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GROWTH OF SOLUTIONS OF SOME SECOND ORDER DIFFERENTIAL EQUATIONS WITH ENTIRE COEFFICIENTS

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Abstract. In this paper, we consider the differential equation

$$f'' + Af' + Bf = 0, \tag{$*$}$$

where $A(z)$ and $B(z) \not\equiv 0$ are entire functions. Assume that $A(z)$ is a transcendental solution of $\omega'' + P(z)\omega = 0$, where $P(z)$ is a polynomial. If $B(z)$ satisfies extremal for Yang's inequality and other conditions, then every transcendental solution f of equation (*) has $\mu(f) = \infty$. We also investigate the relation between a small function and a differential polynomial of f.

1. Introduction and main results

We assume that the reader is familiar with fundamental results and standard notations of the Nevanlinna value distribution theory of meromorphic functions (see [7, 21, 22]). In this paper, we use $\rho(f)$ to denote the order of an entire function $f(z)$, $\lambda(f)$ (resp. $\overline{\lambda}(f)$), to denote the exponent of convergence of zeros (resp. of distinct zeros) of $f(z)$, and $\mu(f)$, $\rho_2(f)$ to denote the lower order and hyper-order of $f(z)$(see [22]), respectively. Moreover, we use the standard notations $S(\alpha, \beta) := \{z : \alpha < \arg z < \beta\}$ and $S(\alpha, \beta; r) := S(\alpha, \beta) \cap \{z : |z| < r\}$ frequently in what follows.

In this paper, we are treating second order linear differential equations of type

$$f'' + A(z)f' + B(z)f = 0, \tag{1}$$

where $A(z)$ and $B(z)(\not\equiv 0)$ are entire functions. It is well known that every solution of (1) is an entire function. Every nonconstant solution of (1) is of infinite order, whenever either $A(z)$ and $B(z)$ are entire functions with $\rho(A) <$

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$\rho(B)$, or $A(z)$ is a polynomial and $B(z)$ is transcendental, or $\rho(B) < \rho(A) \leq \frac{1}{2}$, see e.g., [5, 8, 14].

Recently, a number of papers appear to proving that, under some conditions upon $B(z)$, every transcendental solution to (1) is of infinite order, whenever the coefficient $A(z)$ in (1) is a nonconstant solution to equation

$$w'' + P(z)w = 0, \tag{2}$$

where $P(z) = a_m z^m + \cdots + a_0$ is a polynomial of degree $m \geq 1$, see e.g., [12, 13, 18, 19, 24]. It is well-known that all nonconstant solutions to (2) are of order $(m+2)/2$. We first recall a result of this type, see e.g., [18]:

Theorem A. *Let $A(z)$ be a nonconstant solution of* (2)*, and let $B(z)$ be a transcendental entire function with $\rho(B) < 1/2$. Then every transcendental solution of* (1) *is of infinite order.*

As for another result of this type, we first recall the notion of an *accumulation ray for the zero sequence* of a meromorphic function $f(z)$, see e.g., [13, 16, 17]: Let $\gamma = re^{i\theta}$ be a ray from origin. For each $\varepsilon > 0$, the exponent of convergence of the zero sequence of $f(z)$ at the ray $\gamma = re^{i\theta}$ is denoted by $\lambda_\theta(f) = \lim_{\varepsilon \to 0^+} \lambda_{\theta,\varepsilon}(f)$, where

$$\lambda_{\theta,\varepsilon}(f) = \varlimsup_{r\to\infty} \frac{\log^+ n(S(\theta-\varepsilon, \theta+\varepsilon; r), 1/f)}{\log r}.$$

Here, $n(S(\theta-\varepsilon, \theta+\varepsilon; r), 1/f)$ counts the number of zeros of $f(z)$ with multiplicities in the angular sector $S(\theta-\varepsilon, \theta+\varepsilon; r)$. The ray $\gamma = re^{i\theta}$ is now called an accumulation ray of the zero sequence of $f(z)$ if $\lambda_\theta(f) = \rho(f)$.

We now recall another result of this type, see e.g., [13]:

Theorem B. *Suppose that $A(z)$ and $B(z)$ are two linearly independent solutions of* (2)*. If the number of accumulation rays of the zero sequence of $A(z)$ is less than $m+2$, then every transcendental solution of* (1) *is of infinite order.*

A natural related question is now to find conditions that ensure every transcendental solution to (1) be of infinite order, whenever the number of accumulation rays of the zero sequence of solutions to (2) equals to $m+2$. Indeed, it follows from Lemma 2.1 below that the number of accumulation rays of the zero sequence of every nonconstant solution of (2) is $\leq m+2$, and the set of the accumulation rays of the zero sequence of every nonconstant solution of (2) is a subset of $\{\theta_j : 0 \leq j \leq m+1\}$, where $\theta_j = \frac{2j\pi - \arg(a_m)}{m+2}$, $j = 0, 1, \ldots, m+1$.

For the convenience of the reader, we next recall the notion of Borel direction for meromorphic functions of finite order, see, e.g., [23]: A ray $\arg z = \theta$ from the origin is called a *Borel direction of order $\rho(f)$*, if for any $\varepsilon > 0$ and for any complex value $a \in \mathbb{C} \cup \{\infty\}$ with at most two exceptions, we have

$$\varlimsup_{r\to\infty} \frac{\log n(S(\theta-\varepsilon, \theta+\varepsilon; r), a, f)}{\log r} = \rho(f).$$

Let now an entire function $f(z)$ be of finite order $\rho(f)$. Suppose that $f(z)$ has p distinct Borel directions and q is the number of its finite deficient values. Then it is well-known that $2q \leq p$, see [20]. This inequality is called as the *Yang's inequality*. An entire function f is called *extremal for Yang's inequality*, if f satisfies $2q = p$.

We are now ready to state our main results as follows:

Theorem 1.1. *Suppose $A(z)$ is a nonconstant solution to equation* (2) *such that the number of accumulation rays of the zero sequence of $A(z)$ equals to $m+2$ and that an entire function $B(z)$ of finite order satisfies one of the following conditions*:

(1) *$B(z)$ is extremal for Yang's inequality*;

(2) *$B(z)$ is transcendental with a finite deficient value.*

Then every transcendental solution f of (1) *satisfies $\mu(f) = \infty$ and $\rho_2(f) = \max\{\rho(A), \rho(B)\}$.*

Moreover, let $d_j(z)$ $(j = 0, 1, 2)$ be three polynomials that are not all vanishing and let $\varphi(z)$ $(\not\equiv 0)$ be a meromorphic function of finite order. If $\rho(A) \neq \rho(B)$, then the differential polynomial $g_f = d_2 f'' + d_1 f' + d_0 f$ satisfies $\overline{\lambda}(g_f - \varphi) = \infty$.

Theorem 1.2. *Suppose that $A(z)$ and $\varphi(z)$ satisfy the same conditions as in Theorem 1.1. Let $B(z)$ be an entire function of finite order that has a finite Borel exceptional value. Then every transcendental solution f to* (1) *satisfies $\mu(f) = \infty$.*

Moreover, let $d_j(z)$ $(j = 0, 1, 2)$ be three polynomials that are not all vanishing. Then $\overline{\lambda}(g_f - \varphi) = \infty$, provided $\rho(A) \neq \rho(B)$.

2. Preliminary lemmas

In this section, we collect some lemmas that are used in proving our theorems.

Definition. Suppose that $Q(z) = b_n z^n + b_{n-1} z^{n-1} + \cdots + b_0$, where $b_n \neq 0$ and $\delta(Q, \theta) = \mathrm{Re}(b_n e^{in\theta})$. A ray $\arg z = \theta$ is called a critical ray of $e^{Q(z)}$, if $\delta(Q, \theta) = 0$. Moreover, we fix the following notations:

$$E^+ := \{\theta \in [0, 2\pi] : \delta(Q, \theta) \geq 0\};$$
$$E^- := \{\theta \in [0, 2\pi] : \delta(Q, \theta) \leq 0\}.$$

Critical rays of $e^{Q(z)}$ divide the whole complex plane into $2n$ sectors of equal opening π/n. Suppose that ϕ_i and ψ_i $(1 \leq i \leq n)$ are critical rays of $e^{Q(z)}$ such that $0 \leq \phi_1 < \psi_1 < \phi_2 < \psi_2 < \cdots < \phi_n < \psi_n$ and $\phi_{n+1} = 2\pi + \phi_1$. These critical rays form $2n$ disjoint sectors $S(\phi_i, \psi_i)$ and $S(\psi_i, \phi_{i+1})$; $1 \leq i \leq n$ in which $e^{Q(z)}$ satisfies $\delta(Q, \theta) > 0$ and $\delta(Q, \theta) < 0$, respectively.

Lemma 2.1 ([1], Lemma 3)**.** *Let $B(z) = d(z)e^{Q(z)}$ be an entire function, where $Q(z)$ is a polynomial of degree $n \geq 1$, and $d(z)$ is an analytic function such that*

$\rho(d) < \rho(B) = \deg Q(z)$. Then, for given $\varepsilon > 0$, there exists a set $E \subset [0, 2\pi)$ with linear measure zero, such that

(1) *if $\theta \in E^+ \setminus E$, there exists a $R(\theta) > 1$ such that*

$$|B(re^{i\theta})| \geq \exp((1-\varepsilon)\delta(Q,\theta)r^n)$$

holds for all $r > R(\theta)$ and

(2) *if $\theta \in E^- \setminus E$, there exists a $R(\theta) > 1$ such that*

$$|B(re^{i\theta})| \leq \exp((1-\varepsilon)\delta(Q,\theta)r^n)$$

holds for all $r > R(\theta)$.

Remark 2.2. Observe that

$$E^+ = \bigcup_{i=1}^{i=n} (\phi_i, \psi_i), \quad E^- = \bigcup_{i=1}^{i=n} (\psi_i, \phi_{i+1}).$$

For simplicity, we say that $f(z)$ blows up to infinity (exponentially) in $S(\alpha, \beta)$ if for any $\theta, \alpha < \theta < \beta$,

$$\lim_{r\to\infty} \frac{\log\log |f(re^{i\theta})|}{\log r} = \rho(f)$$

holds and we say that $f(z)$ decays to zero (exponentially) in $S(\alpha, \beta)$ if for any $\theta, \alpha < \theta < \beta$,

$$\lim_{r\to\infty} \frac{\log\log |f(re^{i\theta})|^{-1}}{\log r} = \rho(f)$$

holds.

The following lemma due to Hille, see [9, Section 7.4], plays an important role in what follows:

Lemma 2.3 ([9])**.** *Let $w(z)$ be a solution to (2). Set $\theta_j = \frac{2\pi j - \arg(a_m)}{m+2}; 0 \leq j \leq m+1$. Then $w(z)$ has the following properties in sectors $S_j := S(\theta_j, \theta_{j+1}), j = 0, \dots, m+1$, where $S_{m+1} = S_0$:*

1. *In each sector S_j, $w(z)$ either blows up to infinity or decays to zero (exponentially).*
2. *If, for some j, $w(z)$ decays to zero in S_j, then it must blow up in S_{j-1} and S_{j+1}. However, it is possible for $w(z)$ to blow up in several adjacent sectors;*
3. *If $w(z)$ decays to zero in S_j, then $w(z)$ has at most finitely many zeros in any closed sub-sector within $S_{j-1} \cup \overline{S_j} \cup S_{j+1}$;*
4. *If $w(z)$ blows up in S_{j-1} and S_j, then for each $\varepsilon > 0$, $w(z)$ has infinitely many zeros in each sector $\theta_j - \varepsilon \leq \arg z \leq \theta_j + \varepsilon$.*

Remark 2.4. By Lemma 2.3, it follows that $w(z)$ blows up exponentially in every sector S_j, if the number of accumulation rays of the zero sequence of $w(z)$ is exactly $m+2$, see [15, Lemma 7].

Lemma 2.5 ([10, Lemma 3]). *Let $f(z)$ be a non-constant entire function. Then there exists a real number $R > 0$ such that for all $r \geq R$ we have*

$$\left| \frac{f(z)}{f'(z)} \right| \leq r$$

where $|z| = r$.

Lemma 2.6 ([3, Lemma 5]). *Let $f(z)$ be an entire function with $\rho(f) = \rho < \infty$. Suppose there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, such that for any ray $\arg z = \theta_0 \in [0, 2\pi) \setminus E$, $|f(re^{i\theta_0})| \leq Mr^k$, where $M = M(\theta_0) > 0$ is a constant and $k (> 0)$ is a constant independent of θ_0. Then $f(z)$ is a polynomial with $\deg f \leq k$.*

Lemma 2.7 ([4, Lemma 1]). *Suppose that $f(z)$ is a meromorphic function of finite order ρ. Then for a given $\delta > 0$ and $0 < l < 1/2$, there exists a constant $\kappa(\rho, \delta)$ and a set $E_\delta \subset [0, \infty)$ of lower logarithmic density greater than $1 - \delta$ such that for all $r \in E_\delta$ and for every interval I of length l, we have*

$$r \int_I \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| d\theta < \kappa(\rho, \delta) \left(l \log \frac{1}{l} \right) T(r, f).$$

The following two lemmas provide an upper and lower bound for hyper-order of growth for transcendental solutions f of (1).

Lemma 2.8 ([11, Theorem 7.3]). *Let A, B be entire functions of finite order. If $f(z)$ is a solution of (1), then $\rho_2(f) \leq \max\{\rho(A), \rho(B)\}$.*

Lemma 2.9 ([10, Theorem 1]). *Let $A(z)$ and $B(z)$ be entire functions such that $\rho(A) < \rho(B)$ or $\rho(B) < \rho(A) < \frac{1}{2}$. Then every solution $f \not\equiv 0$ of (1) satisfies $\rho_2(f) \geq \max\{\rho(A), \rho(B)\}$.*

Lemma 2.10 ([2, Lemma 4]). *Let $A_0, \ldots, A_{k-1}, F \not\equiv 0$ be meromorphic functions of finite order. If f is a meromorphic solution of infinite order to*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_0 f = F,$$

then f satisfies $\lambda(f) = \overline{\lambda}(f) = \rho(f) = \infty$.

3. Proof of Theorem 1.1

Proof. We start this section by proving first that $\mu(f) = \rho(f) = \infty$.

(a) Suppose that $B(z)$ is an entire function extremal for Yang's inequality. Let $b_1, b_2, \ldots, b_q$ be the finite deficient values of $B(z)$. Then, $B(z)$ has $2q$ Borel directions, say $\phi_1, \phi_2, \ldots, \phi_{2q}$, dividing the complex plane into $2q$ sectors, say $\Omega_j(\phi_j, \phi_{j+1})$, where $1 \leq j \leq 2q$ and $\phi_{2q+1} = \phi_1 + 2\pi$. As $B(z)$ is extremal for Yang's inequality, so for the alternative sectors, say $\Omega_1, \Omega_3, \ldots, \Omega_{2q-1}$, there exists $\phi \in (\phi_j, \phi_{j+1})$; $j = 1, 3, \ldots, 2q - 1$, such that $B(z)$ satisfies

$$\overline{\lim_{r \to \infty}} \frac{\log \log |B(re^{i\phi})|}{\log r} = \rho(B).$$

For the remaining sectors Ω_j, and for every deficient value b_j, where $j = 1, 2, \dots, q$, there exists a corresponding sectorial domain $\Omega_j, j \in \{2, 4, \dots, 2q\}$ such that

$$\log \frac{1}{|B(z) - b_j|} > CT(r, B)$$

holds for all $|z|$ sufficiently large so that $z \in \Omega(\phi_j + \varepsilon, \phi_{j+1} - \varepsilon)$, where C is a constant depending on $\phi_j, \phi_{j+1}, \varepsilon$ and $\delta(b_j, B)$. Without loss of generality, corresponding to a finite deficient value b_{j_0}, we can take a sector Ω_{2i}; $1 \leq i \leq q$ such that

$$\log \frac{1}{|B(z) - b_{j_0}|} > CT(r, B) \tag{3}$$

holds for $z \in \Omega(\phi_{2i} + \varepsilon, \phi_{2i+1} - \varepsilon)$, whenever $|z|$ is sufficiently large.

Using Lemma 2.3, $A(z)$ blows up exponentially in each sector S_j; $j = 0, 1, \dots, m+1$. Therefore, there exists a sector $S_k(\theta_k, \theta_{k+1})$ such that $A(z)$ blows up exponentially for any $\theta \in (\theta_k, \theta_{k+1}) \cap (\phi_{2i} + \varepsilon, \phi_{2i+1} - \varepsilon)$ for some $1 \leq i \leq q$ and we have

$$\lim_{r \to \infty} \frac{\log \log |A(z)|}{\log r} = \rho(A) \tag{4}$$

for all sufficiently large r.

By [6, Theorem 2], there exists a set $F \subset [0, 2\pi)$ with $m(F) = 0$ such that if $\theta_0 \in [0, 2\pi) \setminus F$, then there is a constant $R_1 = R_1(\theta_0) > 1$, we have

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq M \left(T(2r, f) \right)^{2(k-j)}, (0 \leq j < k \leq 2), \tag{5}$$

for all z satisfying $\arg z = \theta_0$ and $|z| \geq R_1$, where M is a constant.

Combining (1), (3), (4), (5) and Lemma 2.5, there exists a sequence $z = re^{i\theta}$ such that for $\theta \in (\theta_k, \theta_{k+1}) \cap (\phi_{2i} + \varepsilon, \phi_{2i+1} - \varepsilon) \setminus F$ for some $1 \leq i \leq q$ and $r > \max\{R, R_1, r_0\}$, we have

$$\begin{aligned} |A(z)| &\leq \left| \frac{f''(z)}{f'(z)} \right| + |(B(z) - b_{j_0}) + b_{j_0}| \left| \frac{f(z)}{f'(z)} \right|, \\ \exp\{r^{\rho(A) - \varepsilon'}\} &\leq M \left(T(2r, f) \right)^2 + r \left(\exp\{-CT(r, B)\} + |b_{j_0}| \right) \\ &\leq M(T(2r, f))^2(1 + o(1)). \end{aligned} \tag{6}$$

Therefore, we obtain $\mu(f) = \infty$.

We now prove that

$$\rho_2(f) = \max\{\rho(A), \rho(B)\}.$$

From Lemma 2.8, we obtain $\rho_2(f) \leq \max\{\rho(A), \rho(B)\}$. Thus, we just need to prove that $\rho_2(f) \geq \max\{\rho(A), \rho(B)\}$.

(i) If $\rho(A) < \rho(B)$, then we have $\rho_2(f) \geq \max\{\rho(A), \rho(B)\}$ by Lemma 2.9.

(ii) If $\rho(B) \leq \rho(A)$, we get by (6) that

$$\rho(A) - \varepsilon' \leq \overline{\lim_{r\to\infty}} \frac{\log\log T(r,f)}{\log r},$$

where $\varepsilon' > 0$ is arbitrary. This means that $\rho_2(f) \geq \rho(A) = \max\{\rho(A), \rho(B)\}$ and so $\rho_2(f) = \max\{\rho(A), \rho(B)\}$.

(b) Suppose next that $c \in \mathbb{C}$ is a finite deficient value of $B(z)$. Then

$$\underline{\lim}_{r\to\infty} \frac{m\left(r, \frac{1}{B(z)-c}\right)}{T(r,B)} = \alpha > 0,$$

which gives

$$m\left(r, \frac{1}{B(z)-c}\right) \geq \alpha T(r,B)$$

for all sufficiently large r. Thus, for r sufficiently large, there exists $z_r = re^{i\theta_r}$ such that

$$\log|B(z_r) - c| \leq -\alpha T(r,B).$$

From Lemma 2.7, we may choose $\delta > 0$ and $0 < l < \frac{1}{2}$ in such a way that $\kappa(\rho(B),\delta)(l\log(1/l))$ is sufficiently small. We can also choose $\phi > 0$, $|\theta_r - \phi| \leq l$ such that

$$\begin{aligned}\log|B(re^{i\theta}) - c| &= \log|B(re^{i\theta_r}) - c| + \int_{\theta_r}^{\theta} \frac{d}{dt}\log|B(re^{it}) - c|dt \\ &\leq -\alpha T(r,B) + r\int_{\theta_r}^{\theta} \left|\frac{(B-c)'(re^{it})}{(B-c)(re^{it})}\right| dt \\ &\leq -\alpha T(r,B) + \kappa(\rho(B),\delta)(l\log(1/l))T(r,B) \leq 0\end{aligned}$$

holds for all $\theta \in [\theta_r - \phi, \theta_r + \phi]$ and for all sufficiently large $r \in E_\delta$, where $\underline{\log dens}(E_\delta) > 1 - \delta$. Thus we have

$$B(re^{i\theta}) \leq 1 + c \tag{7}$$

for all sufficiently large $r \in E_\delta$ and for all $\theta \in [\theta_r - \phi, \theta_r + \phi]$.

Using Lemma 2.3, there exists a sector $S_k(\theta_k, \theta_{k+1})$ such that $A(z)$ blows up exponentially for any $\theta \in [\theta_r - \phi, \theta_r + \phi] \cap (\theta_k, \theta_{k+1})$ and we have

$$\lim_{r\to\infty} \frac{\log\log|A(z)|}{\log r} = \rho(A). \tag{8}$$

Combining (1), (5), (7), (8) and Lemma 2.5, there exists a sequence $z = re^{i\theta}$ such that for $\theta \in [\theta_r - \phi, \theta_r + \phi] \cap (\theta_k, \theta_{k+1}) \setminus F$, we have

$$\begin{aligned}|A(re^{i\theta})| &\leq \left|\frac{f''(z)}{f'(z)}\right| + |B(z)|\left|\frac{f(z)}{f'(z)}\right|, \\ \exp\{r^{\rho(A)-\varepsilon'}\} &\leq M\left(T(2r,f)\right)^2 + r(1+c) < M(T(2r,f))^2(1+o(1))\end{aligned}$$

for all r sufficiently large and $r \in E_\delta$. This now implies that $\mu(f) = \infty$.

(c) To proceed, we now define $w := g_f - \varphi$. Substituting $f'' = -Af' - Bf$ into w, we have

$$w = (d_1 - d_2A)f' + (d_0 - d_2B)f - \varphi. \tag{9}$$

Differentiating both sides of (9), and replacing f'' with $f'' = -Af' - Bf$, we get

$$\begin{aligned} w' = &\left[d_2A^2 - d_1A - d_2'A - d_2A' - d_2B + d_1' + d_0\right] f' \\ &+ \left(d_2BA - d_1B - d_2'B - d_2B' + d_0'\right) f - \varphi'. \end{aligned} \tag{10}$$

We then rewrite (9) and (10) into

$$\begin{cases} \alpha_1 f' + \alpha_0 f = w + \varphi, \\ \beta_1 f' + \beta_0 f = w' + \varphi', \end{cases} \tag{11}$$

where

$$\begin{cases} \alpha_1 = d_1 - d_2A, \\ \alpha_0 = d_0 - d_2B, \\ \beta_1 = d_2A^2 - (d_2A' + d_2'A + d_1A) + d_0 + d_1' - d_2B, \\ \beta_0 = d_2AB - d_1B - d_2B' - d_2'B + d_0'. \end{cases} \tag{12}$$

We next define $h_1 := \alpha_1\beta_0 - \alpha_0\beta_1$. Then

$$\begin{aligned} h_1 = &- d_0d_2A^2 + (d_0d_1 + d_1d_2B + d_2^2B' - d_0'd_2 + d_0d_2')A \\ &+ (d_0d_2 - d_2^2B)A' - d_2^2B^2 - (d_1^2 - 2d_0d_2 + d_1d_2' - d_1'd_2)B \\ &- d_1d_2B' - d_0^2 - d_0d_1' + d_0'd_1. \end{aligned} \tag{13}$$

We now show that h_1 does not vanish. This has to be proved in six subcases below.

(d) (i) Suppose first that $d_2 \neq 0$, $d_0 \neq 0$. If now $\rho(A) < \rho(B)$, and $h_1 = 0$, we may write (13) in the form $s_2B^2 + s_1B + s_0 = 0$, where $s_2 = -d_2^2$, and s_1, s_0 are polynomials in $d_0, d_2, A, B'/B$. Therefore,

$$T(r, B) = m(r, B) \leq O(r^{\rho(A)+\varepsilon}) + O(\log r),$$

a contradiction, provided ε is small enough. If next $\rho(B) < \rho(A)$, and $h_1 = 0$, we obtain $\sigma_2A^2 + \sigma_1A + \sigma_0 = 0$, where $\sigma_2 = -d_0d_2$, and σ_1, σ_0 are polynomials in $d_0, d_2, B, A'/A$, and a contradiction follows as in the preceding case. Therefore, h_1 is not vanishing in this case.

(ii) Suppose that $d_2 = d_0 = 0, d_1 \neq 0$. Then $h_1 = -d_1^2B \not\equiv 0$.

(iii) Suppose that $d_1 = d_2 = 0, d_0 \neq 0$. Then $h_1 = -d_0^2 \not\equiv 0$.

(iv) Suppose that $d_0 = d_1 = 0, d_2 \neq 0$. Then

$$h_1 = d_2^2AB' - d_2^2A'B - d_2^2B^2.$$

If $h_1 = 0$, we get $AB' - A'B - B^2 = 0$. It's not hard to see that $\rho(A) = \rho(B)$, a contradiction.

(v) Suppose that $d_0 \neq 0, d_1 \neq 0$ and $d_2 = 0$. Then

$$h_1 = d_0d_1A - d_1^2B - d_0d_1' + d_0'd_1 - d_0^2.$$

Clearly, the leading coefficient is d_0d_1A when $\rho(A) > \rho(B)$, and $-d_1^2B$ when $\rho(B) > \rho(A)$. Therefore, $h_1 \not\equiv 0$.

(vi) Suppose finally that $d_1 \neq 0, d_2 \neq 0$ and $d_0 = 0$. Then

$$\begin{aligned} h_1 &= d_1d_2AB + d_2^2AB' - d_2^2A'B \\ &\quad - (d_1^2B + d_1d_2B' + d_2^2B^2 + d_1d_2'B - d_1'd_2B). \end{aligned} \tag{14}$$

This may also be written as

$$\begin{aligned} h_1 = \varsigma_1 A - \varsigma_0 &= \left(d_1d_2B + d_2^2B' - d_2^2B\frac{A'}{A}\right)A \\ &\quad - (d_1^2B + d_1d_2B' + d_2^2B^2 + d_1d_2'B - d_1'd_2B). \end{aligned} \tag{15}$$

If $\rho(A) < \rho(B)$, and $h_1 = 0$, we may easily write (14) in the form $\tau_2B^2 + \tau_1B = 0$, where $\tau_2 = -d_2^2$ and τ_1 is a polynomial in $d_1, d_2, A, B'/B$. Therefore,

$$T(r,B) = m(r,B) \leq O(r^{\rho(A)+\varepsilon}) + O(\log r),$$

a contradiction, and we have $h_1 \neq 0$.

Assume then that $\rho(B) < \rho(A)$, and $h_1 = 0$. By (15), $\varsigma_1A = \varsigma_0$. We first see that $\varsigma_0 \neq 0$. Indeed, if this is not the case, then

$$d_2^2B = d_1'd_2 - d_1^2 - d_1d_2' - d_1d_2\frac{B'}{B}.$$

Using [6], Corollary 1 and the fact that d_1, d_2 are polynomials, there exists a set $E_1 \subset [0, 2\pi)$ with $m(E_1) = 0$ such that for $\theta_0 \in [0, 2\pi) \setminus E_1$, and for constants $R_0 > 1$ and $d \neq 0$, we have

$$|B(re^{i\theta_0})| \leq |z|^d$$

for all z satisfying $\arg z = \theta_0$ and $|z| \geq R_0$. By Lemma 2.6, we get that B is a polynomial, a contradiction.

Obviously, $\rho(\varsigma_0) \leq \rho(B) < \rho(A)$, and so $\rho(\varsigma_1) = \rho(A)$. From

$$\varsigma_1 = d_1d_2B + d_2^2B' - d_2^2B\frac{A'}{A},$$

we immediately conclude that

$$m(r, \varsigma_1) = O(r^{\rho(B)+\varepsilon}) + O(\log r).$$

Moreover, all possible poles of ς_1 are simple. Recalling that $\varsigma_1A = \varsigma_0$ and writing

$$\varsigma_1 = d_2^2B\left(\frac{d_1}{d_2} + \frac{B'}{B} - \frac{\varsigma_0'}{\varsigma_0} + \frac{\varsigma_1'}{\varsigma_1}\right) = \tau + d_2^2B\frac{\varsigma_1'}{\varsigma_1},$$

which may be written in the form

$$d_2^2B\varsigma_1' = \varsigma_1^2 - \tau\varsigma_1. \tag{16}$$

In order to estimate $N(r, \varsigma_1)$, the poles z_0 of ς_1 divide in two groups: If τ has a pole at z_0, the contribution of these poles to $N(r, \varsigma_1)$ is $\leq O(r^{\rho(B)+\varepsilon}) + O(\log r)$, since $\rho(\tau) \leq \rho(B)$. On the other hand, if $\tau(z_0)$ is finite, then the double poles in (16) must cancel, and we must have $d_2^2(z_0)B(z_0) = -1$. This means that the contribution of these poles to $N(r, \varsigma_1)$ is $\leq N(r, 1/(d_2^2 B + 1)) \leq O(r^{\rho(B)+\varepsilon}) + O(\log r)$. Therefore, altogether, we have

$$T(r, \varsigma_1) = O(r^{\rho(B)+\varepsilon}) + O(\log r), \; \rho(\varsigma_1) \leq \rho(B) + 2\varepsilon < \rho(A),$$

provided ε is small enough, a contradiction. Thus, we have that h_1 does not vanish.

(e) We now have that h_1 is a non-vanishing entire function of finite order. Consider now f, by (11), in the form

$$f = \frac{1}{h_1}[(w' + \varphi')\alpha_1 - (w + \varphi)\beta_1] = \frac{w'\alpha_1 - w\beta_1}{h_1} + \psi, \tag{17}$$

where

$$\psi = \frac{\varphi'\alpha_1 - \varphi\beta_1}{h_1}$$

is a meromorphic function of finite order. If $\rho(w)$ is finite, then $\rho(f)$ is finite as well, a contradiction. Therefore, $\rho(w) = \rho(g_f) = \infty$.

Substituting now (17) into (1), we have

$$\frac{\alpha_1}{h_1} w''' + \phi_2 w'' + \phi_1 w' + \phi_0 w = -(\psi'' + A\psi' + B\psi), \tag{18}$$

where ϕ_j $(j = 0, 1, 2)$ are meromorphic functions of finite order. Since every transcendental solution of (1) is of infinite order and $\rho(\psi) < \infty$, we have $\psi'' + A\psi' + B\psi \not\equiv 0$. Thus, by $h_1 \not\equiv 0, \alpha_1 \not\equiv 0$ and Lemma 2.10, we obtain $\lambda(w) = \overline{\lambda}(w) = \rho(w) = \infty$. □

Remark 3.1. We remark that (1) cannot have non-constant solutions in the case of $B(z)$ being extremal for Yang's inequality. Suppose for a while that $f(z)$ is such a solution of (1). Then we may write

$$f(z) = z^k(1 + o(1)), \quad k \geq 1. \tag{19}$$

By (1), (3), (4) and (19), there exists a sequence $z = re^{i\theta}$ such that $\theta \in (\theta_k, \theta_{k+1}) \cap (\phi_{2i} + \varepsilon, \phi_{2i+1} - \varepsilon)$, for some $1 \leq i \leq q$ and $r > r_0$, we have

$$\begin{aligned} |A(z)f'(z)| &\leq |f''(z)| + |(B(z) - b_{j_0}) + b_{j_0}||f(z)|, \\ \exp\{r^{\rho(A)-\varepsilon'}\}r^{k-1}(1 + o(1)) &\leq r^{k-2}(1 + o(1)) \\ &\quad + \left(\exp\{-CT(r, B)\} + |b_{j_0}|\right)r^k(1 + o(1)\right), \\ &\leq r^k(1 + \exp\{-CT(r, B)\} + |b_{j_0}|)(1 + o(1)), \end{aligned}$$

which is a contradiction. Hence, every nonconstant solution f of (1) is transcendental.

4. Proof of Theorem 1.2.

Proof. Let $f(z)$ be a transcendental solution of (1).

(a) We first proceed to proving that $\mu(f)=\infty$. Since a is a Borel exceptional value of $B(z)$, we may apply Weierstrass factorization theorem to obtain

$$B(z)-a=d(z)e^{Q(z)},$$

where $Q(z)=b_nz^n+\cdots+b_0$, $b_n\neq 0$ and $\rho(d)<\rho(B)=\deg Q(z)$. This implies that

$$|B(z)-a|=|d(z)e^{Q(z)}|=|d(z)|e^{\mathrm{Re}\{Q(z)\}}.$$

Applying Lemma 2.1, for $\theta\in E^-\setminus E$, there exists a $R_0(\theta)>1$ such that

$$|B(re^{i\theta})-a|\leq \exp((1-\varepsilon)\delta(Q,\theta)r^n) \tag{20}$$

holds for all $r>R_0(\theta)$. For the convenience to the reader, we say that (20) holds for $\theta\in\bigcup_{i=1}^{n}(\psi_i,\phi_{i+1})\setminus E$ and $r>R_0(\theta)$.

By Lemma 2.3, there exists a sector $S_k(\theta_k,\theta_{k+1})$ such that $A(z)$ blows up exponentially for any $\theta\in(\psi_i,\phi_{i+1})\cap(\theta_k,\theta_{k+1})\setminus E$, for some $1\leq i\leq n$ and we then have

$$\lim_{r\to\infty}\frac{\log\log|A(z)|}{\log r}=\rho(A) \tag{21}$$

in these sectors for all r sufficiently large.

By [6, Theorem 2], there exists a set $F\subset[0,2\pi)$ with $m(F)=0$ such that if $\theta_0\in[0,2\pi)\setminus F$, then there is a constant $R_1=R_1(\theta_0)>1$, we have

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right|\leq M\left(T(2r,f)\right)^{2(k-j)},(0\leq j<k\leq 2), \tag{22}$$

for all z satisfying $\arg z=\theta_0$ and $|z|\geq R_1$, where M is a constant.

Together with (1), (20)-(22) and Lemma 2.5, there exists a sequence $z=re^{i\theta}$ such that for $\theta\in(\psi_i,\phi_{i+1})\cap(\theta_k,\theta_{k+1})\setminus(E\cup F)$ for some $1\leq i\leq n$ and for all sufficient large $r>\max\{R,R_0,R_1\}$, we have

$$\begin{aligned}\exp\{r^{\rho(A)-\varepsilon'}\}&\leq\left|\frac{f''(z)}{f'(z)}\right|+|(B(z)-a)+a|\left|\frac{f(z)}{f'(z)}\right|\\&\leq M(T(2r,f))^2+r\left(\exp((1-\varepsilon)\delta(Q,\theta)r^n)+|a|\right)\\&\leq M(T(2r,f))^2(1+o(1)).\end{aligned}$$

Therefore, we obtain $\mu(f)=\infty$.

(b) Similarly as in proving Theorem 1.1, we proceed to proving that an entire function h, to be defined below, is not vanishing. This will be done in six subcases.

(i) Suppose first that $d_2\neq 0, d_0\neq 0$. Substituting $f''=-Af'-(de^Q+a)f$ into g_f, we have

$$g_f=(d_1-d_2A)f'+(d_0-d_2de^Q-d_2a)f. \tag{23}$$

Differentiating both sides of (23), and eliminating f'' with $f'' = -Af' - (de^Q + a)f$, we obtain

$$\begin{aligned} g_f' = \; & \left[-d_2 de^Q + (d_1' - d_2 A' - d_2' A - d_1 A + d_2 A^2 + d_0 - d_2 a)\right] f' \\ & + \left[(-d_2' d - d_2 d' - d_2 dQ' - d_1 d + d_2 dA)e^Q \right. \\ & \left. +(d_0' - d_2' a - d_1 a + d_2 aA)\right] f. \end{aligned} \tag{24}$$

We then rewrite (23) and (24) into

$$\begin{cases} \alpha_1 f' + \alpha_0 f = g_f, \\ \beta_1 f' + \beta_0 f = g_f'. \end{cases} \tag{25}$$

Here,

$$\begin{cases} \alpha_1 = d_1 - d_2 A, \\ \alpha_0 = d_0 - d_2 de^Q - d_2 a, \\ \beta_1 = -d_2 de^Q + d_1' - d_2 A' - d_2' A - d_1 A + d_2 A^2 + d_0 - d_2 a, \\ \beta_0 = (-d_2' d - d_2 d' - d_2 dQ' - d_1 d + d_2 dA)e^Q + (d_0' - d_2' a \\ \qquad -d_1 a + d_2 aA). \end{cases} \tag{26}$$

Define now $h := \alpha_1 \beta_0 - \alpha_0 \beta_1$.

If $\rho(B) < \rho(A)$, then

$$\begin{aligned} h = & - d_2^2 d^2 e^{2Q} + (-d_1 d_2' d - d_1 d_2 d' - d_1 d_2 dQ' - d_1^2 d + d_1 d_2 dA \\ & + d_2^2 d' A + d_2^2 dQ' A + d_1' d_2 d - d_2^2 dA' + 2d_0 d_2 d - 2d_2^2 da)e^Q \\ & + (d_0' d_1 - d_1 d_2' a - d_1^2 a - d_0' d_2 A + d_1 d_2 aA - d_0 d_1' + d_0 d_2 A' + d_0 d_2' A \\ & + d_0 d_1 A - d_0 d_2 A^2 - d_0^2 + 2d_0 d_2 a + d_1' d_2 a - d_2^2 aA' - d_2^2 a^2). \end{aligned} \tag{27}$$

If $h = 0$, then (27) may be written as $s^2 A^2 + s_1 A + s_0 = 0$. Here, the proximity functions of the coefficients are $\le O(r^{\rho(B)+\varepsilon}) + O(\log r)$, and so we easily get $\rho(A) \le \rho(B)$, a contradiction. So, we complete the conclusion that $h \ne 0$.

If then $\rho(A) < \rho(B)$, it follows that h is a second order polynomial in e^Q with the leading coefficient $-d_2^2 d^2 \ne 0$. Thus, $h \ne 0$.

(ii) Suppose that $d_2 = d_0 = 0, d_1 \ne 0$. Then $h = -d_1^2 B \ne 0$.

(iii) Suppose that $d_2 = d_1 = 0, d_0 \ne 0$. Then $h = -d_0^2 \ne 0$.

(iv) Suppose that $d_0 = d_1 = 0, d_2 \ne 0$, then

$$h = -d_2^2 d^2 e^{2Q} + (d_2^2 d' A + d_2^2 dQ' A - d_2^2 dA' - 2d_2^2 da)e^Q - d_2^2 aA' - d_2^2 a^2.$$

It's not hard to see that $\rho(A) = \deg Q = \rho(B)$ if $h = 0$, a contradiction.

(v) Suppose that $d_0 = 0, d_1 \ne 0, d_2 \ne 0$. Then we may proceed similarly as (vi) of Part (d) in the proof of Theorem 1.1. Therefore, $h \ne 0$.

(vi) Suppose that $d_2 = 0, d_1 \ne 0$ and $d_0 \ne 0$, then

$$h = -d_1^2 de^Q + d_0' d_1 + d_0 d_1 A - d_0^2 - d_0 d_1' - d_1^2 a.$$

If now $h = 0$ and $\rho(A) < \rho(B)$, the leading coefficient $-dd_1^2$ does not vanish, and we get a contradiction. Finally, if $\rho(B) < \rho(A)$, then $d_0 d_1 \ne 0$ and a contradiction again follows. Hence, $h \ne 0$.

(c) To complete the proof of Theorem 1.2, we may proceed exactly as in Part (e) of the proof of Theorem 1.1. Thus $\overline{\lambda}(g_f - \varphi) = \infty$. $\square$

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