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DETECTING THE SLOWLY GROWING SOLUTIONS OF SECOND ORDER LINEAR DIFFERENCE EQUATIONS

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Abstract By using asymptotic method, we verify the existence on the slowly growing solutions to second order difference equations discussed by Ishizaki-Yanagihara's Wiman-Valiron method and Ishizaki-Wen's binomial series method. The classical problem on finding conditions on the polynomial coefficients $P_j(z)$ $(j=0,1,2)$ and $F(z)$ to guarantee that all nontrivial solutions of complex second order difference equation $P_2(z)f(z+2)+P_1(z)f(z+1)+P_0(z)f(z)=F(z)$ has slowly growing solutions with order $1/2$ is detected.

Keywords complex difference equation; slowly growing solution; asymptotic method; Wiman-Valiron method; binomial Series method

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1 Introduction and main results

We assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution of meromorphic functions, see, e.g. [7, 8, 14, 20]. For a meromorphic function $f(z)$ in the complex plane $\mathbb{C}$, the order $\rho(f)$ and the lower order $\mu(f)$ are defined by, respectively,

$$\rho(f)=\limsup_{r\to\infty}\frac{\log^+ T(r,f)}{\log r}\quad\text{and}\quad \mu(f)=\liminf_{r\to\infty}\frac{\log^+ T(r,f)}{\log r}.$$

If f is entire, then the Nevanlinna characteristic $T(r,f)$ can be replaced with $\log M(r,f)$, where $M(r,f)=\max\{|f(z)|:|z|\le r\}$.

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It is well known that if the coefficients $P_j(z)$ $(j = 0, 1, \cdots, n)$ are polynomials, then the order of "most" solutions of equation

$$P_n(z)f(z+n) + \cdots + P_1(z)f(z+1) + P_0(z)f(z) = 0 \tag{1.1}$$

are not less than 1, see e.g. [2–4, 6]. On the other hand, there are equations of the form (1.1) that possess a nontrivial solution with order less than 1. For example, using Wiman-Valiron method, Ishizaki and Yanagihara [11] paid attention to difference equation

$$Q_n(z)\Delta^n f(z) + \cdots + Q_1(z)\Delta f(z) + Q_0(z)f(z) = 0, \tag{1.2}$$

where $Q_n(z), \cdots, Q_0(z)$ are polynomials, and obtained the following theorem.

Theorem 1.1 ([11, Theorem 1.1]) Let $f(z)$ be a transcendental entire solution of (1.2) and of order $\chi < 1/2$. Then we have

$$\log M(r, f) = Lr^{\chi}(1 + o(1)),$$

where a rational number χ is a slope of Newton polygon for the equation (1.2), and $L > 0$ is a constant. In particular, we have $\chi > 0$.

Putting a formal solution of the form

$$f(z) = \sum_{n=0}^{\infty} \alpha_n z_\lambda(n), \quad z_\lambda(n) = \frac{\Gamma(z+1)}{\Gamma(z+1-\lambda+n)}, \quad \alpha_0 \neq 0,$$

Ishizaki et.al. proved that the difference equation

$$b_3 z(z-1)(z-2)\Delta^3 f(z-3) + b_2 z(z-1)\Delta^2 f(z-2) + b_1 z\Delta f(z-1) + (c_0 z + b_0)f(z) = 0$$

has an entire solution with order 1/3.

Define

$$\Delta f(z) = f(z+1) - f(z) \ \text{ and } \ \Delta^n f(z) = \Delta(\Delta^{n-1} f(z)) \ (n = 2, 3, \cdots),$$

and then

$$\Delta^n f(z) = \sum_{j=0}^{n} \tbinom{n}{j}(-1)^{n-j} f(z+j) \text{ and } f(z+n) = \sum_{j=0}^{n} \tbinom{n}{j}\Delta^j f(z).$$

Therefore, equation (1.1) and equation (1.2) can be expressed in terms of each other.

Ishizaki and Wen [12] considered the convergence of binomial series $\sum\limits_{n=0}^{\infty} a_n z^{\underline{n}}$, where $z^{\underline{0}} = 1$ and $z^{\underline{n}} = z(z-1)\cdots(z-n+1)$ $n = 1, 2, \cdots$, and constructed a difference Riccati equation possessing a transcendental entire solution of order 1/2, which is represented by a binomial series.

Naturally, two questions arise:

Question 1.2 Under what conditions on polynomial coefficients, does difference equation (1.1) admit a transcendental meromorphic solution $f(z)$ with order $\rho(f) < 1$?

Question 1.3 Which values may be taken by the order of transcendental meromorphic solutions $f(z)$ of difference equation (1.1) with polynomial coefficients?

The above two questions are very important. But so far, there are just two papers involving the Question 1.3. By Wiman-Valiron theory[11], 1/3 and 1/5 can be the numbers of order for entire solutions to difference equation (1.2). By binomial series[12], 1/2 can be the number of order for entire solution to a certain type of difference Riccati equation. However, Chen considered a first order linear difference equation and obtained the following theorem.

Theorem 1.4 ([5, Theorem 1.3]) Let $p_2(z) \not\equiv 0, p_1(z), F(z)$ be polynomials, $c_2, c_1(\neq c_2)$ be constants. Suppose that $f(z)$ is a finite order transcendental meromorphic solution of difference equation

$$p_2(z)f(z+c_2) + p_1(z)f(z+c_1) = F(z),$$

then $\rho(f) \geq 1$.

Remark 1.5 Theorem 1.4 shows that a first order linear difference equation with polynomial coefficients does not admit any transcendental meromorphic solution with order $\rho(f) < 1$, which gives a negative answer to Question 1.2.

The study of linear homogeneous difference equations in the complex plane was active in the early 19's, e.g., Jordan[13], Landau[15], Milne-Thomson[16], Nörlund[17] and Whittaker[19]. They have considered different methods to construct meromorphic solutions deeply. The result due to Praagman [18] in 1986 contributed to the existence of meromorphic solutions, which is important in this area. Asymptotic method is definitely powerful tools for non-existence of solutions, however somewhat weak for existence of solutions. Thus, it is an important and interesting problem to detect the slowly growing solutions of second order linear difference equations by asymptotic method, and try to answer Question 1.2 and Question 1.3. We obtain the following theorems.

Theorem 1.6 Suppose that $f(z)$ is a transcendental meromorphic solution of difference equation

$$P_2(z)f(z+2) + P_1(z)f(z+1) + P_0(z)f(z) = 0, \tag{1.3}$$

where $P_j(z)$ $(j = 0, 1, 2)$ are polynomials. Then every transcendental solution $f(z)$ satisfies $\rho(f) \geq 1$, except for the slowly growing order $\rho(f) = 1/2$ when

(i)

$$P_j(z) = a_j z^n + \beta_j(z), \ (j = 0, 1, 2),$$

where $n \in \mathbb{N}_+$, $a_j(\neq 0)$ are constants with $a_2 = a_0$, and $\beta_j(z)$ $(j = 0, 1, 2)$ are polynomials of degree $\deg(\beta_j) \leq n-1$ satisfying

$$\deg(P_0 + P_1 + P_2) = \deg(\beta_0 + \beta_1 + \beta_2) = n - 1,$$

and

(ii) $f(z)$ has finitely many poles at most.

Theorem 1.7 Suppose that $f(z)$ is a transcendental meromorphic solution of difference equation

$$P_2(z)f(z+2) + P_1(z)f(z+1) + P_0(z)f(z) = F(z), \tag{1.4}$$

where $P_j(z)$ $(j = 0, 1, 2)$, $F(z)$ are polynomials. Then the same statements of Theorem 1.6 still hold.

Example 1.8 Consider the equation

$$(4z+6)f(z+2) - (8z+9)f(z+1) + (4z+4)f(z) = 0, \tag{1.5}$$

where the coefficients $P_2(z) = 4z + 6$, $P_1(z) = -(8z + 9)$, $P_0(z) = 4z + 4$ satisfy the exception (i) of Theorem 1.6 with

$$\deg P_0 = \deg P_1 = \deg P_2 = 1, \qquad a_0 = a_2 = 4,$$
$$\deg(P_0 + P_1 + P_2) = n - 1 = 0 < 1.$$

Therefore, we see that equation (1.5) admits an slowly growing entire solution $f(z)$ with $\rho(f) = 1/2$. In fact, equation (1.5) can be rewritten as

$$(4z + 6)\Delta^2 f(z) + 3\Delta f(z) + f(z) = 0. \tag{1.6}$$

Ishizaki and Wen [12, Example 4.3] have shown that equation (1.6) has an entire solution in the form of binomial series which has slowly growing order 1/2.

Example 1.9 The equation

$$f(z + 2) + z^2 f(z + 1) - (z^2 + 1)f(z) = z^2 + 2 \tag{1.7}$$

has solutions $f_1(z) = z$ and $f_2(z) = \tan(\pi z) + z$, where $P_2(z) = 1$, $P_1(z) = z^2$ and $P_0(z) = -(z^2 + 1)$ don't satisfy the exceptions of Theorem 1.7. Thus, we see that every transcendental meromorphic solution $f(z)$ of equation (1.7) satisfies $\rho(f) \geq 1$ since f_1 and f_2 are linear independent with $\rho(f_2) = 1$.

2 Preliminary lemmas

Following Hayman[9, p.77], we define an $\varepsilon-$set to be a countable union of open discs not containing the origin and subtending angles at the origin whose sum is finite. If E is an $\varepsilon-$set, then the set of $r \geq 1$ for which the circle $S(0, r)$ meets E has finite logarithmic measure, and for almost all real θ the intersection of E with the ray $\arg z = \theta$ is bounded. Now, we state some lemmas which are important for the proofs of theorems. Lemmas 2.1−2.3 are the asymptotic formulas among derivatives, shifts and difference operators of meromorphic functions with order less than one, which are given by Bergweiler and Langley [1].

Lemma 2.1 ([1, Lemma 3.3]) Let $g(z)$ be a function transcendental and meromorphic in the plane of order less than one. Let $h > 0$. Then there exists an ε-set E such that

$$\frac{g'(z + c)}{g(z + c)} \to 0 \quad \text{and} \quad \frac{g(z + c)}{g(z)} \to 1, \quad \text{as} \quad z \to \infty \quad \text{in} \quad \mathbb{C} \setminus E,$$

uniformly in c for $|c| \leq h$. Further, E may be chosen so that for large z not in E the function $g(z)$ has no zeros and poles in $|\zeta - z| \leq h$.

Lemma 2.2 ([1, Lemma 3.5]) Let $f(z)$ be a transcendental meromorphic function of order less than one. Let $h > 0$. Then there exists an ε-set E' such that

$$f(z + c) - f(z) = cf'(z)(1 + o(1)) \quad \text{as} \quad z \to \infty \quad \text{in} \quad \mathbb{C} \setminus E'$$

uniformly in c for $|c| \leq h$.

Lemma 2.3 ([1, Lemma 4.2]) Let $n \in \mathbb{N}_+$. Let $f(z)$ be a transcendental meromorphic function of order less than 1. Then there exists an ε-set E_n such that

$$\Delta^n f(z) \sim f^{(n)}(z) \quad (n = 1, \cdots) \quad \text{as} \quad z \to \infty \quad \text{in} \quad \mathbb{C} \setminus E_n.$$

Lemma 2.4 Suppose that $f(z)$ is a transcendental meromorphic solution of equation (1.3) with only finitely many poles. Set $f(z)=\frac{f_1(z)}{P(z)}$, where $f_1(z)$ is an entire function, $P(z)$ is a polynomial formed by the poles of $f(z)$. Then, equation (1.3) can be rewritten as

$$D_2(z)f_1(z+2)+D_1(z)f_1(z+1)+D_0(z)f_1(z)=0, \tag{2.1}$$

where $D_j(z)$ $(j=0,1,2)$ are polynomials. Suppose that, for equation (1.3), $P_j(z)$ $(j=0,1,2)$ satisfy the exception (i) of Theorem 1.6. Then $D_j(z)$ $(j=0,1,2)$ also satisfy the exception (i) of Theorem 1.6.

Proof Substituting $f(z)=\frac{f_1(z)}{P(z)}$ into equation (1.3), we obtain equation (2.1), where

$$D_2(z)=P_2(z)P(z+1)P(z), \tag{2.2}$$

$$D_1(z)=P_1(z)P(z+2)P(z), \tag{2.3}$$

$$D_0(z)=P_0(z)P(z+2)P(z+1). \tag{2.4}$$

Since $P_j(z)$ $(j=0,1,2)$ satisfy the exception (i) of Theorem 1.6, we see that

$$P_j(z)=a_jz^n+\beta_j(z),\quad a_2=a_0,\quad (a_j\neq 0,\ j=0,1,2\ \text{ are }\ \text{ constants}),$$

and $\beta_j(z)$ $(j=0,1,2)$ are polynomials satisfying

$$\deg\beta_j\leq n-1,$$

and

$$\deg(P_0+P_1+P_2)=\deg(\beta_0+\beta_1+\beta_2)=n-1. \tag{2.5}$$

Set

$$\beta_j(z)=b_{j(n-1)}z^{n-1}+\cdots, \tag{2.6}$$

$$P(z)=d_mz^m+d_{m-1}z^{m-1}+\cdots,\quad d_m\neq 0, \tag{2.7}$$

where d_k $(k=m,m-1,\cdots,0)$, $b_{j(n-1)},\cdots,b_{j0}$ $(j=0,1,2)$ are constants. By (2.5), we see that

$$b_{2(n-1)}+b_{1(n-1)}+b_{0(n-1)}\neq 0. \tag{2.8}$$

Thus, by (2.2), (2.6) and (2.7), we obtain

$$\begin{aligned}D_2(z)&=(a_2z^n+b_{2(n-1)}z^{n-1}+\cdots)(d_mz^m+(md_m+d_{m-1})z^{m-1}+\cdots)\\&\quad\cdot(d_mz^m+d_{m-1}z^{m-1}+\cdots)\\&=a_2d_m^2z^{n+2m}+(2a_2d_md_{m-1}+a_2md_m^2+b_{2(n-1)}d_m^2)z^{n+2m-1}+\cdots.\end{aligned} \tag{2.9}$$

By (2.3), (2.6) and (2.7), we obtain

$$\begin{aligned}D_1(z)&=(a_1z^n+b_{1(n-1)}z^{n-1}+\cdots)(d_mz^m+(2md_m+d_{m-1})z^{m-1}+\cdots)\\&\quad\cdot(d_mz^m+d_{m-1}z^{m-1}+\cdots)\\&=a_1d_m^2z^{n+2m}+(2a_1d_md_{m-1}+2a_1md_m^2+b_{1(n-1)}d_m^2)z^{n+2m-1}+\cdots.\end{aligned} \tag{2.10}$$

By (2.4), (2.6) and (2.7), we obtain

$$\begin{aligned} D_0(z) &= (a_0z^n + b_{0(n-1)}z^{n-1} + \cdots)(d_mz^m + (2md_m + d_{m-1})z^{m-1} + \cdots) \\ &\quad \cdot (d_mz^m + (md_m + d_{m-1})z^{m-1} + \cdots) \\ &= a_0d_m^2z^{n+2m} + (2a_0d_md_{m-1} + 3a_0md_m^2 + b_{0(n-1)}d_m^2)z^{n+2m-1} + \cdots . \end{aligned} \tag{2.11}$$

By (2.9)−(2.11), we obtain

$$\begin{aligned} &D_2(z) + D_1(z) + D_0(z) = (a_0 + a_1 + a_2)d_m^2z^{n+2m} \\ &+\{2d_md_{m-1}(a_0 + a_1 + a_2) + md_m^2(3a_0 + 2a_1 + a_2) \\ &+d_m^2(b_{0(n-1)} + b_{1(n-1)} + b_{2(n-1)})\}z^{n+2m-1} + \cdots . \end{aligned} \tag{2.12}$$

In what follows, we show that $D_j(z)$ $(j = 0, 1, 2)$ satisfy the exception (i) of Theorem 1.6. Since $P_j(z)$ $(j = 0, 1, 2)$ satisfy the exception (i) of Theorem 1.6, we see that

$$a_0 + a_1 + a_2 = 0 \text{ and } a_2 = a_0, \tag{2.13}$$

so, $a_1 = -2a_2$, and

$$(a_0 + a_1 + a_2)d_m^2 = 0. \tag{2.14}$$

Hence

$$a_2d_m^2 = a_0d_m^2, \tag{2.15}$$

and

$$\deg(D_0 + D_1 + D_2) < n + 2m.$$

By (2.8) and (2.13), we see from (2.12) that

$$2d_md_{m-1}(a_0 + a_1 + a_2) + md_m^2(3a_0 + 2a_1 + a_2) = 0,$$

and

$$d_m^2(b_{0(n-1)} + b_{1(n-1)} + b_{2(n-1)}) \neq 0.$$

Then

$$\deg(D_0 + D_1 + D_2) = n + 2m - 1. \tag{2.16}$$

Hence, for equation (2.1), by (2.9)−(2.12) and (2.14)−(2.16), we obtain that $D_j(z)$ $(j = 0, 1, 2)$ satisfy the exception (i) of Theorem 1.6. □

Remark 2.5 From the proof of Lemma 2.4, we can prove that if for equation (1.3), $P_j(z)$ $(j = 0, 1, 2)$ do not satisfy the exception of Theorem 1.6, then for equation (2.1), $D_j(z)$ $(j = 0, 1, 2)$ also do not satisfy the exception of Theorem 1.6. This means that in the proofs of Theorem 1.6 and Theorem 1.7, without loss of generality, we can assume a transcendental meromorphic solution of finitely many poles to be an entire solution.

Lemma 2.6 ([3, 4]) Let $F(z)$ and $P_j(z)$ $(j = 0, 1, \cdots, n)$ be polynomials such that $FP_nP_0 \not\equiv 0$. Suppose that $f(z)$ is a meromorphic solution with infinitely many poles of equation

$$P_n(z)f(z + n) + P_{n-1}(z)f(z + n - 1) + \cdots + P_0(z)f(z) = F(z),$$

(or equation (1.1)), then $\rho(f) \geq 1$.

Lemma 2.7 ([6, Theorem 9.4]) Let $P_j(z)$ $(j = 0, 1, \cdots, n)$ be polynomials such that there exists an integer $l, 0 \leq l \leq n$ satisfying

$$\deg P_l > \max_{0\leq j\leq n, j\neq l}\{\deg P_j\}.$$

Suppose $f(z)$ is a meromorphic solution of (1.1), then $\rho(f) \geq 1$.

Lemma 2.8 ([3, 4]) Let $P_j(z)$ $(j = 0, 1, \cdots, n)$ be polynomials such that $P_nP_0 \not\equiv 0$ and

$$\deg(P_n + \cdots + P_0) = \max\{\deg P_j,\ j = 0, \cdots, n\} \geq 1.$$

Then every finite order transcendental meromorphic solution $f(z)(\not\equiv 0)$ of equation (1.1) satisfies $\rho(f) \geq 1$, and $f(z)$ assumes every non-zero value $a \in \mathbb{C}$ infinitely often and $\lambda(f - a) = \rho(f)$.

3 Asymptotic method

In this section, we mainly introduce asymptotic method, see, e.g. [7, P.183-184], [10, P.227-229]). We discuss the propositions of asymptotic method and apply them to difference equations.

Suppose that $f(z) = \sum\limits_{n=0}^{\infty} a_nz^n$. The maximum term $\mu(r, f)$ of $f(z)$ is defined by

$$\mu(r, f) := \max\{|a_n|r^n : n \geq 0\},$$

and the central index $\nu(r, f)$ of $f(z)$ is defined by

$$\nu(r, f) = \max\{n : |a_n|r^n = \mu(r, f)\}.$$

Theorem 3.1 ([10, 14]) Suppose that $f(z)$ is a transcendental entire function, for any given $0 < \delta < 1/8$, there exists a set H of finite logarithmic measure such that

$$\frac{f^{(n)}(z)}{f(z)} = \Big(\frac{\nu(r, f)}{z}\Big)^n(1 + o(1)), \quad |z| = r \notin H, \tag{3.1}$$

whenever $|f(z)| \geq M(r, f)\nu(r, f)^{-\frac{1}{8}+\delta}$.

Suppose that $P_j(z)$ $(j = 0, 1, \cdots, n), F(z)$ are polynomials, and consider linear differential equation

$$P_n(z)f^{(n)}(z) + P_{n-1}(z)f^{(n-1)}(z) + \cdots + P_0(z)f(z) = F(z), \tag{3.2}$$

and corresponding homogeneous linear differential equation

$$P_n(z)f^{(n)}(z) + P_{n-1}(z)f^{(n-1)}(z) + \cdots + P_0(z)f(z) = 0. \tag{3.3}$$

If a solution $f(z)$ of (3.2) (or (3.3)) is a transcendental entire function, then from Theorem 3.1, for a set $H \subset (1, +\infty)$ having finite logarithmic measure, we can choose z satisfying $|z| = r \notin [0, 1] \cup H$ and $|f(z)| = M(r, f)$ such that

$$\frac{f^{(j)}(z)}{f(z)} = \Big(\frac{\nu(r, f)}{z}\Big)^j(1 + o(1)),\ (j = 1, 2, \cdots, n). \tag{3.4}$$

Substituting (3.4) into (3.2) and (3.3), respectively, we obtain that, for all $r \notin [0, 1] \cup H$,

$$P_n(z)\Big(\tfrac{\nu(r,f)}{z}\Big)^n(1 + o(1)) + P_{n-1}(z)\Big(\tfrac{\nu(r,f)}{z}\Big)^{n-1}(1 + o(1)) + \cdots + P_0(z) = o(1), \tag{3.5}$$

and

$$P_n(z)\Big(\frac{\nu(r,f)}{z}\Big)^n(1+o(1))+P_{n-1}(z)\Big(\frac{\nu(r,f)}{z}\Big)^{n-1}(1+o(1))+\cdots+P_0(z)=0. \tag{3.6}$$

Suppose that $A_j\neq 0$ and $P_j(z)=A_jz^{m_j}(1+o(1))$ $(j=0,1,\cdots,n)$ as $r\to\infty$. By (3.5) and (3.6), we obtain that, for all $r\notin[0,1]\cup H$,

$$\begin{aligned}&A_n\nu(r,f)^nz^{m_n-n}(1+o(1))+A_{n-1}\nu(r,f)^{n-1}z^{m_{n-1}-(n-1)}(1+o(1))\\&+\cdots+A_0z^{m_0}(1+o(1))=o(1),\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&A_n\nu(r,f)^nz^{m_n-n}(1+o(1))+A_{n-1}\nu(r,f)^{n-1}z^{m_{n-1}-(n-1)}(1+o(1))\\&+\cdots+A_0z^{m_0}(1+o(1))=0.\end{aligned} \tag{3.8}$$

Since solutions of algebraic equations (3.7) and (3.8) are continuous functions of coefficients, solutions $\nu(r,f)$ of equations (3.7) and (3.8) must asymptotically equal to the solution of equation

$$A_n\nu(r,f)^nz^{m_n-n}+A_{n-1}\nu(r,f)^{n-1}z^{m_{n-1}-(n-1)}+\cdots+A_0z^{m_0}=0. \tag{3.9}$$

Since the solution $\nu(r,f)$ of (3.9) is algebraic function of z, we set the principal part of $\nu(r,f)$ to be $a(\rho)z^\rho$ (a, ρ are nonzero real numbers) in the neighborhood of $z=\infty$, i.e.,

$$\nu(r,f)=a(\rho)z^\rho(1+o(1))\ \ \text{in the neighborhood of } z=\infty. \tag{3.10}$$

By (3.9) and (3.10), it is easy to see that the degrees of all terms of the left side of (3.9) are, respectively,

$$n\rho+m_n-n,\quad (n-1)\rho+m_{n-1}-(n-1),\quad \cdots,\quad m_0. \tag{3.11}$$

Since $\nu(r,f)$ is the solution of (3.9), we see from (3.11) that there are two terms at least such that they are the largest numbers and equal, and the sum of coefficients of their corresponding terms in (3.9) is zero. Hence, ρ satisfies

$$i\rho+m_i-i=j\rho+m_j-j\ \ \text{for}\ \ i<j\ \ (i,j=0,1,\cdots,n). \tag{3.12}$$

Thus, we see that ρ is a rational number, at most n such rational numbers which are not less than $1/n$. Thus, we further have

Proposition 3.2 Suppose that $f(z)$ is a transcendental entire solution of difference equation

$$R_k(z)\Delta^kf(z)+R_{k-1}(z)\Delta^{k-1}f(z)+\cdots+R_0(z)f(z)=0, \tag{3.13}$$

where

$$R_j(z)=b_{jn_j}z^{n_j}+b_{j(n_j-1)}z^{n_j-1}+\cdots+b_{j0},\quad (j=0,1,\cdots,k) \tag{3.14}$$

are polynomials, and $b_{jn_j}(\neq 0),b_{j(n_j-1)},\cdots,b_{j0}$ are constants.

If the order $\rho(f)=\sigma<1$, then σ can be obtained from

$$i(\sigma-1)+n_i=j(\sigma-1)+n_j,\quad (i<j,\quad i,j=0,1,\cdots,k), \tag{3.15}$$

where $i(\sigma-1)+n_i$ and $j(\sigma-1)+n_j$ are two largest numbers in numbers

$$s\sigma+n_s-s,\quad (s=0,1,\cdots,k). \tag{3.16}$$

Proof Since the order $\rho(f)=\sigma<1$, we see from Lemma 2.3 that there exist ε-sets E_j $(j=1,\cdots,k)$ such that

$$\Delta^j f(z)=f^{(j)}(z)(1+o(1)) \quad \text{as } z\to\infty \quad \text{in } \mathbb{C}\backslash E_j. \tag{3.17}$$

Set $H_1=\left\{|z|=r: z\in\bigcup_{j=1}^k E_j\right\}$. Then H_1 is a set of finite logarithmic measure.

Since $f(z)$ is a transcendental entire function, then by Theorem 3.1, there exists a set H_2 of finite logarithmic measure such that

$$\frac{f^{(j)}(z)}{f(z)}=\Big(\frac{\nu(r,f)}{z}\Big)^j(1+o(1)), \quad (j=1,\cdots,k), \tag{3.18}$$

where $|f(z)|=M(r,f)$, $|z|=r\notin H_1\cup H_2$, $\nu(r,f)$ is the central index of $f(z)$.

Substituting (3.17) and (3.18) into (3.13), we obtain

$$\begin{aligned}&R_k(z)\Big(\tfrac{\nu(r,f)}{z}\Big)^k(1+o(1))+R_{k-1}(z)\Big(\tfrac{\nu(r,f)}{z}\Big)^{k-1}(1+o(1))\\&+\cdots+R_0(z)=0, \qquad |z|=r\notin H_1\cup H_2.\end{aligned} \tag{3.19}$$

By (3.14) and (3.19), we obtain

$$\begin{aligned}&b_{kn_k}z^{n_k}\Big(\tfrac{\nu(r,f)}{z}\Big)^k(1+o(1))+b_{(k-1)n_{k-1}}z^{n_{k-1}}\Big(\tfrac{\nu(r,f)}{z}\Big)^{k-1}(1+o(1))\\&+\cdots+b_{0n_0}z^{n_0}(1+o(1))=0, \qquad |z|=r\notin H_1\cup H_2,\end{aligned} \tag{3.20}$$

By asymptotic method, we see that solutions $\nu(r,f)$ of equation (3.20) must be asymptotically equal to solutions of equation

$$b_{kn_k}z^{n_k}\Big(\frac{\nu(r,f)}{z}\Big)^k+b_{(k-1)n_{k-1}}z^{n_{k-1}}\Big(\frac{\nu(r,f)}{z}\Big)^{k-1}+\cdots+b_{0n_0}z^{n_0}=0. \tag{3.21}$$

Again by asymptotic method and $\rho(f)=\sigma<1$, we may assume that

$$\nu(r,f)=a(\sigma)z^\sigma(1+o(1)) \quad \text{in the neighborhood of} \quad z=\infty, \tag{3.22}$$

where a is a nonzero real number, σ is a rational number satisfying $\sigma\geq\frac{1}{k}$.

From (3.21) and (3.22), it is easy to see that the degrees of all terms of the left of (3.21) are, respectively,

$$k\sigma+n_k-k, \quad (k-1)\sigma+n_{k-1}-(k-1), \quad \cdots, \quad n_0. \tag{3.23}$$

Thus, σ can be obtained from

$$i(\sigma-1)+n_i=j(\sigma-1)+n_j, \quad (i<j, \quad i,j\in\{0,1,\cdots,k\}), \tag{3.24}$$

where $i(\sigma-1)+n_i$ and $j(\sigma-1)+n_j$ are equal and the largest numbers in numbers

$$s\sigma+n_s-s, \quad (s=0,1,\cdots,k).$$

□

Example 3.3 Suppose that $f(z)$ is a transcendental entire solution of difference equation

$$(6z^2+19z+15)\Delta^3 f(z)+(z+3)\Delta^2 f(z)-\Delta f(z)-f(z)=0. \tag{3.25}$$

If $\rho(f)=\sigma<1$, we obtain that σ can be represented in

$$3(\sigma-1)+2, \quad 2(\sigma-1)+1, \quad 1(\sigma-1)+0, \quad 0$$

by similar calculation to Proposition 3.2. Obviously,

$$3(\sigma-1)+2>2(\sigma-1)+1,\quad 3(\sigma-1)+2>1(\sigma-1)+0.$$

Thus, we get $\sigma=\frac{1}{3}$ by $3(\sigma-1)+2=0$.

Remark 3.4 The method of Proposition 3.2 can only roughly give the orders for entire solutions with $\rho(f)<1$ for linear difference equations with polynomial coefficients. It can not determine under what conditions, linear difference equation with polynomial coefficients admits an entire solution with $\rho(f)<1$.

However, by adding some assumptions to the coefficients $R_j(z)$ $(j=0,1,\cdots,k)$, we can determine a linear difference equation with polynomial coefficients does not admit an entire solution with $\rho(f)<1$.

Proposition 3.5 Consider difference equations (3.13) and

$$R_k(z)\Delta^k f(z)+R_{k-1}(z)\Delta^{k-1}f(z)+\cdots+R_0(z)f(z)=F(z), \tag{3.26}$$

where $F(z)$ is a polynomial, $R_j(z)$ $(j=0,\ \cdots,\ k)$ are defined as in (3.14). If $\deg R_j(z)=n_j$ satisfy

$$\max\{n_j,\ j=s+1,\ \cdots,k\}\le n_s, \tag{3.27}$$

and

$$n_s\ge n_d+(s-d),\quad d=s-1,\quad\cdots,\quad 0, \tag{3.28}$$

then difference equations (3.13) and (3.26) do not admit any transcendental entire solution with order $\rho(f)<1$.

Proof We see from (3.13) that the corresponding numbers in (3.16) of Proposition 3.2 are

$$\begin{aligned}&j(\sigma-1)+n_j\quad(j=k,\cdots,s+1),\\&s(\sigma-1)+n_s,\quad d(\sigma-1)+n_d\ (d=s-1,\cdots,0).\end{aligned} \tag{3.29}$$

By $j>s$ and (3.27), we have

$$j(\sigma-1)+n_j<s(\sigma-1)+n_s,\quad j=k,\cdots,s+1. \tag{3.30}$$

By $d<s$ and (3.28), we have

$$s(\sigma-1)+n_s-[d(\sigma-1)+n_d]=(\sigma-1)(s-d)+n_s-n_d\ge(s-d)\sigma>0.$$

These yield

$$s(\sigma-1)+n_s>d(\sigma-1)+n_d. \tag{3.31}$$

Thus, by (3.30) and (3.31), we see from (3.29) that there exists only one term $s(\sigma-1)+n_s$ which is the largest number, a contradiction.

Hence, difference equation (3.13) do not admit any entire solution with order $\rho(f)<1$. The same assertion can be reached for equation (3.26). □

Corollary 3.6 Consider difference equations (3.13) and (3.26), where $F(z)$ is a polynomial, $R_j(z)$ $(j=0,\ \cdots,\ k)$ are defined as in (3.14). If $\deg R_j(z)=n_j$ satisfy

$$\max\{n_j,\ j=1,\ \cdots,k\}\le n_0, \tag{3.32}$$

then difference equations (3.13) and (3.26) do not admit any transcendental entire solution with order $\rho(f) < 1$.

Example 3.7 Consider again (3.25), we see that

$$\deg R_3 = n_3 = 2 \quad \text{and} \quad \deg R_0 = n_0 = 0,$$

do not satisfy (3.28) in Proposition 3.5. Indeed, Example 3.3 shows that equation (3.25) admits a transcendental entire solution of order 1/3.

Example 3.8 Consider equation

$$(4z+6)\Delta^2 f(z) + 3\Delta f(z) + f(z) = 0. \tag{3.33}$$

We see that

$$\deg R_2 = n_2 = 1, \quad \deg R_1 = n_1 = 0, \quad \text{and} \quad \deg R_0 = n_0 = 0,$$

do not satisfy (3.28) in Proposition 3.5. We note that equation (3.33) can turn into (1.5). Indeed, Example 1.8 shows that equation (3.33) admits a transcendental entire solution of order 1/2.

Example 3.9 Consider equation

$$\Delta^2 f(z) + (z^2+2)\Delta f(z) + 0 = z^2 + 2. \tag{3.34}$$

We see that

$$\deg R_2 = n_2 = 0, \quad \deg R_1 = n_1 = 2, \quad \text{and} \quad \deg R_0 = n_0 = 0,$$

satisfy (3.27) and (3.28) in Proposition 3.5. Hence, all transcendental entire solutions of (3.34) are of order not less than one. Example 1.9 also shows that equation (1.7), which is the form of (3.34), does not admit a transcendental entire solution of order $\rho(f) < 1$.

Proposition 3.10 Consider difference equation

$$B_3(z)\Delta^3 f(z) + B_2(z)\Delta^2 f(z) + B_1(z)\Delta f(z) + B_0(z)f(z) = 0 \tag{3.35}$$

and

$$B_3(z)\Delta^3 f(z) + B_2(z)\Delta^2 f(z) + B_1(z)\Delta f(z) + B_0(z)f(z) = F(z), \tag{3.36}$$

where $B_j(z)$ $(j = 0,1,2,3)$ and $F(z)$ are polynomials, $\deg B_j(z) = n_j$.

If $f(z)$ is an entire solution of (3.35) or (3.36) with order $\rho(f) = \sigma < 1$, then $\sigma \in \{\frac{1}{3}, \ \frac{1}{2}, \ \frac{2}{3}\}$. Further,

(1) if $n_3 = n_1 + 1$ or $n_2 = n_0 + 1$, then $\sigma = \frac{1}{2}$;

(2) if $n_3 = n_0 + 1$, then $\sigma = \frac{1}{3}$;

(3) if $n_3 = n_0 + 2$, then $\sigma = \frac{2}{3}$.

Proof When $n = 3$ and $\rho(f) = \sigma < 1$, the numbers in (3.23) are

$$3\sigma + n_3 - 3, \quad 2\sigma + n_2 - 2, \quad \sigma + n_1 - 1, \quad n_0,$$

where n_j $(j = 0,1,2,3)$ are nonnegative integers.

If $j\sigma + n_j - j = (j-1)\sigma + n_{j-1} - (j-1)$ for $j \in \{1,2,3\}$, then we have $\sigma = 1 + n_{j-1} - n_j$, a contradiction.

If $3\sigma+n_3-3=j\sigma+n_j-j$ for $j\in\{0,1\}$, then $\sigma=(n_0-n_3+3)/3$, $\sigma=(n_1-n_3+2)/2$ respectively.

Noting $\frac{1}{3}\le\sigma<1$. Thus, we have $\sigma=\frac{2}{3}$ if $n_0-n_3=-1$ and $\sigma=\frac{1}{3}$ if $n_0-n_3=-2$. We also have $\sigma=\frac{1}{2}$ if $n_1-n_3=-1$.

If $2\sigma+n_2-2=n_0$, then we have $\sigma=(n_0-n_2+2)/2$. This case comes from $n_0-n_2=-1$, and so $\sigma=\frac{1}{2}$. □

4 Proofs of Theorem 1.6 and Theorem 1.7

In this section, we will give a complete proof of Theorem 1.6. Theorem 1.7 can be reached similarly.

Proof **Firstly**, we prove the exception of Theorem 1.6 holds.

Assume that

$$P_j(z)=a_jz^n+\beta_j(z),\ a_2=a_0,\ (a_j(\neq 0,\ j=0,1,2)\text{ are constants}),\tag{4.1}$$

and $\beta_j(z)$ $(j=0,1,2)$ are polynomials satisfying

$$\deg(\beta_j)\le n-1,\ \deg(P_0+P_1+P_2)=\deg(\beta_0+\beta_1+\beta_2)=n-1.\tag{4.2}$$

Then we have $a_1=-2a_0=-2a_2$. Set

$$P_0(z)+P_1(z)+P_2(z):=P_*(z)=bz^{n-1}+\cdots,\ (b\neq 0\text{ is a constant}),\tag{4.3}$$

and

$$\beta_j(z)=b_jz^{n-1}+\cdots,\ (j=0,1,2),\tag{4.4}$$

where b_j $(j=0,1,2)$ are constants. Since $\deg P_*=n-1$, we have

$$b_0+b_1+b_2\neq 0.$$

Suppose that equation (1.3) possesses a meromorphic solution $f(z)$ with $\rho(f)<1$ and finitely many poles at most. Without loss of generality, we may suppose that $f(z)$ is an entire function with $\sigma(f)<1$ by Lemma 2.4.

We note that equation (1.3) can be written as

$$P_2(z)\Delta^2f(z)+(P_1(z)+2P_2(z))\Delta f(z)+(P_2(z)+P_1(z)+P_0(z))f(z)=0.\tag{4.5}$$

Thus, applying Lemma 2.3 to (4.5), there exists an ε-set E such that

$$P_2(z)f''(z)(1+o(1))+(P_1(z)+2P_2(z))f'(z)(1+o(1))+P_*(z)f(z)=0,\tag{4.6}$$

for $z\notin E$. Here, $H=\{|z|=r:z\in E\}$ is a set with finite logarithmic measure.

We see from Theorem 3.1 that there exists a set H' with finite logarithmic measure, such that

$$\frac{f''(z)}{f(z)}=\Big(\frac{\nu(r,f)}{z}\Big)^2(1+o(1)),\quad \frac{f'(z)}{f(z)}=\frac{\nu(r,f)}{z}(1+o(1)),\tag{4.7}$$

where $|f(z)|=M(r,f),|z|=r\notin[0,1]\cup H'$. Obviously, (4.6) and (4.7) yield

$$P_2(z)\Big(\frac{\nu(r,f)}{z}\Big)^2(1+o(1))+(P_1(z)+2P_2(z))\frac{\nu(r,f)}{z}(1+o(1))+P_*(z)=0,\tag{4.8}$$

where $|f(z)| = M(r,f)$, $|z| = r \notin [0,1] \cup H \cup H'$. Since $a_2 = a_0$ and $a_2 + a_1 + a_0 = 0$, then $a_1 = -2a_2$, and for some constant c,

$$2P_2(z) + P_1(z) = 2\beta_2(z) + \beta_1(z) = cz^h + \cdots, \quad h \leq n-1. \tag{4.9}$$

By applying asymptotic Method to (4.8), we see from (4.3)and (4.9) that the solution $\nu(r,f)$ of (4.8) is asymptotically equal to the solution $\nu(r,f)$ of algebraic equation

$$a_2 z^{n-2}\nu(r,f)^2 + cz^{h-1}\nu(r,f) + bz^{n-1} = 0, \tag{4.10}$$

where b is a constant.

Since $\rho(f) = \sigma < 1$, σ is a rational number not less than $\frac{1}{2}$ and $\nu(r,f) \sim ar^\sigma$ as $r \to \infty$ for some nonzero real number a. Three terms in the left hand side of (4.10) are, respectively,

$$a_2 a^2 r^{n-2+2\sigma}, \quad car^{h-1+\sigma}, \quad br^{n-1}. \tag{4.11}$$

We have $h - 1 + \sigma < n - 2 + 2\sigma$ since $h \leq n-1$. Thus, $\sigma = \frac{1}{2}$ by $n - 2 + 2\sigma = n - 1$.

Secondly, we prove that all other meromorphic solutions satisfy $\rho(f) \geq 1$. Contradicting to the exception, we will have the following three steps.

Step 1. f has infinitely many poles. It follows from Lemma 2.6 that $\rho(f) \geq 1$.

Step 2. If there exists some $P_j(z) \equiv 0$ $(j \in \{0,\ 1,\ 2\})$, then by Theorem 1.4, we have that $\rho(f) \geq 1$.

If $P_j(z)(j = 0,1,2)$ satisfy $\deg(P_0 + P_1 + P_2) = \max\{\deg P_j, j = 0,1,2\} \geq 1$, then by Lemma 2.7 and Lemma 2.8, we have $\rho(f) \geq 1$.

If $P_j(z)$ $(j = 0,1,2)$ all are constants. On the contrary, we suppose that $f(z)$ is an entire solution of equation (1.3) with order $\rho(f) < 1$. Equation (1.3) can be rewritten as

$$a\Delta^2 f(z) + b\Delta f(z) + cf(z) = 0, \tag{4.12}$$

where $a = P_2(z)$, $\quad b = P_1(z) + 2P_0(z)$, $\quad c = P_0(z) + P_1(z) + P_2(z)$. Using the same method as in the proof of the exception of Theorem 1.6, we obtain

$$af''(z)(1 + o(1)) + bf'(z)(1 + o(1)) + cf(z) = 0, \tag{4.13}$$

where $z \notin F$, F is an ε-set. Set $G = \{|z| = r : z \in F\}$. Then G is a set of finite logarithmic measure, and

$$a\Big(\frac{\nu(r,f)}{z}\Big)^2(1 + o(1)) + b\Big(\frac{\nu(r,f)}{z}\Big)(1 + o(1)) + c = 0, \tag{4.14}$$

where $|f(z)| = M(r,f)$, $|z| = r \notin [0,1] \cup G \cup G'$, where G' is of finite logarithmic measure. Again by applying asymptotic method to (4.14), we see that the solution $\nu(r,f)$ of (4.14) is asymptoticly equal to the solution $\nu(r,f)$ of equation

$$az^{-2}\nu(r,f)^2 + bz^{-1}\nu(r,f) + c = 0. \tag{4.15}$$

Since $\rho(f) = \sigma < 1$, then σ is a rational number not less than $\frac{1}{2}$ and $\nu(r,f) \sim dr^\sigma$ as $r \to \infty$ for some nonzero real number d. Thus, three terms in the left hand side of (4.15) are, respectively,

$$ad^2 r^{2\sigma-2},\ bdr^{\sigma-1},\ cr^0. \tag{4.16}$$

Obviously, we have $0 > \sigma - 1 > 2\sigma - 2$, a contradiction. Hence, when $P_j(z)$ $(j = 0,1,2)$ are all constants, all entire solutions of equation (1.3) satisfy $\rho(f) \geq 1$.

Step 3. In what follows, we consider

$$\deg(P_0 + P_1 + P_2) < \max\{\deg P_j,\ j = 0, 1, 2\} = n,$$

and $f(z)$ is an entire solution with $\rho(f) < 1$. We divide this proof into the following two cases.

Case 1. $\deg P_2 = \deg P_1 = \deg P_0 = n > \deg P_* := \deg(P_2 + P_1 + P_0)$.

If $a_2 = a_0$ and $\deg(P_0 + P_1 + P_2) = \deg P_* = n - 1$, then the exception of Theorem 1.6 holds. Hence, we only need to consider the following two subcases.

Subcase 1.1. $a_2 = a_0$ and $\deg(P_*) = k_1 \le n - 2$.

Using the same method as in the proof of the exception of Theorem 1.6, we see (4.5)-(4.8) still hold.

Since $a_2 = a_0, a_2 + a_1 + a_0 = 0$, we have $a_1 = -2a_2$. Thus, we assume

$$P_1 + 2P_2 = b_s z^s + \cdots,\ (s \le n-1), \quad P_* = c_{k_1} z^{k_1} + \cdots (k_1 \le n-2),$$

where b_s, c_{k_1} are nonzero constants, and so we obtain from (4.8) that

$$a_2 z^{n-2} \nu(r,f)^2 (1 + o(1)) + b_s z^{s-1} \nu(r,f)(1 + o(1)) + c_{k_1} z^{k_1}(1 + o(1)) = 0. \tag{4.17}$$

By asymptotic Method, $\nu(r,f)$ is asymptotically equal to the solution $\nu(r,f)$ of algebraic equation

$$a_2 z^{n-2} \nu(r,f)^2 + b_s z^{s-1} \nu(r,f) + c_{k_1} z^{k_1} = 0. \tag{4.18}$$

Since $\rho(f) = \sigma < 1$, σ is a rational number not less than 1/2, and $\nu(r,f) \sim ar^\sigma$ as $r \to \infty$ for some nonzero real number a. Thus, three terms in the left hand side of (4.18)(or (4.17)) are

$$a_2 a^2 r^{n-2+2\sigma},\ b_s a r^{s-1+\sigma},\ c_{k_1} r^{k_1}. \tag{4.19}$$

Since $k_1 \le n-2, s \le n-1, \sigma \ge \frac{1}{2}$, we conclude from (4.19) that there exists only one term $a_2 a^2 r^{n-2+2\sigma}$ with the highest degree, a contradiction.

Hence, if $P_j(z)$ $(j = 0, 1, 2)$ satisfy conditions of Subcase 1.1, every meromorphic solutions of equation (1.3) satisfies $\rho(f) \ge 1$.

Subcase 1.2. $a_2 \ne a_0$ and $\deg(P_*) = k_2 \le n - 1$.

Since $a_2 + a_1 + a_0 = 0$ and $a_2 \ne a_0$, we have

$$a_1 + 2a_2 \ne 0,$$

which yields $\deg(P_1 + 2P_2) = n$. Thus, we assume, for a constant $b(\ne 0)$,

$$P_1 + 2P_2 = bz^n + \cdots,$$

Similarly as in the proof of Subcase 1.1, (4.5)-(4.8) still hold, and the solution $\nu(r,f)$ of (4.8) is asymptoticly equal to the solution $\nu(r,f)$ of algebraic equation

$$a_2 z^{n-2} \nu(r,f)^2 + bz^{n-1} \nu(r,f) + c_{k_2} z^{k_2} = 0, \tag{4.20}$$

where $P_0(z) + P_1(z) + P_2(z) = c_{k_2} z^{k_2} + \cdots,\ c_{k_2}(\ne 0)$ and $b \ne 0$ are nonzero constants.

Since $\rho(f) = \sigma < 1$, σ is a rational number not less than 1/2 and $\nu(r,f) \sim ar^\sigma$ as $r \to \infty$ for nonzero real number a. Thus, three terms in the left hand side of (4.20) are, respectively,

$$a_2 a^2 r^{n-2+2\sigma}, \quad bar^{n-1+\sigma}, \quad c_{k_2} r^{k_2}. \tag{4.21}$$

We easily obtain from $\frac{1}{2}\le\sigma<1$ and $k_2\le n-1$ that

$$n-2+2\sigma<n-1+\sigma \ \text{ and } \ k_2<n-1+\sigma.$$

Thus, there exists only one term with the highest degree $n-1+\sigma$ in (4.21), which contradicts with (4.20).

Case 2. Assume that there are only two polynomials $P_j(z)$, $(j=0,1,2)$ with highest degree and $\deg(P_2+P_1+P_0):=\deg P_*<n$. We would consider the following six subcases.

Subcase 2.1. $\deg P_2=\deg P_1=n>\deg P_0$, $\deg P_0\le\deg(P_1+P_2)<\deg P_1$.

Equation (1.3) can be rewritten as

$$P_2(z)[f(z+2)-f(z+1)]+(P_1(z)+P_2(z))f(z+1)+P_0(z)f(z)=0. \tag{4.22}$$

By Lemmas 2.1-2.3 and the hypothesis that $f(z)$ is an entire solution with $\rho(f)<1$, we obtain there exists ε-sets E_1 and $E_1^{'}$ such that

$$f(z+2)-f(z+1)=f^{'}(z+1)(1+o(1)),\quad z\notin E_1 \text{ and } z\to\infty, \tag{4.23}$$

$$f(z)=f(z+1)(1+o(1)),\quad z\notin E_1^{'} \text{ and } z\to\infty. \tag{4.24}$$

Substituting (4.23) and (4.24) into (4.22), we obtain

$$P_2(z)\frac{f^{'}(z+1)}{f(z+1)}(1+o(1))+P_1^*(z)+P_0(z)(1+o(1))=0,\ z\notin E_1\cup E_1^{'} \text{ and } z\to\infty. \tag{4.25}$$

where $P_1^*(z)=P_1(z)+P_2(z)$.

Set $H_1=\{|z|=r, z\in E_1\cup E_1^{'}\}$. Then H_1 is a set of finite logarithmic measure. By applying Theorem 3.1 to (4.25), we obtain

$$P_2(z)\frac{\nu(r,f(z+1))}{z+1}(1+o(1))+P_1^*(z)+P_0(z)(1+o(1))=0, \tag{4.26}$$

where $|f(z+1)|=M(r,f(z+1))$, $|z|\notin[0,1]\cup H_1^{'}$, $\nu(r,f(z+1))$ is the central index of $f(z+1)$.

Rewrite (4.26) as

$$\frac{P_2(z)}{(z+1)P_1^*(z)}\nu(r,f(z+1))(1+o(1))+1+\frac{P_0(z)}{P_1^*(z)}(1+o(1))=0. \tag{4.27}$$

Since $\deg P_1^*\ge\deg P_0$, we have, for a constant d,

$$\Big|\frac{P_0(z)}{P_1^*(z)}\Big|\le d<\infty \text{ as } z\to\infty. \tag{4.28}$$

Since $\deg P_2\ge\deg((z+1)P_1^*)$, we have, for a constant d_1,

$$\Big|\frac{P_2}{(z+1)P_1^*}\Big|\ge d_1>0 \text{ and } z\to\infty. \tag{4.29}$$

Since $\nu(r,f(z+1))\to\infty$ as $r\notin H_1\cup H_1^{'}$ and $r\to\infty$, (4.27), (4.28) and (4.29) yield a contradiction.

Subcase 2.2. $\deg P_2=\deg P_1>\deg P_0>\deg(P_1+P_2)$.

Similarly as in the proof of Subcase 2.1, we also have (4.22)-(4.26), where E_1, $E_1^{'}$, H_1 and $H_1^{'}$ are defined as in the proof of Subcase 2.1.

Rewrite (4.26) as

$$\frac{P_2(z)}{(z+1)P_0(z)}\nu(r,f(z+1))(1+o(1))+\frac{P_1^*(z)}{P_0(z)}+(1+o(1))=0, \tag{4.30}$$

where $P_1^*(z)=P_1(z)+P_2(z)$ and $|f(z+1)|=M(r,f(z+1))$, $|z+1|\notin H_1\cup H_1^{'}$, ($H_1\cup H_1^{'}$ is of finite logarithmic measure), $\nu(r,f(z+1))$ is the central index of $f(z+1)$.

Since $\deg P_2>\deg P_0$ and $\deg P_1^*<\deg P_0$, we have

$$\Big|\frac{P_2(z)}{(z+1)P_0(z)}\Big|\geq d_2>0,\ \Big|\frac{P_1^*(z)}{P_0(z)}\Big|\to 0 \text{ as } z\to\infty. \tag{4.31}$$

Thus, we again deduce a contradiction from (4.30) and (4.31) since $\nu(r,f(z+1))\to\infty$ as $r\notin H_1\cup H_1^{'}$ and $r\to\infty$.

Subcase 2.3. $\deg P_2=\deg P_0>\deg P_1$ and $\deg P_1\leq\deg(P_2+P_0)<\deg P_2$.

Equation (1.3) can be rewritten as

$$P_2(z)(f(z+2)-f(z))+P_1(z)f(z+1)+(P_2(z)+P_0(z))f(z)=0. \tag{4.32}$$

By Lemmas 2.1-2.3 and the hypothesis that $f(z)$ is an entire solution with $\rho(f)<1$, we see that there exists ε-sets E_3 and $E_3^{'}$, such that for $z\notin E_3\cup E_3^{'}$,

$$f(z+2)-f(z)=2f^{'}(z)(1+o(1)),\quad f(z+1)=f(z)(1+o(1)) \text{ as } z\to\infty. \tag{4.33}$$

Set $H_3=\{|z|=r,z\in E_3\cup E_3^{'}\}$. Then H_3 is a set of finite logarithmic measure.

Substituting (4.33) into (4.32), we have

$$2P_2(z)\frac{f^{'}(z)}{f(z)}(1+o(1))+P_1(z)(1+o(1))+P_3^*(z)=0,\quad |z|=r\notin H_3, \tag{4.34}$$

where $P_3^*(z)=P_2(z)+P_0(z)$. By Theorem 3.1, we obtain

$$\frac{2P_2(z)}{zP_3^*(z)}\nu(r,f)(1+o(1))+\frac{P_1(z)}{P_3^*(z)}(1+o(1))+1=0 \tag{4.35}$$

holds outside an exceptional set $H_3^{'}$ of finite logarithmic measure, $|f(z)|=M(r,f)$ and $\nu(r,f)$ is the central index of $f(z)$.

Together with the hypotheses of Subcase 2.3, we have

$$\deg P_3^*\geq\deg P_1,\ \ \deg P_3^*<\deg P_2,$$

and so

$$\Big|\frac{P_1(z)}{P_3^*(z)}\Big|\leq a<\infty,\ \Big|\frac{2P_2(z)}{zP_3^*(z)}\Big|\geq b>0 \text{ as } z\to\infty. \tag{4.36}$$

for positive constants a and b .

Equations (4.35) and (4.36) again yield a contradiction since $\nu(r,f)\to\infty$ as $r\notin H_3\cup H_3^{'}$ and $r\to\infty$.

Subcase 2.4. $\deg P_2=\deg P_0>\deg P_1>\deg(P_2+P_0)$.

Similarly as in the proof of Subcase 2.3, we also have (4.32)-(4.34).

Again by Theorem 3.1, we obtain

$$\frac{2P_2(z)}{zP_1(z)}\nu(r,f)(1+o(1))+1+o(1)+\frac{P_3^*(z)}{P_1(z)}=0 \tag{4.37}$$

holds outside a exceptional set $H_3^{'}$ of finite logarithmic measure, $|f(z)|=M(r,f)$, $\nu(r,f)$ is the central index of $f(z)$.

From the assumption of Subcase 2.4, we have

$$\deg P_2(z)>\deg P_1(z),\ \ \deg P_1(z)>\deg P_3^*,$$

and so

$$\Big|\frac{2P_2(z)}{zP_1(z)}\Big| \geq a > 0, \quad \Big|\frac{P_3^*(z)}{P_1(z)}\Big| \to 0 \text{ as } z \to \infty. \tag{4.38}$$

Equations (4.37) and (4.38) again yield a contradiction since $\nu(r,f) \to \infty$ as $r \notin H_3 \cup H_3'$ and $r \to \infty$.

Subcase 2.5. $\deg P_0 = \deg P_1 > \deg P_2$ and $\deg P_2 \leq \deg(P_1 + P_0) < \deg P_1$.

Rewrite (1.3) as

$$P_2(z)f(z+2) + P_1(z)(f(z+1) - f(z)) + (P_1(z) + P_0(z))f(z) = 0. \tag{4.39}$$

Similarly as in the proof of Subcase 2.3, by Lemmas 2.1-2.3, there exists ε-sets E_5 and E_5', such that for $z \notin E_5 \cup E_5'$ and $z \to \infty$,

$$f(z+1) - f(z) = f'(z)(1 + o(1)), \quad f(z+2) = f(z)(1 + o(1)). \tag{4.40}$$

Set $H_5 = \{|z| = r, z \in E_5 \cup E_5'\}$. Then H_5 is a set of finite logarithmic measure.

Substituting (4.40) into (4.39), we obtain

$$P_2(z)f(z)(1 + o(1)) + P_1(z)f'(z)(1 + o(1)) + P_5^*(z)f(z) = 0, \quad |z| = r \notin H_5', \tag{4.41}$$

where $P_5^*(z) = P_1(z) + P_0(z)$.

By Theorem 3.1, there exists a set H_5' of finite logarithmic measure, such that for $|z| = r \notin H_5'$, $|f(z)| = M(r,f)$,

$$\frac{P_2(z)}{P_5^*(z)}(1 + o(1)) + \frac{P_1(z)}{zP_5^*(z)}\nu(r,f)(1 + o(1)) + 1 = 0, \tag{4.42}$$

where $\nu(r,f)$ is the central index of $f(z)$.

From the assumption of Subcase 2.5, we have

$$\deg P_5^* \geq \deg P_2, \;\; \deg P_1 > \deg P_5^*,$$

and so for positive constants a and b,

$$\Big|\frac{P_2(z)}{P_5^*(z)}\Big| \leq a < \infty, \quad \Big|\frac{P_1(z)}{zP_5^*(z)}\Big| \geq b > 0 \text{ as } z \to \infty. \tag{4.43}$$

Equations (4.42) and (4.43) yield a contradiction since $\nu(r,f) \to \infty$ as $|z| = r \notin H_5 \cup H_5'$ and $|f(z)| = M(r,f)$, $r \to \infty$.

Subcase 2.6. $\deg P_0 = \deg P_1 > \deg P_2 > \deg(P_1 + P_0)$.

Similarly as in the proof of Subcase 2.5, we see (4.39)-(4.41) also hold.

By Theorem 3.1, we obtain that

$$(1 + o(1)) + \frac{P_1(z)}{zP_2(z)}\nu(r,f)(1 + o(1)) + \frac{P_5^*(z)}{P_2(z)} = 0 \tag{4.44}$$

holds outside a exceptional set H_6' of finite logarithmic measure, $\nu(r,f)$ is the central index of $f(z)$. From the assumption of Subcase 2.6, we easily obtain

$$\Big|\frac{P_1(z)}{zP_2(z)}\Big| \geq a > 0, \; \Big|\frac{P_5^*(z)}{P_2(z)}\Big| \to 0 \text{ as } z \to \infty. \tag{4.45}$$

Equations (4.44) and (4.45) yield a contradiction since $\nu(r,f) \to \infty$ as $r \notin H_6 \cup H_6'$ and $r \to \infty$.

□

5 Discussions

We discuss the propositions of asymptotic method and apply them to difference equations. We verify the existence of the slowly growing solutions to difference equations obtained by Ishizaki-Yanagihara's Wiman-Valiron method and Ishizaki-Wen's binomial series method. By asymptotic method, we give positive answers to Questions 1.2 and 1.3 for first order and second order difference equations. It is also worth discussing how to detect the slowly growing solutions of higher order linear difference equations by asymptotic method.

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