) >$. Then u, v is the unique pair of duplicate vertices in T, and $d(u) = d(v) = 1$ since there will be a cycle induced by u, v and their common neighbours if $d(u) = d(v) > 1$. Let $V' = V(T) \backslash (N(w) \cup \{w\})$ where $u \sim w$. Clearly, $N(w) \backslash \{u, v\} \neq \emptyset$ by $n \geq 4$ and $V' \neq \emptyset$ by $\mathrm{diam}(T) \geq 3$. Moreover, the subgraph induced by $N(w)$ is null, otherwise, there is at least a cycle C_3 induced by w and $N(w)$. By Proposition 3.3 and $\mathrm{diam}(\overline{T}) \neq 2$, T has an edge $xy \in E(T)$ such that $N(x) \cup N(y) = V(T)$. Since u, v are pendant vertices, we suppose $xy = ww_1$ such that $N(w) \cup N(w_1) = V(T)$ where $w_1 \in N(w) \backslash \{u, v\}$. Then $d(w_1) > 1$, otherwise

$(w_1, u) \in \mathrm{Aut}(T)$ and $(w_1, v) \in \mathrm{Aut}(T)$, which contradicts with $|\mathrm{Aut}(T)| = 2$. Let $p \sim w_1$ where $p \in V'$.

Next we show that $N(w) = \{u, v, w_1\}$. If there is a vertex $w_2 \in N(w) \backslash \{u, v, w_1\}$, then $d(w_2) > 1$ otherwise $(w_2, u) \in \mathrm{Aut}(T)$. We can suppose $w_2 \sim t$, then $t \sim w$ or $t \sim w_1$, and thus there will be a cycle induced by w, w_1, w_2, t, a contradiction.

Finally, we show that $V(T) = \{u, v, w, w_1, p\}$. Suppose to the contrary, there is a vertex $r \in V(T) \backslash \{u, v, w, w_1, p\}$, then $r \sim w_1$ by $N(w) \cup N(w_1) = V(T)$ and $|N(w)| = 3$. If $d(r) > 1$, then $r \sim p$ or $r \sim r'$ where $r' \in V(T) \backslash \{u, v, w, w_1, p, r\}$. If $r \sim p$, then r, p, w_1 will induce a cycle C_3, a contradiction. If $r \sim r'$, then $r' \sim w_1$ and r, r', w_1 will induce a cycle C_3, a contradiction. Hence $d(r) = 1$. Similarly, we can deduce that $d(p) = 1$. Thus $(p, r) \in \mathrm{Aut}(T)$, a contradiction. Thus $V(T) = \{u, v, w, w_1, p\}$ and $T \cong T_1$. □

Let D_n denote the Dynkin graph of order $n \geq 4$, a tree obtained from the path of order $n-1$ by adding a pendant edge at the second vertex [24]. Then we show that there exist almost controllable tree T such that $\mathrm{diam}(\overline{T}) = 2$ by Example 3.7.

Example 3.7 *([24]).* The tree D_n has $n - 2$ main eigenvalues if $4|n$ and $n - 1$ main eigenvalues otherwise.

It is easy to check that $\mathrm{diam}(\overline{D_5}) = 3$ by $D_5 \cong T_1$, and D_n is an almost controllable tree with $\mathrm{diam}(\overline{D_n}) = 2$ if $4 \nmid n$ and $n \neq 5$.

Next we consider the diameters of the complements of unicyclic graphs.

Lemma 3.8. *Let G be a unicyclic graph with the unique cycle C. If G has an edge uv such that $N(u) \cup N(v) = V(G)$, then we have:*

(i) *$G - \{u, v\}$ has no cycle;*
(ii) *$C \cong C_3$ or $C \cong C_4$, furthermore, if $C \cong C_4$, then $uv \in E(C)$.*

Proof. (i) Suppose to the contrary, $G - \{u, v\}$ has a cycle C'. Since $|V(C')| \geq 3$, for any two vertices $w_1 \sim w_2$ of C', u, v, w_1, w_2 will induce a cycle C_3 or C_4, which contradicts with G being unicyclic.

(ii) Let $C \cong C_i$ for $i \geq 3$. Next we prove $i \in \{3, 4\}$. Clearly, $u \in V(C)$ or $v \in V(C)$ or $uv \in E(C)$ by (i).

Firstly, we show that if $u \in V(C)$ and $v \notin V(C)$, then $i = 3$. Otherwise, if $i \geq 4$ then there exists a vertex $w \in V(C)$ satisfying $d_C(w, u) = 2$. Since w is adjacent to u or v, which implies G has another cycle C_3 or C_4, and this contradicts G being unicyclic. Secondly, if $u \notin V(C)$ and $v \in V(C)$, then $i = 3$ is similar. Finally, we show that if $uv \in E(C_i)$, then $i \leq 4$. Otherwise, if $i \geq 5$, then there exists a vertex $s \in V(C)$ satisfying $d_C(s, u) = 2$, and s is adjacent to u or v, which implies G has another cycle C_3 or C_4, a contradiction.

Thus $C \cong C_3$ or $C \cong C_4$, and if $C \cong C_4$, then $uv \in E(C)$. □

Lemma 3.9. *Let G be a unicyclic graph of order n (≥ 5). If $|\mathrm{Aut}(G)| = 1$, then* $\mathrm{diam}(\overline{G}) = 2$.

Proof. By Proposition 3.3, we just need to show that any two adjacent vertices of G have at least one common non-adjacent vertex in G.

Suppose to the contrary, there are two vertices $u \sim v$ in G such that $N(u) \cup N(v) = V(G)$. Since u, v and each pair of adjacent vertices of $G - \{u, v\}$ will induce a cycle C_3 or C_4, we have $|E(G - \{u, v\})| \leq 1$ by G is unicyclic. Let C be the unique cycle of G. Then $C \cong C_3$ or $C \cong C_4$ by Lemma 3.8.

If $uv \notin E(C)$, then $C \cong C_3$ by Lemma 3.8. Let $w_1, w_2 \in V(C)$. Then $d(w_1) = d(w_2) = 2$ by $|E(G - \{u, v\})| \leq 1$. Now w_1, w_2 is a pair of co-duplicate vertices in G, which contradicts with $|\mathrm{Aut}(G)| = 1$.

If $uv \in E(C_3)$, then $|E(G - \{u, v\})| = 0$. Therefore, for any vertex $p_i \in V(G)\backslash\{u, v, w\}$ where $w \sim u, w \sim v$, we have $d(p_i) = 1$ by G is unicyclic. For $n = 5$, let $V(G) = \{u, v, w, p_1, p_2\}$. Then $(u, v)(p_1, p_2) \in \mathrm{Aut}(G)$ if $p_1 \sim u$, $p_2 \sim v$, $(u, v)(p_1, p_2) \in \mathrm{Aut}(G)$ if $p_1 \sim v$, $p_2 \sim u$, and $(p_1, p_2) \in \mathrm{Aut}(G)$ if both p_1 and p_2 are adjacent to u (or v), all cases contradict with $|\mathrm{Aut}(G)| = 1$. For $n \geq 6$, G has at least one pair of duplicate vertices p_i, p_j where both p_i and p_j are adjacent to u (or v), which contradicts with $|\mathrm{Aut}(G)| = 1$.

If $uv \in E(C_4)$, then $|E(G - \{u, v\})| = 1$. Similarly, for any vertex $p_i \in V(G)\backslash\{u, v, w_1, w_2\}$ where $w_1 \sim u, w_2 \sim v, w_1 \sim w_2$, we have $d(p_i) = 1$ by G is unicyclic. For $n = 5$, w_1, v (if $p_i \sim u$) or w_2, u (if $p_i \sim v$) is a pair of duplicate vertices in G, a contradiction. For $n = 6$, let $V(G) = \{u, v, w_1, w_2, p_1, p_2\}$. Then G has an automorphism $(u, v)(w_1, w_2)(p_1, p_2)$ if $p_1 \sim u$, $p_2 \sim v$ (or $p_1 \sim v$, $p_2 \sim u$), and has an automorphism (p_1, p_2) if both p_1 and p_2 are adjacent to u (or v), all cases contradict with $|\mathrm{Aut}(G)| = 1$. For $n \geq 7$, G has at least one pair of duplicate vertices p_i, p_j where both p_i and p_i are adjacent to u (or v), a contradiction.

Hence G has no edge uv such that $N(u) \cup N(v) = V(G)$, and we complete the proof. □

Similar to Corollary 3.5, we have the following corollary by Lemmas 3.2 and 3.9.

Corollary 3.10. *Let G be a controllable unicyclic graph of order n (≥ 5). Then $\overline{G}$ is a controllable graph with* $\mathrm{diam}(\overline{G}) = 2$.

Theorem 3.11. *Let G be an almost controllable unicyclic graph of order n (≥ 5). Then* $\mathrm{diam}(\overline{G}) \in \{2, 3\}$. *Moreover,* $\mathrm{diam}(\overline{G}) = 3$ *if and only if $G \cong U_i$ $(i = 1, 4, 5, 6)$* (see Figure 6).

Proof. We consider the following two cases according to the automorphism groups by Lemma 2.1. Let C be the unique cycle of G. It is easy to check that U_1,U_4,U_5,U_6 are almost controllable, and $\mathrm{diam}(\overline{U_i}) = 3$ for $i \in \{1, 4, 5, 6\}$.

Case 1: $|\mathrm{Aut}(G)| = 1$.

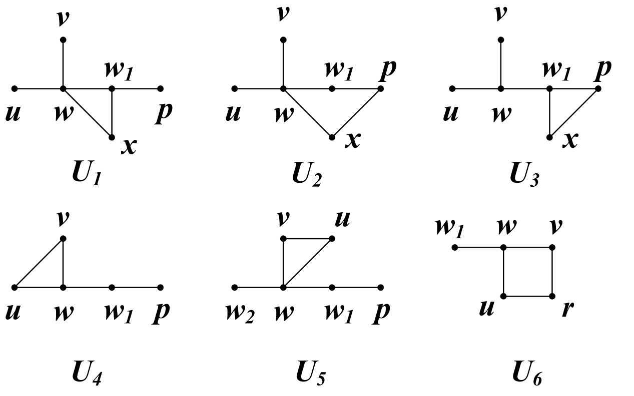

Fig. 6. The graphs U_i $(i = 1, 2, 3, 4, 5, 6)$.

We have $\mathrm{diam}(\overline{G}) = 2$ by Lemma 3.9 directly.

Case 2: $|\mathrm{Aut}(G)| = 2$.

By Lemma 2.1, we can suppose $\mathrm{Aut}(G) =< (u, v) >$. Then u, v is the unique pair of (co-)duplicate vertices in G. Clearly, we have $d(u) = d(v) \leq 2$, otherwise there will be at least three cycles induced by $\{u, v\}$ and $N(u)$, a contradiction.

Subcase 2.1: $d(u) = d(v) = 1$.

We will show that G can only be isomorphic to U_1. First we state a fact: if H is a unicyclic graph that is obtained by a unicyclic graph H' after adding a new vertex x, then $d(x) = 1$.

Let $V' = V(G)\backslash(\{w\} \cup N(w))$ where $u \sim w, v \sim w$. Note that $N(w)\backslash\{u, v\} \neq \emptyset$ since $|V(G)| \geq 5$. We claim that $V' \neq \emptyset$, otherwise $G \cong K^+_{1,n-1}$ where $K^+_{1,n-1}$ is obtained by $K_{1,n-1}$ after adding an edge, but $|\mathrm{Aut}(K^+_{1,n-1})| > 2$ for $n \geq 5$, this contradicts with $|\mathrm{Aut}(G)| = 2$. Let $w_1 \sim w, p \sim w_1$ where $p \in V'$.

For $n = 5$, it is evident that there is no such unicyclic graph G of order 5.

For $n = 6$, G can only be obtained by T_1 after adding a vertex x and two edges. Moreover, w, w_1, p and x must induce the cycle C, which will result in G possibly being isomorphic to U_1, U_2, U_3 (see Fig. 6). It is easy to check that $|\mathrm{Aut}(U_1)| = 2$, $|\mathrm{Aut}(U_2)| > 2$ and $|\mathrm{Aut}(U_3)| > 2$. By calculation, U_1 is almost controllable with $\mathrm{diam}(\overline{U_1}) = 3$. Thus in this subcase, G can only be isomorphic to U_1, and $\mathrm{diam}(\overline{G}) = 3$.

For $n = 7$, G can only be obtained by T_1 after adding 2 vertices s_1, s_2 and three edges. We now show $\mathrm{diam}(\overline{G}) = 2$ by proving that G has no edge $e = xy \in E(G)$ such that $N(x) \cup N(y) = V(G)$.

Suppose to the contrary, G has an edge $e = xy$ such that $N(x) \cup N(y) = V(G)$, then e must incident to w. Without loss of generality, we suppose $e = xy = ww_1$ since $d(u) = d(v) = 1$, $N(u) = N(v) = \{w\}$ and $p \in V'$.

We claim that if $s_i \sim w$ for $i \in \{1, 2\}$, then $d(s_i) > 1$. Otherwise, $(s_i, u), (s_i, v) \in \mathrm{Aut}(G)$, a contradiction. Therefore if $s_1 \sim w$, then $s_1 \sim w_1$ or $s_1 \sim p$ or $s_1 \sim s_2$.

If $s_1 \sim w$ and $s_1 \sim w_1$, then w, w_1, s_1 induce a cycle C_3. Now s_2 can only be adjacent to w_1 since $d(s_2) = 1$. Then $(s_2, p) \in \mathrm{Aut}(G)$, a contradiction.

If $s_1 \sim w$ and $s_1 \sim p$, then w, w_1, s_1, p induce a cycle C_4. Then $s_2 \sim w_1$, and G is isomorphic to the graph that is obtained by U_2 after joining a pendant vertex to w_1. By calculation, G has 5 main eigenvalues in this case, a contradiction.

If $s_1 \sim w$ and $s_1 \sim s_2$, then $s_2 \sim w$ or $s_2 \sim w_1$. It is easy to check that $(s_1, s_2) \in \mathrm{Aut}(G)$ if $s_2 \sim w$, or G has 5 main eigenvalues if $s_2 \sim w_1$, a contradiction.

If $s_1 \nsim w$, then $s_1 \sim w_1$ and we consider the case $s_2 \nsim w, s_2 \sim w_1$ since all other cases are as same as the above. Now $s_1 \sim s_2$ or $s_1 \sim p$ or $d(s_1) = 1$, which implies $(s_1, s_2) \in \mathrm{Aut}(G)$ or $(s_1, p) \in \mathrm{Aut}(G)$ or $(p, s_2) \in \mathrm{Aut}(G)$, a contradiction.

Thus in this subcase, $\mathrm{diam}(\overline{G}) = 2$.

For $n \geq 8$, G can only be obtained by T_1 after adding $n-5$ vertices and $n-4$ edges. Let $V(G)\backslash V(T_1) = \{s_1, \cdots, s_{n-5}\}$ where $n-5 \geq 3$, and $H = G[s_1, \cdots, s_{n-5}]$ be the subgraph of G that is induced by the vertex set $\{s_1, \cdots, s_{n-5}\}$. We now show $\mathrm{diam}(\overline{G}) = 2$ by proving that G has no edge $e = xy \in E(G)$ such that $N(x) \cup N(y) = V(G)$.

Suppose to the contrary, G has an edge $e = xy$ such that $N(x) \cup N(y) = V(G)$. Similarly, we can suppose $e = xy = ww_1$. Then s_i $(1 \leq i \leq n-5)$ is adjacent to at least one of $\{w, w_1\}$. Clearly, $d(s_i) \geq 2$ if $s_i \sim w$ for $1 \leq i \leq n-5$, otherwise $(s_i, u), (s_i, v) \in \mathrm{Aut}(G)$, a contradiction. Besides, we have $|E(H)| \leq 1$ since G is unicyclic and each pair of adjacent vertices $s_i \sim s_j$ and w, w_1 will induce a cycle in G.

If $|E(H)| = 0$, then s_i, w, w_1, p must induce a cycle for some $i \in \{1, \cdots, n-5\}$, and we suppose $i = 1$. Then $d(s_j) = 1$ and $s_j \sim w_1$ for $j \in \{2, \cdots, n-5\}$ by the above discussion. Now $(s_j, s_k) \in \mathrm{Aut}(G)$ for $j, k \in \{2, \cdots, n-5\}$, a contradiction.

If $E(H) = 1$, we can suppose $E(H) = \{s_1 s_2\}$, then $s_1 s_2 \in E(C_3)$ or $s_1 s_2 \in E(C_4)$ by Lemma 3.8. In each case, we have $d(s_j) = 1$ and $s_j \sim w_1$ for $j \in \{3, \cdots, n-5\}$, and then $(s_j, p) \in \mathrm{Aut}(G)$, a contradiction.

Thus in this subcase, $\mathrm{diam}(\overline{G}) = 2$.

Subcase 2.2: $d(u) = d(v) = 2$.

Then $u, v \in V(C)$, $C \cong C_3$ if $u \sim v$ and $C \cong C_4$ if $u \nsim v$. We will show that $\mathrm{diam}(\overline{G}) = 3$ if and only if $G \cong U_i$ $(i = 4, 5, 6)$, and $\mathrm{diam}(\overline{G}) = 2$ otherwise. Clearly, U_4, U_5, U_6 are almost controllable with $\mathrm{diam}(\overline{U_4}) = \mathrm{diam}(\overline{U_5}) = \mathrm{diam}(\overline{U_6}) = 3$, and we only need show $\mathrm{diam}(\overline{G}) = 2$ if $G \notin \{U_4, U_5, U_6\}$ by proving that G has no edge $e = xy$ such that $N(x) \cup N(y) = V(G)$.

Subcase 2.2.1: $u \sim v$.

Let $V' = V(G)\backslash(\{w\} \cup N(w))$, where $u \sim w, v \sim w$. Then $N(w)\backslash\{u, v\} \neq \emptyset$ by $n \geq 5$, and $V' \neq \emptyset$ by a similar reason to Subcase 2.1. We can suppose $w \sim w_1, w_1 \sim p$ where $p \in V'$.

Let $G \notin \{U_4, U_5, U_6\}$. If there exists an edge $e = xy \in E(G)$ such that $N(x) \cup N(y) = V(G)$, without loss of generality, we can suppose $e = xy = ww_1$ since $d(u) = d(v) = 2$, $u \sim v, u \sim w, v \sim w$ and $p \in V'$. Let $V_1 = V(G)\backslash\{u, v, w, w_1, p\}$.

If $V_1 = \emptyset$, then $G \cong U_4$, it is a contradiction by $G \notin \{U_4, U_5, U_6\}$.

If $V_1 \neq \emptyset$, then for any $s \in V_1$, $d(s) = 1$ by G is unicyclic. We note that $d(p) = 1$, otherwise w, w_1, p, p' will induce a cycle where $p' \sim p$ for $p' \in V_1$, a contradiction. Thus $s \nsim w_1$ where $s \in V_1$, otherwise $(s, p) \in \mathrm{Aut}(G)$, a contradiction. Therefore $d(w_1) = 2$

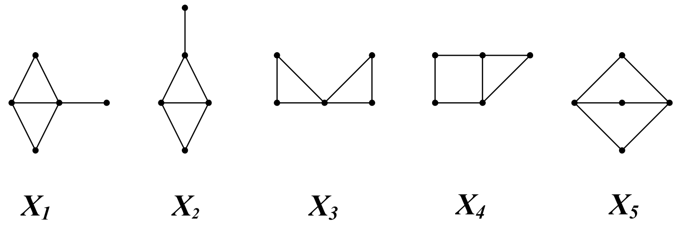

Fig. 7. All bicyclic graphs of order 5.

and $s \sim w$ for any $s \in V_1$. For $n = 6$, we have $G \cong U_5$, and this contradicts with $G \notin \{U_4, U_5, U_6\}$. For $n \geq 7$, we have $(s_i, s_j) \in \mathrm{Aut}(G)$ where $s_i, s_j \in V_1$, a contradiction.

Thus in this subcase, $\mathrm{diam}(\overline{G}) = 2$ if $G \notin \{U_4, U_5, U_6\}$, and then $\mathrm{diam}(\overline{G}) = 3$ if and only if $G \cong U_i$ $(i = 4, 5)$.

Subcase 2.2.2: $u \nsim v$.

Then $C \cong C_4$ and let $V(C) = \{u, v, w, r\}$ where $N(u) = N(v) = \{w, r\}$. Let $V' = V(G)\backslash(\{w, r\} \cup N(w) \cup N(r))$. Since $n \geq 5$, $N(w)\backslash\{u, v\} \neq \emptyset$ or $N(r)\backslash\{u, v\} \neq \emptyset$, we can suppose $w_1 \in N(w)\backslash\{u, v\}$.

Let $G \notin \{U_4, U_5, U_6\}$. If there exists an edge $e = xy \in E(G)$ such that $N(x) \cup N(y) = V(G)$, without loss of generality, we can suppose $e = xy = uw$ since $d(u) = d(v) = 2$ and $u \sim w, u \sim r, v \sim w, w_1 \sim w$. Furthermore, $d(w_1) = 1$ and $d(s) = 1$ for any $s \in V(G)\backslash\{u, v, w, r, w_1\}$ by G is unicyclic, and thus $s \sim w$ by $d(u) = 2$.

If $N(r)\backslash\{u, v\} = \emptyset$, then $G \cong U_6$ if $d(w) = 3$, and $(s, w_1) \in \mathrm{Aut}(G)$ if $d(w) \geq 4$ for any $s \in V(G)\backslash\{u, v, w, r, w_1\}$, a contradiction.

If $N(r)\backslash\{u, v\} \neq \emptyset$, then we have another cycle induced by u, w, r, s' where $s' \in N(r)\backslash\{u, v\}$, and this contradicts the fact that G is unicyclic.

Thus in this subcase, $\mathrm{diam}(\overline{G}) = 2$ if $G \notin \{U_4, U_5, U_6\}$, and then $\mathrm{diam}(\overline{G}) = 3$ if and only if $G \cong U_6$.

Combining the above arguments, we have $\mathrm{diam}(\overline{G}) \in \{2, 3\}$, and $\mathrm{diam}(\overline{G}) = 3$ if and only if $G \cong U_i$ $(i = 1, 4, 5, 6)$. $\square$

4. Almost controllable bicyclic graphs

In this section, we show $\mathrm{diam}(\overline{G}) \in \{2, 3\}$ if G is an almost controllable bicyclic graph of order n (≥ 6), and determine the graphs with $\mathrm{diam}(\overline{G}) = 3$. By the way, the result about the controllable bicyclic graphs is obtained.

All bicyclic graphs of order 5 are isomorphic to X_i $(i = 1, 2, 3, 4, 5)$ (see Fig. 7) and only X_1, X_2 are almost controllable. Moreover, $\mathrm{diam}(\overline{X_2}) = 3$ but $\overline{X_1}$ is disconnected. Thus we consider bicyclic graphs of order greater than 5.

Lemma 4.1. *Let G be a bicyclic graph with circumference k and order n (≥ 6). If G has an edge $uv \in E(G)$ such that $N(u) \cup N(v) = V(G)$, then we have:*

(i) *$G - \{u, v\}$ has no cycle;*

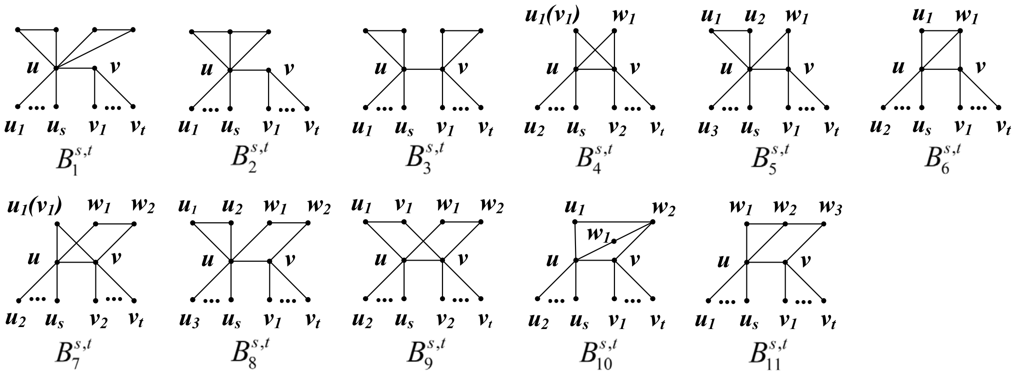

Fig. 8. The graphs $B_i^{s,t}$, $1 \le i \le 11$.

(ii) *The circumference* $k \in \{3, 4, 5, 6\}$*, and there exist integers* s (≥ 0)*,* t (≥ 0)*,* i $(1 \le i \le 11)$ *such that* $G \cong B_i^{s,t}$ (see Fig. 8).

Proof. (i) The proof is similar to (i) of Lemma 3.8 and we omit it.

(ii) Let C_k be the cycle of G with the longest length k. We complete the proof by the following three cases.

Case 1: $u \in V(C_k)$ and $v \notin V(C_k)$.

Let $V_1 = V(C_k) \cup \{v\}$, $G_1 = G[V_1]$ be the subgraph of G that is induced by V_1. Without loss of generality, we suppose $w_1u, w_2u \in E(C_k)$. Then for any $w \in V_1 \backslash \{u, v, w_1, w_2\}$, $wu \in E(G_1)$ or $wv \in E(G_1)$ by $N(u) \cup N(v) = V(G)$, and thus $|E(G_1)| \ge |V_1| + (|V_1| - 4)$ since $E(C_k) \cup \{uv\} \subseteq E(G_1)$ and $V_1 = V(C_k) \cup \{v\}$. On the other hand, $|E(G_1)| \le |E(G)| - (|V(G)| - |V_1|)$ since for any vertex $w' \in V(G) \backslash V_1$, $w'u \in E(G)$ or $w'v \in E(G)$. Then we have $|V_1| \le 5$ and thus $|V(C_k)| \le 4$ by $|E(G)| = |V(G)| + 1$. In this case, there exist s (≥ 0), t (≥ 0) such that $G \cong B_i^{s,t}$ for $i \in \{1, 2, 3\}$.

Case 2: $u \notin V(C_k)$ and $v \in V(C_k)$.

Similar to Case 1 by the symmetry of u, v, and we omit it.

Case 3: $u \in V(C_k)$ and $v \in V(C_k)$.

Now we show that the length of the longest path between u and v is less than or equal to 4. Let $P = uw_1w_2 \cdots w_{l-1}v$ be the longest path with length l $(l \ge 2)$ between u and v. Then $C = P + uv$ is a cycle with length $l + 1$, and thus $(l + 1) + (l - 4) \le |E(G[P])| \le |E(G)| - (|V(G)| - |V(P)|) = l + 1$ by $|E(G)| = |V(G)| + 1$ and for any vertex $w \in V(G) \backslash \{u, v, w_1, w_{l-1}\}$, $wu \in E(G)$ or $wv \in E(G)$, it implies $l \le 4$.

Let $\{u_1, \cdots, u_s\} = N(u) \backslash \{w_1, v\}$, $\{v_1, \cdots, v_t\} = N(v) \backslash \{w_{l-1}, u\}$. By G is a bicyclic graph of order n (≥ 6) and $N(u) \cup N(v) = V(G)$, we complete the proof by the following three subcases.

Subcase 3.1: $l = 2$.

Then $uv \in E(C_3)$, where $V(C_3) = \{u, v, w_1\}$. If there are some i $(1 \le i \le s)$ and some j $(1 \le j \le t)$ such that $u_i = v_j$, then $G \cong B_4^{s,t}$ for $s \ge 1, t \ge 1$, $s + t \ge 4$; if there are some i, j $(1 \le i, j \le s)$ such that $u_i \sim u_j$ (or some i, j such that $v_i \sim v_j$, where

$1 \le i, j \le t$), then $G \cong B_5^{s,t}$ for $s \ge 2, t \ge 0$, $s + t \ge 3$; if there is some i ($1 \le i \le s$) such that $u_i \sim w_1$ (or some j such that $v_j \sim w_1$, where $1 \le j \le t$), then $G \cong B_6^{s,t}$ for $s \ge 1, t \ge 0$, $s + t \ge 3$. In this subcase, $k \in \{3, 4\}$.

Subcase 3.2: $l = 3$.

Then $uv \in E(C_4)$, where $V(C_4) = \{u, v, w_1, w_2\}$. If $w_1 \sim v$ or $w_2 \sim u$, then $G \cong B_6^{s,t}$ for $s \ge 1, t \ge 0$, $s + t \ge 3$; if there are some i ($1 \le i \le s$) and some j ($1 \le j \le t$) such that $u_i = v_j$, then $G \cong B_7^{s,t}$ for $s \ge 1, t \ge 1$, $s + t \ge 3$; if there are some i, j ($1 \le i, j \le s$) such that $u_i \sim u_j$ (or some i, j such that $v_i \sim v_j$, where $1 \le i, j \le t$), then $G \cong B_8^{s,t}$ for $s \ge 2, t \ge 0$; if there are some i ($1 \le i \le s$) and some j ($1 \le j \le t$) such that $u_i \sim v_j$, then $G \cong B_9^{s,t}$ for $s \ge 1, t \ge 1$; if there is some i ($1 \le i \le s$) such that $u_i \sim w_2$ (or some j such that $v_j \sim w_2$, where $1 \le j \le t$), then $G \cong B_{10}^{s,t}$ for $s \ge 1, t \ge 0$, $s + t \ge 2$. In this subcase, $k \in \{4, 5, 6\}$.

Subcase 3.3: $l = 4$.

Then $uv \in E(C_5)$, where $V(C_5) = \{u, v, w_1, w_2, w_3\}$, and $w_2u \in E(G)$ or $w_2v \in E(G)$, and thus $G \cong B_{11}^{s,t}$ for $s \ge 0, t \ge 0$, $s + t \ge 1$. In this subcase, $k = 5$. □

Lemma 4.2. *Let G be an almost controllable bicyclic graph of order n (≥ 6). If $|\mathrm{Aut}(G)| = 1$, then $\mathrm{diam}(\overline{G}) \in \{2, 3\}$. Moreover, $\mathrm{diam}(\overline{G}) = 3$ if and only if $G \cong B_{11}^{1,1}$.*

Proof. If $\mathrm{diam}(\overline{G}) = 2$, then the result follows.

Now we show $\mathrm{diam}(\overline{G}) \ne 2$ if and only if $G \cong B_{11}^{1,1}$. By calculation, $B_{11}^{1,1}$ is almost controllable with $\mathrm{diam}(\overline{B_{11}^{1,1}}) = 3$.

If $\mathrm{diam}(\overline{G}) \ne 2$, then G has an edge $e = uv$ such that $N(u) \cup N(v) = V(G)$ by Proposition 3.3, thus we have $G \cong B_i^{s,t}$ for some s (≥ 0), t (≥ 0) and i ($1 \le i \le 11$) by Lemma 4.1.

Since $|\mathrm{Aut}(G)| = 1$, G has no (co-)duplicate vertices or other automorphisms except for the identity transformation. However, it is easy to check that graphs $B_1^{s,t}$, $B_2^{s,t}$, $B_3^{s,t}$, $B_4^{s,t}$, $B_5^{s,t}$, $B_8^{s,t}$, $B_{10}^{s,t}$ all have at least one pair of (co-)duplicate vertices of degree 2, and $B_9^{s,t}$ has an automorphism $(w_1, u_1)(w_2, v_1)$, which contradicts with $|\mathrm{Aut}(G)| = 1$. Then we consider $G \cong B_j^{s,t}$ for $j \in \{6, 7, 11\}$, and some $s \ge 0$, $t \ge 0$.

Case 1: $G \cong B_6^{s,t}$ for $s \ge 1$, $t \ge 0$, $s + t \ge 3$.

For $n = 6$, G can only be isomorphic to $B_6^{3,0}$, $B_6^{2,1}$, $B_6^{1,2}$. However, both $B_6^{3,0}$ and $B_6^{1,2}$ have one pair of duplicate vertices of degree 1, $B_6^{2,1}$ is controllable by calculation. For $n \ge 7$, G has at least one pair of duplicate vertices of degree 1 that are both adjacent to u (or v). Then we have $G \ncong B_6^{s,t}$ for $s \ge 1$, $t \ge 0$, $s + t \ge 3$.

Case 2: $G \cong B_7^{s,t}$ for $s \ge 1$, $t \ge 1$, $s + t \ge 3$.

For $n = 6$, $G \cong B_7^{2,1}$, but $B_7^{2,1}$ is controllable by calculation. For $n = 7$, G can only be isomorphic to $B_7^{2,2}$ or $B_7^{3,1}$. However, $(u, v)(w_1, w_2)(u_2, v_2) \in \mathrm{Aut}(B_7^{2,2})$ and $(u_2, u_3) \in \mathrm{Aut}(B_7^{3,1})$. For $n \ge 8$, G has at least one pair of duplicate vertices of degree 1 that are both adjacent to u (or v). Then $G \ncong B_7^{s,t}$ for $s \ge 1$, $t \ge 1$, $s + t \ge 3$.

Case 3: $G \in B_{11}^{s,t}$ for $s \ge 0, t \ge 0$, $s + t \ge 1$.

For $n = 6$, G can only be isomorphic to $B_{11}^{1,0}$ or $B_{11}^{0,1}$. However, $B_{11}^{1,0} (\cong B_7^{2,1})$ and $B_{11}^{0,1}$ are controllable by calculation. For $n = 7$, G can only be isomorphic to $B_{11}^{2,0}$, $B_{11}^{1,1}$, $B_{11}^{0,2}$. However, both $B_{11}^{2,0}$ and $B_{11}^{0,2}$ have one pair of duplicate vertices of degree 1, and $B_{11}^{1,1}$ is almost controllable with $\mathrm{diam}(\overline{B_{11}^{1,1}}) = 3$. For $n \geq 8$, G has at least one pair of duplicate vertices of degree 1 that are both adjacent to u (or v). Thus we have $G \cong B_{11}^{1,1}$.

Combining the above discussion, we have $\mathrm{diam}(\overline{G}) \in \{2, 3\}$, and $\mathrm{diam}(\overline{G}) = 3$ if and only if $G \cong B_{11}^{1,1}$. □

Let G be a controllable graph. Then $|\mathrm{Aut}(G)| = 1$ by [23]. By the proof of Lemma 4.2, we have the following result.

Theorem 4.3. *Let G be a controllable bicyclic graph of order n (≥ 6). Then* $\mathrm{diam}(\overline{G}) \in \{2, 3\}$. *Moreover,* $\mathrm{diam}(\overline{G}) = 3$ *if and only if* $G \in \{B_6^{2,1}, B_7^{2,1}, B_{11}^{0,1}\}$.

Lemma 4.4. *Let G be an almost controllable bicyclic graph of order $n \geq 6$, $\mathcal{B} = \{B_2^{0,1}, B_2^{1,1}, B_5^{2,1}, B_5^{3,1}, B_6^{3,1}, B_6^{2,2}, B_7^{3,1}, B_8^{2,1}, B_8^{3,1}, B_{10}^{2,1}, B_{11}^{0,2}\}$. If $|\mathrm{Aut}(G)| = 2$, then* $\mathrm{diam}(\overline{G}) \in \{2, 3\}$. *Moreover,* $\mathrm{diam}(\overline{G}) = 3$ *if and only if $G \in \mathcal{B}$.*

Proof. By calculation, each graph in $\mathcal{B}$ is an almost controllable bicyclic graph with $\mathrm{diam}(\overline{G}) = 3$.

If $\mathrm{diam}(\overline{G}) = 2$, then the result follows.

Suppose $\mathrm{diam}(\overline{G}) \neq 2$, now we show $G \in \mathcal{B}$. Firstly, G has an edge uv such that $N(u) \cup N(v) = V(G)$ by Proposition 3.3, and then there exist some s (≥ 0), t (≥ 0) and i $(1 \leq i \leq 11)$ such that $G \cong B_i^{s,t}$ by Lemma 4.1.

By Lemma 2.1, we can suppose $\mathrm{Aut}(G) = <(x, y)>$. Then x, y is the unique pair of (co-)duplicate vertices in G, and thus it is easy to check that $d(x) = d(y) \leq 3$, otherwise G has $|E(G)| > |V(G)| + 1$ which contradicts with G being bicyclic. Now we consider the following three cases.

Case 1: $d(x) = d(y) = 1$.

It is easy to check that there exist some s (≥ 0), t (≥ 0) such that $G \cong B_i^{s,t}$ for $i \in \{6, 7, 11\}$ since $\mathrm{Aut}(G) = <(x, y)>$.

If $G \cong B_6^{s,t}$ for $s \geq 1$, $t \geq 0$, $s + t \geq 3$, then $G \in \{B_6^{3,1}, B_6^{2,2}\}$.

If $G \cong B_7^{s,t}$ for $s \geq 1$, $t \geq 1$, $s + t \geq 3$, then $G \in \{B_7^{3,1}, B_7^{3,2}\}$. However, $B_7^{3,2}$ has 6 main eigenvalues with $|V(B_7^{3,2})| = 8$ by calculation, a contradiction. Thus $G \cong B_7^{3,1}$.

If $G \cong B_{11}^{s,t}$ for $s \geq 0, t \geq 0$, $s+t \geq 1$, then $G \in \{B_{11}^{2,0}, B_{11}^{2,1}, B_{11}^{0,2}, B_{11}^{1,2}\}$. However, both $B_{11}^{2,1}$ and $B_{11}^{1,2}$ have 6 main eigenvalues with $|V(B_{11}^{2,1})| = |V(B_{11}^{1,2})| = 8$ by calculation, a contradiction. Thus $G \in \{B_{11}^{2,0}, B_{11}^{0,2}\}$, where $B_{11}^{2,0} \cong B_7^{3,1}$.

Case 2: $d(x) = d(y) = 2$.

It is easy to check that there exist some s (≥ 0), t (≥ 0) such that $G \cong B_i^{s,t}$ for $i \in \{2, 4, 5, 8, 10\}$ since each of the other graphs either has no such pair of vertices x, y or satisfies $|\mathrm{Aut}(G)| > 2$.

If $G \cong B_2^{s,t}$ for $s \geq 0, t \geq 0$, $s + t \geq 1$, then $G \in \{B_2^{0,1}, B_2^{1,1}\}$.

Table 1
Number of connected almost controllable graphs G with $\operatorname{diam}(\overline{G}) > 2$.

Order	Tree	Unicyclic	Bicyclic	Tricyclic	Tetracyclic	Pentacyclic
5	1	2	2(1)	0	0	0
6	0	2	2	2	5(2)	3(2)
7	0	0	9	17	30(1)	38(3)
8	0	0	1	18	68	150
9	0	0	0	10	101	386
10	0	0	0	0	25	293
11	0	0	0	0	6	165
12	0	0	0	0	0	?

If $G \cong B_4^{s,t}$ for $s \geq 1, t \geq 1$, $s+t \geq 4$, then there is no such G since $|V(G)| \geq 6$ and $|\operatorname{Aut}(G)| = 2$.

If $G \cong B_5^{s,t}$ for $s \geq 2, t \geq 0$, $s+t \geq 3$, then $G \in \{B_5^{2,1}, B_5^{3,1}\}$.

If $G \cong B_8^{s,t}$ for $s \geq 2, t \geq 0$, then $G \in \{B_8^{2,1}, B_8^{3,1}\}$.

If $G \cong B_{10}^{s,t}$ for $s \geq 1, t \geq 0$, $s+t \geq 2$, then $G \cong B_{10}^{2,1}$.

Case 3: $d(x) = d(y) = 3$.

It is easy to check that there is no such G by $|V(G)| \geq 6$ and $|\operatorname{Aut}(G)| = 2$.

Combining the above arguments, we have $G \in \mathcal{B}$ if $\operatorname{diam}(\overline{G}) \neq 2$, and thus $\operatorname{diam}(\overline{G}) = 3$ if and only if $G \in \mathcal{B}$, then $\operatorname{diam}(\overline{G}) \in \{2, 3\}$. $\square$

By Lemmas 2.1, 4.2 and 4.4, we have the following result.

Theorem 4.5. *Let G be an almost controllable bicyclic graph of order n (≥ 6). Then* diam$(\overline{G}) \in \{2, 3\}$. *Moreover,* $\operatorname{diam}(\overline{G}) = 3$ *if and only if* $G \in \{B_{11}^{1,1}\} \cup \mathcal{B}$.

5. Some problems for further research

In this section, based on the results in Sections 3 and 4, two conjectures about the (almost) controllable graphs are proposed.

For (almost) controllable trees, unicyclic and bicyclic graphs, we note that the diameters of their complements are less than 4 by the discussion in Section 3 and Section 4. Naturally, we want to know if this result also holds for tricyclic, tetracyclic and pentacyclic (almost) controllable graphs. Then we calculate the number of tricyclic, tetracyclic and pentacyclic (almost) controllable graphs G of order less than 13 and $\operatorname{diam}(\overline{G}) > 2$ (see Table 1, Table 2), where the number $a(b)$ in Table 1 indicates that b of the a graphs have disconnected complements.

During the calculation, we find that there are only four almost controllable pentacyclic graphs G_1, G_2, G_3, G_4 and one controllable tetracyclic graph G_5 has connected complements with $\operatorname{diam}(\overline{G_i}) > 3$ (actually all equal to 4) for $i \in \{1, 2, 3, 4, 5\}$ (see Fig. 9).

From Table 1 and Table 2, we propose two conjectures as follows.

Conjecture 1. *Let $t \geq -1$ and G be an almost controllable connected graph of order n (≥ 5). If $|E(G)| = n + t$ and* $\operatorname{diam}(\overline{G}) > 2$, *then* $n \leq \lfloor \frac{3t+13}{2} \rfloor$.

Table 2
Number of connected controllable graphs G with $\text{diam}(\overline{G}) > 2$.

Order	Tree	Unicyclic	Bicyclic	Tricyclic	Tetracyclic	Pentacyclic
5	0	0	0	0	0	0
6	0	0	3	3	1	0
7	0	0	0	8	14	17
8	0	0	0	5	31	108
9	0	0	0	0	6	100
10	0	0	0	0	2	51
11	0	0	0	0	0	2
12	0	0	0	0	?	?

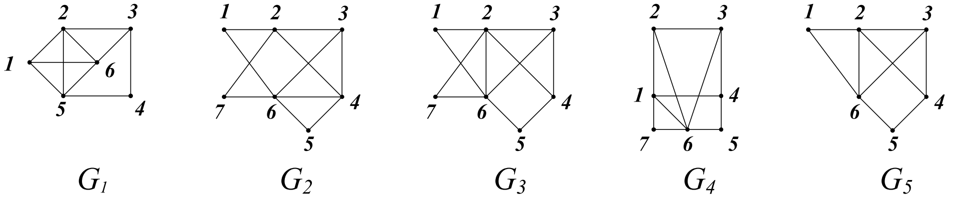

Fig. 9. Five (almost) controllable graphs G_i with $\text{diam}(\overline{G_i}) = 4$ $(i = 1, 2, 3, 4, 5)$.

Conjecture 2. *Let $t \geq -1$ and G be a controllable connected graph of order n (≥ 6). If $|E(G)| = n + t$ and $\text{diam}(\overline{G}) > 2$, then $n \leq 2t + 4$.*

Obviously, Conjecture 1 holds for $t \in \{-1, 0, 1\}$ by Theorem 3.6, Theorem 3.11 and Theorem 4.5. Conjecture 2 holds for $t \in \{-1, 0, 1\}$ by Corollary 3.5, Corollary 3.10 and Theorem 4.3.

6. Integral almost controllable graphs

A graph is called *integral* if its spectrum consists entirely of integers. In this section, all integral almost controllable graphs are determined.

The following lemmas are useful and interesting.

Lemma 6.1 *(([6]). The only integral controllable graph is graph K_1.*

Lemma 6.2 *([6]). Let G be a graph with n vertices. If G has n distinct integral eigenvalues, then $n \leq 10$.*

Lemma 6.3. *Let G be a graph of order n (≥ 2) with integral spectra. If G has $n - 1$ distinct eigenvalues, then $n \leq 13$.*

Proof. By G is an integral graph of order n with $n-1$ distinct eigenvalues, we know that G has $n - 2$ simple eigenvalues and an eigenvalue with multiplicity 2. Let $\text{Spec}(G) = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. Then we complete the proof in the following two cases.

Case 1: $n = 2t$.

It is evident that $\sum_{1\le i\le n} \lambda_i^2$ has the minimum value when $\mathrm{Spec}(G) = \{0^2, \pm 1, \cdots, \pm(t-1)\}$. Then we have $2 \le n \le 14$ by solving inequalities

$$2 \times \frac{n(n-1)}{2} \ge 2|E(G)| = \sum_{1\le i\le n} \lambda_i^2 \ge \frac{t(t-1)(2t-1)}{3}. \tag{1}$$

If $n = 14$, then $|E(G)| = 91$ by inequalities (1), and thus $G \cong K_{14}$. However, K_{14} has exactly two distinct eigenvalues, a contradiction. Hence $n \le 12$ when n is even.

Case 2: $n = 2t + 1$.

Similarly, $\sum_{1\le i\le n} \lambda_i^2$ has the minimum value when $\mathrm{Spec}(G) = \{0^2, \pm 1, \cdots, \pm(t-1), t\}$. Then we have $2 \le n \le 13$ by solving inequalities

$$2 \times \frac{n(n-1)}{2} \ge 2|E(G)| = \sum_{1\le i\le n} \lambda_i^2 \ge \frac{t(t-1)(2t-1)}{3} + t^2. \quad \square \tag{2}$$

Lemma 6.4. *There is no connected integral almost controllable graph of order* 13.

Proof. Let G be an integral almost controllable graph of order 13. Then we have $t = 6$ and $78 > |E(G)| \ge 73$ by inequalities (2) of Lemma 6.3. Thus G can only be obtained by K_{13} after deleting at most 5 edges e_i $(i = 1, 2, 3, 4, 5)$.

If e_1, e_2, e_3, e_4, e_5 have no common vertices in $\overline{G}$, then there will be 3 vertices of degree 12 in G, and thus each two of the 3 vertices is a pair of co-duplicate vertices, which contradicts with $|\mathrm{Aut}(G)| \in \{1, 2\}$.

If e_1, e_2, e_3, e_4, e_5 have common vertices in $\overline{G}$, or G is obtained by K_{13} after deleting less than 5 edges, then there are at least 4 vertices of degree 12 in G, and thus each two of these vertices of degree 12 is a pair of co-duplicate vertices, which contradicts with $|\mathrm{Aut}(G)| \in \{1, 2\}$. $\square$

Now we determine all integral almost controllable graphs.

Theorem 6.5. *Let G be an integral almost controllable graph of order n. Then $G \in \{P_2, 2K_1, P_2 \cup K_1\}$.*

Proof. We complete the proof by the following two cases.

Case 1: G is connected.

Since the main eigenvalues are different from each other, G has $n-1$ or n distinct eigenvalues by G is almost controllable.

If G has n distinct eigenvalues, then $n \le 10$ by Lemma 6.2. All connected integral graphs of order up to 10 have been listed in [1], and only P_2 is almost controllable with all distinct eigenvalues by calculation.

If G has $n-1$ distinct eigenvalues, then $n \le 13$ by Lemma 6.3. All connected integral graphs of order up to 12 have been listed in [1–3], and there is no almost controllable graph with $n-1$ distinct eigenvalues with $n \leq 12$ by calculation. Moreover, we have $n \neq 13$ by Lemma 6.4.

Thus in this case, P_2 is the unique connected almost controllable graph with integral spectra.

Case 2: G is disconnected.

Let $l \geq 2$, $G \cong H_1 \cup H_2 \cup \cdots \cup H_l$ where H_i is a connected integral graph for $i \in \{1, \cdots, l\}$. By the definition of main eigenvalues, it is easy to check that $|\text{MainSpec}(G)| \leq \sum_{i=1}^{l} |\text{MainSpec}(H_i)|$ where $\text{MainSpec}(G), \text{MainSpec}(H_i)$ denote the set of all main eigenvalues of G, H_i, respectively. Then there is at most one H_i $(1 \leq i \leq l)$ that is almost controllable and all others are controllable since G is almost controllable.

If all $H_1, H_2, \cdots, H_l$ are controllable, then $G \cong 2K_1$ by Lemma 6.1.

If there is exactly one H_i is almost controllable, then $G \cong P_2 \cup K_1$ by Lemma 6.1 and Case 1. □

Declaration of competing interest

The authors declare that they have no conflict of interest.

Data availability

No data was used for the research described in the article.

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