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The inner radius of univalency of certain types of convex quadrilaterals

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ABSTRACT

Calvis has showed that the inner radius of univalency is $2k^2$ for a normal circular triangle with the smallest interior angle $k\pi$, and $2[(n-2)/n]^2$ for a regular n-sided polygon. In this paper, using these results and ideas introduced by Calvis, we study the inner radius of univalency of certain types of convex quadrilateral. We calculate that the inner radius of univalency of a convex quadrilateral P with side sequences $aabb$ and interior angles $k\pi, 2k\pi, k\pi, 2\pi-4k\pi$. The inner radius of univalency of an isosceles trapezoid with side sequences $aaab$ and smallest interior angle $k\pi$ is also discussed.

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1. Introduction

Let D be a domain in the complex plane $\mathbb{C}$ with at least two boundary points and ρ_D be its *Poincaré density* whose curvature is -4. Let $\mathcal{H}$ be the family of simply connected domains in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ of hyperbolic type. For $D \in \mathcal{H}$, by Schwarz-Pick lemma, we define the Poincaré density $\rho_D(z)$ of D by

$$\rho_D(z) = \frac{|g'(z)|}{1-|g(z)|^2},$$

where $g(z)$ is any conformal map from D onto the unit disk $\Delta = \{z : |z| < 1\}$, and also denote the family of functions which are locally injective and meromorphic in D by $\mathcal{M}(D)$. For $f(z) \in \mathcal{M}(D)$, we define the *Schwarzian derivative* of $f(z)$ by

$$S_f(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

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and the norm of $S_f(z)$ by

$$\|S_f\|_D = \sup_{z \in D}\{|S_f(z)|/\rho_D(z)^2\}.$$

The *inner radius of univalency* of domain D by Schwarzian derivative is defined by

$$\sigma(D) = \sup\{a : \|S_f\|_D \leq a \Rightarrow f \text{ univalent in } D\}.$$

Nehari [15] has proved that if $f(z) \in \mathcal{M}(\Delta)$ with $\|S_f\|_\Delta \leq 2$, then f is univalent in Δ. Hill [7] has showed that the constant $C = 2$ is sharp. Moreover, whenever $\|S_f\|_\Delta < 2$, then not only f is univalent in Δ but also admits a quasiconformal extension in $\overline{\mathbb{C}}$ (see [2] and [5]).

Several properties concerning $S_f(z)$ are as follows.

(i) If T is a Möbius transformation, then $S_T \equiv 0$.
(ii) For $f, g \in \mathcal{M}(D)$, we have

$$S_{f\circ g} = S_{f(g)}(g')^2 + S_g.$$

Therefore we have $\sigma(D)$ is Möbius equivalent since $S_{f\circ T^{-1}} = S_f(T^{-1})$ and $\|S_{f\circ T^{-1}}\|_{T(D)} = \|S_f\|_D$.
(iii) For each $D \in \mathcal{H}$, $\sigma(D) \leq 2$. Moreover $\sigma(D) = 2$ if and only if $D = T(\Delta)$, where T is a Möbius transformation (see [9]).
(iv) For each $D \in \mathcal{H}$, then D is a quasidisk if and only if $\sigma(D) > 0$ (see [1] and [6]).

For more results related to Schwarzian derivative and pre-Schwarzian derivative (see [1], [4] and [8]).

For an angular domain $A_k = \{z : 0 < \arg z < k\pi\}$, Lehto [13] and Lehtinen [9] have shown that

$$\sigma(A_k) = \begin{cases} 2k^2, & 0 < k \leq 1, \\ 4k - 2k^2, & 1 < k < 2. \end{cases}$$

For distinct points $z_1, z_2 \in \mathbb{C}$, we let (z_1, z_2) denote the open line segment joining z_1, z_2 and $[z_1, z_2]$ its closure.

We first define P. For any $a, b > 0$, make a circle of radius b with O as center. Fix three points w_1, w_2, w_3 on the circle in clockwise direction such that the segments $[w_1, w_2]$, $[w_2, w_3]$ both have length a. Replace O as w_4, similarly, $[w_1, w_4]$ and $[w_3, w_4]$ both have length b. Consider the convex quadrilateral P with vertexes w_1, w_2, w_3, w_4 and side sequences *aabb*. Suppose that the interior angles of the convex quadrilateral P at vertexes w_1, w_2, w_3, w_4 are $k\pi, 2k\pi, k\pi, 2\pi - 4k\pi$ respectively. The convex quadrilateral P is well-defined, if $0 < 2\pi - 4k\pi < \pi$, i.e. $\frac{1}{4} < k < \frac{1}{2}$.

In this paper, we study the inner radius of univalency for the above convex quadrilateral P, and give the following results.

Theorem 1.1. *Suppose that P is a convex quadrilateral with side sequences aabb and interior angles $k\pi, 2k\pi, k\pi, 2\pi - 4k\pi$ for $\frac{1}{4} < k < \frac{1}{2}$. Then*

$$\sigma(P) = \begin{cases} 2k^2, & \frac{1}{4} < k \leq \frac{2}{5}, \\ 2(2-4k)^2, & \frac{2}{5} \leq k < \frac{1}{2}. \end{cases}$$

Corollary 1.2. *Suppose that P is a convex quadrilateral with side sequences aabb and interior angles $k\pi, 2k\pi, k\pi, 2\pi - 4k\pi$ for $\frac{1}{4} < k < \frac{1}{2}$. Assuming that its smallest angle is $\alpha\pi$ where $0 < \alpha \leq \frac{2}{5}$, then $\sigma(P) = 2\alpha^2$.*

2. Preliminaries

Define an *open* (resp. *closed*) *circular arc* γ as the image in $\overline{\mathbb{C}}$ of an open (resp. closed) line segment under a Möbius transformation. The *support circle* $C(\gamma)$ is the circle or line in $\overline{\mathbb{C}}$ which contains γ. A *circular triangle* is a Jordan domain $D \subset \overline{\mathbb{C}}$ whose boundary is a union of three closed circular arcs and is not a union of any two circular arcs. Generally, we say that a circular triangle D is *normal* if for each pair γ_1, γ_2 of distinct sides of D we have

$$C(\gamma_1) \cap C(\gamma_2) = \{v, v'\},$$

where v is a vertex of D and $v' \in \overline{\mathbb{C}} \setminus \overline{D}$.

Lehtinen [12] studied the inner radius of univalency for domains bounded by conic sections by using geometric methods, and then Calvis [3] calculated $\sigma(D)$ when D is a normal circular triangle or a regular n-sided polygon, and obtained the results as follows.

Lemma 2.1 *([3]). If D is a normal circular triangle whose smallest angle is $k\pi$, then*

$$\sigma(D) = 2k^2.$$

Theorem 2.2 *([3]). If D is a regular n-sided polygon, then*

$$\sigma(D) = 2\left(\frac{n-2}{n}\right)^2.$$

Wieren [14] also obtained the inner radius of univalency for a regular n-sided polygon P_n by constructing the Schwarz-Christoffel transformations mapping Δ onto P_n, and proved that if R is an open rectangle with $1 \le \frac{b}{a} \le 1.52346\cdots$, then R is a Nehari disk with $\sigma(R) = \sigma(P_4) = \frac{1}{2}$. Also if H is an equiangular hexagon with side sequence *baabaa* and $1 \le \frac{b}{a} \le 1.67117\cdots$, then H is a Nehari disk with $\sigma(H) = \sigma(P_6) = \frac{8}{9}$. Using these results, Shen complete Wieren's results for the case where H is a hexagon, Zhu obtained the inner radius of univalency for rhombus with smallest interior angle $k\pi$ where $0 < k < \frac{1}{2}$ and proved that all rhombus are Nehari disk (see [16] and [17]).

Lehtinen gave a brief calculation about $\sigma(D)$ when D is a regular n-sided polygon, and also obtained $\sigma(D^*)$ when D^* is the exterior of D (see [10]).

To achieve our result, we also need the following lemmas and propositions.

Lemma 2.3 *([3]). Let $G \in \mathcal{H}$ and $b > 0$. Suppose that for each pair of distinct points $z_1, z_2 \in G$ there exists a $G' \in \mathcal{H}$ with $z_1, z_2 \in \overline{G'}$, $G' \subset G$ and $\sigma(G') \ge b$. Then*

$$\sigma(G) \ge b.$$

Lemma 2.4 *([3]). Let $G_1, G_2 \in \mathcal{H}$, and suppose that for each pair of distinct points $z_1, z_2 \in G_2$ there exists a $T \in \text{Möb}$ with $z_1, z_2 \in \overline{T(G_1)}$ and $T(G_1) \subset G_2$. Then*

$$\sigma(G_2) \ge \sigma(G_1).$$

A direct corollary follows below.

Corollary 2.5 *([11]). Assume A has a boundary angle $k\pi$, $0 < k < 1$, at a boundary point z_0. Then $\sigma(A) \le 2k^2$.*

Recall that γ for a open circular arc and $\overline{\gamma}$ its closure. Denote the set of vertices of $D \in \mathcal{H}$ as $V(D)$. We have the following crucial lemma.

Lemma 2.6 *([3]). Let $v, v_1 \in V(D)$ be adjacent vertexes of D and let $v_2 \in \partial D \setminus \overline{\delta}$, where δ denotes the open edge (v_1, v). Then there exists an open circular arc γ joining v_1 and v_2 and tangent to δ at v_1 such that*

$$\gamma \subset D$$

and

$$C(\gamma) \cap \partial D = \{v_1, v_2\}.$$

Using the ideas introduced by Calvis, we also obtain a result with respect to isosceles trapezoid.

Theorem 2.7. *Let $a < b$. Suppose that P is an isosceles trapezoid with side sequences aaab and smallest interior angle $k\pi$ for $0 < k \leq \frac{1}{2}$, then $\sigma(P) = 2k^2$.*

Based on the above, we have the following propositions.

Proposition 2.8. *Suppose that P is a convex quadrilateral defined as in Theorem 1.1 for $\frac{1}{4} < k \leq \frac{2}{5}$. Then for arbitrary two points $z_1, z_2 \in \partial P$, P contains a domain $D \in \mathcal{H}$ with $z_1, z_2 \in \partial D$ and $\sigma(D) \geq 2k^2$.*

Proposition 2.9. *Suppose that P is a convex quadrilateral defined as in Theorem 1.1 for $\frac{2}{5} \leq k < \frac{1}{2}$. Then for arbitrary two points $z_1, z_2 \in \partial P$, P contains a domain $D \in \mathcal{H}$ with $z_1, z_2 \in \partial D$ and $\sigma(D) \geq 2(2-4k)^2$.*

3. Proof of Theorem 2.7

Proof. Suppose that P is defined as in Theorem 2.7. Since the smallest interior angle of P is $k\pi$, we have $\sigma(P) \leq 2k^2$ by Corollary 2.5. We now claim that for any $z_1, z_2 \in P$, there exists a domain $D \in \mathcal{H}$ with $D \subset P$ such that $z_1, z_2 \in \partial D$ and $\sigma(D) \geq 2k^2$. Therefore, by Lemmas 2.3 and 2.4 we have $\sigma(P) \geq 2k^2$. Thus, it is sufficient to consider the case when $z_1, z_2 \in \partial P$.

Consider the case when z_1, z_2 are on different sides of P, particularly on non-adjacent sides of P. If $z_1 \in (w_1, w_2)$ and $z_2 \in (w_3, w_4)$, take a circular arc $\tau_1 = \widehat{z_1 w_4}$ to be tangent to (w_1, w_4) at point w_4 and another circular arc $\tau_2 = \widehat{z_1 w_3}$ to be tangent to (w_2, w_3) at point w_3. Then $\overline{\tau_1} \cup [w_3, w_4] \cup \overline{\tau_2}$ bounds a circular triangle $D_1 \subset P$ with $z_1, z_2 \in \partial P$ [see Fig. 1 on the left]. Clearly the angles of D_1 at vertexes w_3, w_4 both are $k\pi$. Denote the angle between (z_1, w_4) and (w_1, w_4) to be $k_1\pi$, the angle between (z_1, w_3) and (w_2, w_3) to be $k_2\pi$ and the angle between (z_1, w_3) and (z_1, w_4) to be $k_3\pi$. Then the angle of D_1 at vertex z_1 is $(k_1 + k_2 + k_3)\pi$. We claim that $(k_1 + k_2 + k_3)\pi \geq k\pi$, that is $k_3\pi \geq \frac{\pi - k\pi}{2}$. Denote the circumcircle of P as circle O. Actually, the circumferential angle of chord (w_3, w_4) in circle O is $\pi - \frac{3k\pi}{2}$, thus $k_3\pi \geq \pi - \frac{3k\pi}{2}$. Since $0 < k \leq \frac{1}{2}$, we have $\frac{\pi - k\pi}{2} \leq \pi - \frac{3k\pi}{2}$. Therefore, D_1 is a normal circular triangle with smallest interior angle $k\pi$ and $\sigma(D_1) = 2k^2$ by Lemma 2.1.

If $z_1 \in (w_1, w_4)$ and $z_2 \in (w_2, w_3)$, similarly we take a circular arc $\tau_3 = \widehat{z_1 w_2}$ to be tangent to (w_1, w_2) at point w_2 and another circular arc $\tau_4 = \widehat{z_1 w_3}$ to be tangent to (w_3, w_4) at point w_3. Then $\overline{\tau_3} \cup [w_2, w_3] \cup \overline{\tau_4}$ bounds a circular triangle $D_2 \subset P$ with $z_1, z_2 \in \partial D_2$ [see Fig. 1 on the right]. It is easy to see that the angles of D_2 at vertexes w_2, w_3 are $\pi - k\pi, k\pi$ respectively. Consider the angle of D_2 at vertex z_1. Denote the angle between (w_1, w_2) and (z_1, w_2) to be $k_1\pi$, the angle between (w_3, w_4) and (z_1, w_3) to be $k_2\pi$ and the angle between (z_1, w_2) and (z_1, w_3) to be $k_3\pi$. Thus the angle of D_2 at vertex z_1 is $(k_1 + k_2 + k_3)\pi$. We claim that $(k_1 + k_2 + k_3)\pi \geq k\pi$, that is $k_3\pi \geq \frac{k\pi}{2}$. It can be obtained from the fact that the circumferential

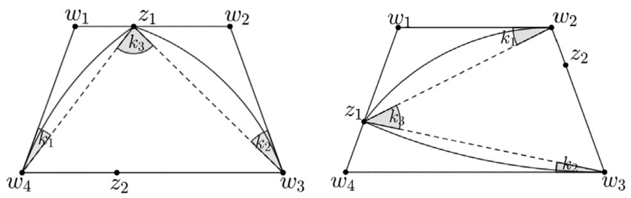

Fig. 1. z_1, z_2 on non-adjacent sides of P.

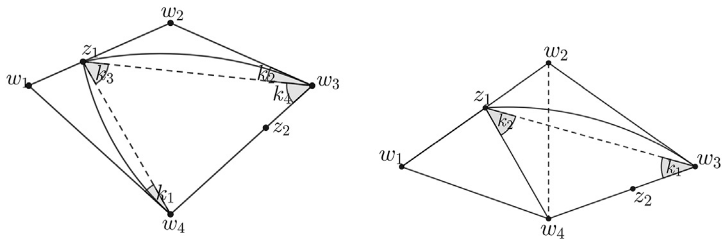

Fig. 2. $z_1 \in (w_1, w_2)$, $z_2 \in (w_3, w_4)$.

angle chord (w_2, w_3) in circle O is $\frac{k\pi}{2}$. Therefore it is easy to show that D_2 is normal with smallest interior angle $k\pi$. By Lemma 2.1, $\sigma(D_2) = 2k^2$.

The other cases that z_1 and z_2 lie on adjacent sides of P are obvious since we can always find a normal circular triangle $D \subset P$ or a domain D Möbius equivalent to an angular domain A_k such that $\sigma(D) \geq 2k^2$ and $z_1, z_2 \in \partial D$. We omit the construction of D here.

In conclusion, we complete the proof of Theorem 2.7. $\square$

4. Proof of Proposition 2.8

Using Lemmas 2.1 and 2.6, we prove Proposition 2.8 as below.

Proof. For $\frac{1}{4} < k \leq \frac{2}{5}$, the smallest interior angle of P is $k\pi$. For any $z_1, z_2 \in P$, it is sufficient to consider the case when $z_1, z_2 \in \partial P$. We consider separately the following three cases: z_1, z_2 are on open sides of P, z_1 or z_2 is a vertex of P and z_1, z_2 are both vertexes of P.

Case A. Firstly, we consider the case where z_1, z_2 are on open sides of P. We separate into the following three subcases: z_1, z_2 are on non-adjacent open sides of P, z_1, z_2 are on adjacent open sides of P, z_1, z_2 are on the same open side of P.

Case A1. z_1, z_2 are on non-adjacent open sides of P. By the symmetry of P, it is sufficient to consider $z_1 \in (w_1, w_2), z_2 \in (w_3, w_4)$. If $\frac{1}{3} < k \leq \frac{2}{5}$, we first take a circular arc $\gamma_1 = \widehat{z_1 w_3}$ to be tangent to (w_2, w_3) at point w_3 and a circular arc $\gamma_2 = \widehat{z_1 w_4}$ to be tangent to (w_1, w_4) at point w_4. We have $\overline{\gamma_1} \cup \overline{\gamma_2} \cup [w_3, w_4]$ bounds a domain $D_1 \subset P$ with $z_1, z_2 \in \partial D_1$ [see Fig. 2 on the left]. Clearly the angles of D_1 at w_3, w_4 are $k\pi, 2\pi - 4k\pi$ respectively. Next we consider the angle of D_1 at vertex z_1. Denote the angle between (z_1, w_4) and (w_1, w_4) to be $k_1\pi$, the angle between (z_1, w_3) and (w_2, w_3) to be $k_2\pi$ and the angle between (z_1, w_3) and (z_1, w_4) to be $k_3\pi$. Then the angle of D_1 at z_1 is $(k_1 + k_2 + k_3)\pi$. Since the length of (z_1, w_4) is always smaller than the length of (w_3, w_4). Denote the angle between (z_1, w_3) and (w_3, w_4) to be $k_4\pi$. According to the law of Sines, $k_3\pi > k_4\pi$ and $k_2\pi + k_3\pi > k\pi$. Thus we have the angle of D_1 at z_1 is greater than $k\pi$. Finally we have D_1 is a normal circular triangle with smallest interior angle $k\pi$. By Lemma 2.1, we have $\sigma(D_1) = 2k^2$.

If $\frac{1}{4} < k \leq \frac{1}{3}$, connect the segment (z_1, w_4). γ_1 is defined as above. Then $[z_1, w_4] \cup \overline{\gamma_1} \cup [w_3, w_4]$ bounds a circular triangle $D'_1 \subset P$ with $z_1, z_2 \in \partial D'_1$ [see Fig. 2 on the right]. The angles of D'_1 at vertexes w_3 and w_4 are not less than $k\pi$. Consider the angle of D'_1 at vertex z_1. Denote the angle between (z_1, w_3) and (w_3, w_4) to be $k_1\pi$ and the angle between (z_1, w_3) and (z_1, w_4) to be $k_2\pi$. Clearly $k_1\pi < k_2\pi$. Thus the angle of D'_1

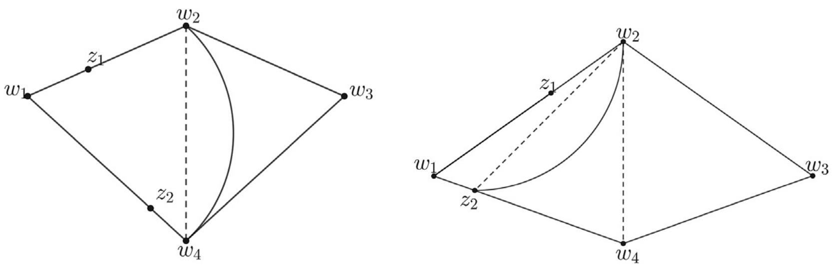

Fig. 3. $z_1 \in (w_1, w_2),\ z_2 \in (w_1, w_4)$.

at z_1 is greater than $k\pi$. We can see that D_1' is a normal circular triangle with smallest interior angle $k\pi$. By Lemma 2.1, we have $\sigma(D_1') \geq 2k^2$.

Case A2. z_1, z_2 are on adjacent open sides. We consider separately the following three cases: $z_1 \in (w_1, w_2)$ and $z_2 \in (w_2, w_3)$; $z_1 \in (w_1, w_2)$ and $z_2 \in (w_1, w_4)$; $z_1 \in (w_1, w_4)$ and $z_2 \in (w_3, w_4)$.

If $z_1 \in (w_1, w_2), z_2 \in (w_2, w_3)$, take a circular arc $\gamma_3 = \widehat{w_1 w_3}$ to be tangent to (w_1, w_4) and (w_3, w_4) at point w_1 and w_3 respectively. Then $[w_1, w_2] \cup \overline{\gamma_3} \cup [w_2, w_3]$ bounds a circular triangle $D_2 \subset P$ with $z_1, z_2 \in \partial D_2$. Clearly the angles of D_2 at vertexes w_1, w_2, w_3 are $k\pi, 2k\pi, k\pi$ respectively. Thus D_2 is a normal circular triangle with smallest interior angle $k\pi$ and $\sigma(D_2) = 2k^2$ by Lemma 2.1.

If $z_1 \in (w_1, w_2), z_2 \in (w_1, w_4)$, for $\frac{1}{3} < k \leq \frac{2}{5}$, take a circular arc $\gamma_4 = \widehat{w_2 w_4}$ to be tangent to (w_3, w_4) at point w_4. Then $[w_1, w_2] \cup \overline{\gamma_4} \cup [w_1, w_4]$ bounds a circular triangle $D_3 \subset P$ with $z_1, z_2 \in \partial D_3$ [see Fig. 3 on the left]. It is clear that the angle of D_3 at vertexes w_1, w_2, w_4 are $k\pi, \pi - k\pi, 2\pi - 4k\pi$ respectively. Therefore D_3 is normal with smallest interior angle $k\pi$ and $\sigma(D_3) = 2k^2$ by Lemma 2.1. For $\frac{1}{4} < k \leq \frac{1}{3}$, consider in the triangle $\Delta w_1 w_2 w_4$. Take a circular arc $\gamma_5 = \widehat{z_2 w_2}$ to be tangent to (w_2, w_4) at point w_2. Then $[z_2, w_1] \cup \overline{\gamma_5} \cup [w_1, w_2]$ bounds a domain $D_3' \subset P$ with $z_1, z_2 \in \partial D_3'$ [see Fig. 3 on the right]. It is easy to see that the angle of D_3' at vertex z_2 is greater than $k\pi$ and the angles of D_3' at vertexes w_1, w_2 are both $k\pi$. Thus D_3' is normal with smallest interior angle $k\pi$ and $\sigma(D_3') = 2k^2$ by Lemma 2.1.

If $z_1 \in (w_1, w_4), z_2 \in (w_3, w_4)$, the proof is similar to the case where $z_1 \in (w_1, w_2), z_2 \in (w_2, w_3)$, and so we omit it.

Case A3. z_1, z_2 are on the same open side of P. We consider the following two cases: $z_1, z_2 \in (w_1, w_2)$ and $z_1, z_2 \in (w_1, w_4)$. All above, we can find a domain $D_4 \subset P$, which is Möbius equivalent to an angular domain A_k, such that $z_1, z_2 \in \partial D_4$. Since the inner radius of univalency is Möbius equivalent, we have $\sigma(D_4) = \sigma(A_k) = 2k^2$.

Case B. Secondly, we consider the case when one of z_1 and z_2 is a vertex of P and the other lies on the open side of P. Similar to the proof of Case A, there always exists a domain $D_5 \subset P$ with $z_1, z_2 \in \partial P$, which is a normal circular triangle with smallest interior angle not less than $k\pi$ or Möbius equivalent to an angular domain A_{k_1} where $k \leq k_1 < \frac{1}{2}$. As above, we have $\sigma(D_5) \geq 2k^2$.

Case C. Finally, we consider the case when z_1, z_2 are both vertexes of P. It is sufficient to consider the case where z_1, z_2 lie on different sides of P. We separate into the following two subcases: $z_1 = w_1, z_2 = w_3$ and $z_1 = w_2, z_2 = w_4$.

If $z_1 = w_1, z_2 = w_3$, we can join z_1 and z_2 by a pair of circular arcs such that the intersection points are w_1, w_3 and the angles at w_1, w_3 are both $k\pi$. Take a circular arc γ_6 connecting z_1 and z_2 to be tangent to (w_1, w_2) and (w_2, w_3) at w_1 and w_3 respectively. Also, take a circular arc γ_7 connecting z_1 and z_2 to be tangent to (w_1, w_4) and (w_3, w_4) at w_1 and w_3 respectively. Then $\overline{\gamma_6} \cup \overline{\gamma_7}$ bounds a domain $D_6 \subset P$ with $z_1, z_2 \in \partial D_6$. It is clear that D_6 is Möbius equivalent to an angular domain A_k, therefore $\sigma(D_6) = 2k^2$.

If $z_1 = w_2, z_2 = w_4$, for $\frac{1}{3} < k \leq \frac{2}{5}$, take circular arcs γ_8 and γ_9 connecting w_2 and w_4 to be tangent to (w_1, w_4) and (w_3, w_4) at point w_4 respectively. Then $\overline{\gamma_8} \cup \overline{\gamma_9}$ bounds a domain $D_7 \subset P$ with $z_1, z_2 \in \partial D_7$ [see Fig. 4 on the left]. In addition, D_7 is Möbius equivalent to an angular domain A_{2-4k}, thus $\sigma(D_7) = 2(2 - 4k)^2 \geq 2k^2$. For $\frac{1}{4} < k \leq \frac{1}{3}$, we can similarly find a domain $D_7' \subset P$ with $z_1, z_2 \in \partial D_7'$ and D_7' is

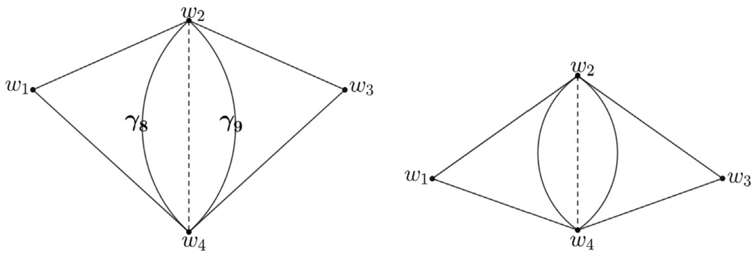

Fig. 4. $z_1 = w_2, z_2 = w_4$.

Möbius equivalent to an angular domain A_{2k} [see Fig. 4 on the right]. Thus we have $\sigma(D_7) = 2(2k)^2 > 2k^2$.

In conclusion, according to Case $A - C$ and Lemma 2.1, we prove that for $z_1, z_2 \in \partial P$, there always exist a normal circular triangle $D \subset P$ with smallest interior angle not less than $k\pi$ or a domain $D \subset P$ Möbius equivalent to an angular domain A_{k_1} $(k \le k_1 < \frac{1}{2})$ such that $z_1, z_2 \in \partial D$ and $\sigma(D) \ge 2k^2$. $\square$

5. Proof of Proposition 2.9

Similar to the proof of Proposition 2.8, we give a brief proof of Proposition 2.9 in the following.

Proof. If $\frac{2}{5} \le k < \frac{1}{2}$, the smallest interior angle of P is $2\pi - 4k\pi$. Similar to the proof of Proposition 2.8, for any $z_1, z_2 \in \partial P$, we consider separately the following three cases: z_1, z_2 are on different open sides of P, z_1, z_2 are on the same open side of P and at least one of z_1 and z_2 is a vertex of P.

Case 1. z_1, z_2 are on different open sides of P. Firstly, we consider the case where z_1, z_2 are on non-adjacent open sides. As in the proceeding section, it is sufficient to consider the case where $z_1 \in (w_1, w_2)$ and $z_2 \in (w_3, w_4)$. Similar to Lemma 2.6, we take a circular arc $\gamma_1 = \widehat{z_1 w_3}$ to be tangent to (w_2, w_3) at point w_3 and a circular arc $\gamma_2 = \widehat{z_1 w_4}$ to be tangent to (w_1, w_4) at point w_4. Then $\overline{\gamma_1} \cup [w_3, w_4] \cup \overline{\gamma_2}$ bounds a circular triangle $D_1 \subset P$ with $z_1, z_2 \in \partial D_1$. Clearly the angles of D_1 at vertexes w_3, w_4 are $k\pi, 2\pi - 4k\pi$ respectively. According to the Law of Sines, the angle of D_1 at vertex z_1 is greater than $k\pi$. Consequently, we have D_1 is a normal circular triangle with smallest interior angle $2\pi - 4k\pi$, thus $\sigma(D_1) = 2(2 - 4k)^2$ by Lemma 2.1.

Secondly, we consider the case where z_1, z_2 are on adjacent open sides of P. We separate into three subcases: $z_1 \in (w_1, w_2)$ and $z_2 \in (w_2, w_3)$; $z_1 \in (w_1, w_2)$ and $z_2 \in (w_1, w_4)$; $z_1 \in (w_1, w_4)$ and $z_2 \in (w_3, w_4)$.

If $z_1 \in (w_1, w_2)$ and $z_2 \in (w_2, w_3)$, take a circular arc $\gamma_3 = \widehat{w_1 w_3}$ to be tangent to (w_1, w_4) and (w_3, w_4) at points w_1 and w_3 respectively. Then it is easy to know that $[w_1, w_2] \cup \overline{\gamma_3} \cup [w_2, w_3]$ bounds a normal circular triangle D_2 with smallest interior angle $k\pi$. Thus $\sigma(D_2) = 2k^2 \ge 2(2 - 4k)^2$ by Lemma 2.1. The other two subcases are similar as above, so we omit them.

Case 2. z_1, z_2 are on the same open side of P. As in the proof of Proposition 2.8, we can always find a domain $D_3 \subset P$ with $z_1, z_2 \in \partial D_3$, which is Möbius equivalent to an angular domain A_{k_1}, where $2 - 4k \le k_1 < \frac{1}{2}$. According to the Möbius equivalent of the inner radius of univalency, we have $\sigma(D_3) = \sigma(A_{k_1}) = 2k_1^2 \ge 2(2 - 4k)^2$.

Case 3. At least one of z_1, z_2 is a vertex of P. Otherwise, it can reduce to the case where z_1, z_2 are on the same open side of P or on adjacent open sides of P. As in Case 1 and Case 2, there exist a normal circular triangle $D_4 \subset P$ with smaller interior angle not less than $2\pi - 4k\pi$ or a domain $D_4 \subset P$ Möbius equivalent to an angular domain A_{k_1} $(2 - 4k \le k_1 < \frac{1}{2})$ such that $z_1, z_2 \in \partial P$ and $\sigma(D_4) \ge 2(2 - 4k)^2$.

According to Cases $1 - 3$, we show that for any $z_1, z_2 \in \partial P$, there always exists a domain $D \subset P$ with $z_1, z_2 \in \partial D$ such that $\sigma(D) \ge 2(2 - 4k)^2$. $\square$

6. Proof of Theorem 1.1

Proof. For $\frac{1}{4} < k \leq \frac{2}{5}$, P has smallest interior angle $k\pi$. By Corollary 2.5, we have $\sigma(P) \leq 2k^2$. By Proposition 2.8, Lemma 2.3 and Lemma 2.4, we have $\sigma(P) \geq 2k^2$. Therefore $\sigma(P) = 2k^2$.

For $\frac{2}{5} \leq k < \frac{1}{2}$, P has smallest interior angle $2\pi - 4k\pi$. By Corollary 2.5, we have $\sigma(P) \leq 2(2-4k)^2$. By Proposition 2.9, Lemma 2.3 and Lemma 2.4, we have $\sigma(P) \geq 2(2-4k)^2$. Therefore $\sigma(P) = 2(2-4k)^2$. Consequently, we show that

$$\sigma(P) = \begin{cases} 2k^2, & \frac{1}{4} < k \leq \frac{2}{5}, \\ 2(2-4k)^2, & \frac{2}{5} \leq k < \frac{1}{2}. \end{cases}$$

Corollary 1.2 follows directly from Theorem 1.1. $\square$

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