On the classification and evolution of bifurcation curves for a quasilinear regularized MEMS model: Corrected version⋆

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Abstract

The quasilinear equation $-\left(\frac{u'(x)}{\sqrt{1+(u'(x))^2}}\right)' = \frac{\lambda}{(1-u)^2} - \frac{\lambda\varepsilon^2}{(1-u)^4}$ with the boundary condition $u(-L) = u(L) = 0$ governs the steady-state solutions of a regularized MEMS model. We prove that for any evolution parameters $\varepsilon \in (0, 1)$ and $L > 0$, the global bifurcation curve of positive solutions is strictly increasing or ⊃-like shaped or S-like shaped in the $(\lambda, \|u\|_\infty)$-plane. The bifurcation curves present a variety of shapes and structures, significantly different from those in non-regularized case (i.e., $\varepsilon = 0$) and in the simplified semilinear case. The main tools are some new time-map techniques, the total positivity theory, and Sturm's Theorem.

Keywords: S-shaped bifurcation curve, Micro-Electro-Mechanical System, Bistability, Total positivity, Exact multiplicity

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1. Introduction

In the last fifteen years, global bifurcation curves and exact multiplicity of positive solutions for the quasilinear problem

$$\begin{cases} -\left(\dfrac{u'(x)}{\sqrt{1+(u'(x))^2}}\right)' = \lambda f(u), \qquad -L < x < L, \\ u(-L) = u(L) = 0, \end{cases} \tag{1.1}$$

with various different types of nonlinearities $f(u)$ have been extensively studied (e.g., [1–17]). One of remarkable features of those research works is that the length L can effect the structure of bifurcation curves. (1.1) is also known as one-dimensional prescribed (mean) curvature problem. Due to the special geometric meaning of the mean curvature operator $\mathcal{M} : u \mapsto \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$, quasilinear equations of mean curvature type naturally appear in many physical models such as liquid drops in capillary theory ([18]), phase transition with high spatial gradients ([2, 19]), electrostatic devices in Micro-Electro-Mechanical System (MEMS, [20, 21]), corneal shapes of eyes ([22, 23]).

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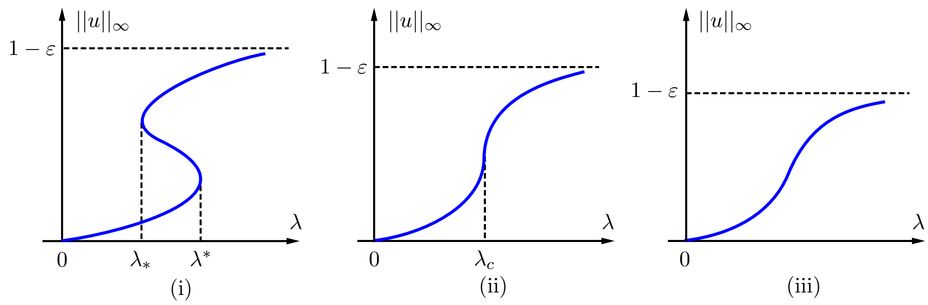

Figure 1: Conjectured global bifurcation curves of (1.3) with varying $\varepsilon \in (0,1)$: (i) $0 < \varepsilon < \varepsilon_c$, (ii) $\varepsilon = \varepsilon_c$, (iii) $\varepsilon_c < \varepsilon < 1$.

In this paper, we are concerned with the classification and evolution of global bifurcation curves of positive solutions for the three-parameter quasilinear problem

$$\begin{cases} -\left(\dfrac{u'(x)}{\sqrt{1+(u'(x))^2}}\right)' = \lambda\left[\dfrac{1}{(1-u)^2} - \dfrac{\varepsilon^2}{(1-u)^4}\right], & -L < x < L, \\ u(-L) = u(L) = 0, \end{cases} \tag{1.2}$$

where λ is a positive bifurcation parameter, $\varepsilon < 1$ and L are two positive evolution parameters. The relevant semilinear case

$$\begin{cases} -u''(x) = \dfrac{\lambda}{(1-u)^2} - \dfrac{\lambda\varepsilon^2}{(1-u)^4}, & -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases} \tag{1.3}$$

arises in a regularized model in Micro-Electro-Mechanical Systems (MEMS); see Lindsay et al. [24, 25]. The model is proposed to describe the post-touchdown configurations in an electrostatic MEMS device. The parameter λ is proportional to the square of the voltage applied to the device. Physically, the new regularizing term $-\frac{\lambda\varepsilon^2}{(1-u)^4}$ induces a potential which simulates the effect of a small insulating layer put on the ground plate to prevent a short circuit. Mathematically, the regularizing term eliminates the singularity at $u = 1$ occurring in the non-regularized model (i.e., the case of $\varepsilon = 0$). The numerical simulation in [24] suggests that there exists a critical value of bistability $\varepsilon_c \in (\tilde{\varepsilon}, \breve{\varepsilon})$ such that the bifurcation curve for (1.3) is S-shaped for $0 < \varepsilon < \varepsilon_c$ and strictly increasing for $\varepsilon_c \leqslant \varepsilon < 1$; see Fig.1(with $\|u\|_2^2$ instead of $\|u\|_\infty$). Recently, Iuorio et al. [26, Theorem 1.2(i)] proved that the bifurcation curve is S-shaped for sufficiently small ε in the $(\lambda, \|u\|_2^2)$-plane, using geometric singular perturbation theory and the blow-up method. Very recently, Lao et al. [27] proved that there exist three constants $\hat{\varepsilon}, \tilde{\varepsilon}$, and $\breve{\varepsilon}(\approx 0.25458, 0.26262$, and 0.29212, respectively) such that the bifurcation curve in the $(\lambda, \|u\|_\infty)$-plane is S-shaped for $0 < \varepsilon \leqslant \hat{\varepsilon}$, S-like shaped for $\hat{\varepsilon} < \varepsilon \leqslant \tilde{\varepsilon}$, and strictly increasing for $\breve{\varepsilon} \leqslant \varepsilon < 1$, using the quadrature method and some time-map techniques.

Problem (1.3) is a simplified case of (1.2). Recall the following prescribed mean curvature problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \dfrac{\lambda}{(1-u)^2}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

governs the positive steady-states from a non-regularized MEMS model built by Brubaker and Pelesko [20]. If it is assumed that the elastic membrane in MEMS capacitor only has a small deflection, i.e., $|\nabla u| \ll 1$, then by dropping the small gradient term, (1.4) is reduced to the classic non-regularized MEMS problem

$$\begin{cases} -\Delta u = \dfrac{\lambda}{(1-u)^2}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.5}$$

which has been deeply studied by researchers; see [28, 29] and references therein. The quasilinear non-regularized MEMS model (1.4) and its generalizations have been investigated in [1, 21, 30–33]. In particular, its one-dimensional case is the following quasilinear problem

$$\begin{cases} -\left(\dfrac{u'(x)}{\sqrt{1+(u'(x))^2}}\right)' = \dfrac{\lambda}{(1-u)^2}, & -L < x < L, \\ u(-L) = u(L) = 0, \end{cases} \tag{1.6}$$

which is also the limit case of problem (1.2) corresponding to $\varepsilon \to 0$. Global bifurcation diagrams and exact multiplicity of positive solutions of (1.6) have been proved in Brubaker and Pelesko [1] and Pan and Xing [32] as follows:

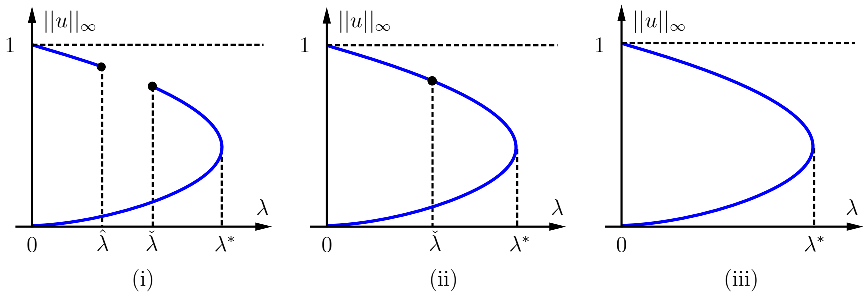

Figure 2: $\supset$-shaped bifurcation curves of (1.6) or (1.2) with $\varepsilon = 0$: (i) $0 < L < L_0$, (ii) $L = L_0$, (iii) $L > L_0$.

Theorem 1.1 ([1, 32])**.** *Consider positive solutions of* (1.6) *in* $C^2(-L, L) \bigcap C[-L.L]$*. Then there exists a positive constant* $L_0(\approx 0.34997)$ *such that the following assertions hold (see Fig.2):*

(i) *If* $0 < L < L_0$*, then there exist* $\lambda^* > \check{\lambda} > \hat{\lambda} > 0$ *such that* (1.6) *has no positive solution for* $\lambda \in (\lambda^*, +\infty)$*, exactly one positive solution for* $\lambda \in (\hat{\lambda}, \check{\lambda}) \cup \{\lambda^*\}$*, exactly two positive solutions for* $\lambda \in (0, \hat{\lambda}] \cup [\check{\lambda}, \lambda^*)$*.*

(ii) *If* $L \geqslant L_0$*, then there exists* $\lambda^* > 0$ *such that* (1.6) *has no positive solution for* $\lambda \in (\lambda^*, +\infty)$*, exactly one positive solution for* $\lambda = \lambda^*$*, exactly two positive solutions for* $\lambda \in (0, \lambda^*)$*.*

(iii) $\check{\lambda}$ *(respectively,* $\hat{\lambda}$*) is strictly decreasing (respectively, increasing) with respect to* $L \in (0, L_0)$*, and* $\lim_{L\to L_0} \check{\lambda} = \lim_{L\to L_0} \hat{\lambda}$*, denoting the limit still by* $\check{\lambda}$*. All solutions belong to* $C^2[-L, L]$*, except for the solutions* u *corresponding to* $\hat{\lambda}$ *and* $\check{\lambda}$ *on the upper solution branch, satisfying* $u'(\pm L) = \mp\infty$*.*

In contrast to (1.6), the bifurcation diagram for the following one-dimensional case of (1.5)

$$\begin{cases} -u''(x) = \dfrac{\lambda}{(1-u)^2}, & -1 < x < 1, \\ u(-1) = u(1) = 0. \end{cases} \tag{1.7}$$

is significantly different and can be depicted in Fig.2(iii) (see e.g., [34, Theorems 2.5, 2.9, and Lemma 3.2] and [29, Fig.7.16]). In particular, the length of the interval does not change the structure of the bifurcation curve for (1.7), but for (1.6). As sketched in Fig.2, the splitting phenomenon happens for (1.6) when the length $L < L_0$. For more research works on quasilinear equations in MEMS models or with singular nonlinearities, we refer the reader to [3, 15, 21, 35, 36]. So far, no research results are known for (1.2) with $\varepsilon > 0$.

Motivated by the results mentioned above, we in this paper will investigate the classification and evolution of bifurcation curves for positive solutions of (1.2) with varying evolution parameters ε and L. By a solution we mean a function $u \in C^2(-L, L) \bigcap C[-L, L]$ satisfying (1.2). For any $\varepsilon \in (0, 1)$, denote the bifurcation curve (or solution curve) of (1.2) by

$$\mathcal{C}_\varepsilon = \left\{ (\lambda, \|u\|_\infty) \mid \lambda > 0 \text{ and } u \text{ is a positive solution of } (1.2)_{\varepsilon,\lambda} \right\}.$$

Note that all positive solutions u of (1.2) are even in $(-L, L)$ and can be parameterized by $u(0) = \max u$.

Similarly as in [4, 37], we introduce several terms to describe shapes of $\mathcal{C}_\varepsilon$ in the $(\lambda, \|u\|_\infty)$-plane for simplicity. We call the bifurcation curve $\mathcal{C}_\varepsilon$ to be $\supset$*-shaped* (respectively, $\subset$*-shaped*), if $\mathcal{C}_\varepsilon$ turns exactly once and to the left (respectively, the right); call $\mathcal{C}_\varepsilon$ to be $\supset$*-like shaped* (respectively, $\subset$*-like shaped*), if $\mathcal{C}_\varepsilon$ turns exactly odd times and eventually to the left (respectively, the right); call $\mathcal{C}_\varepsilon$ to be S*-shaped* (respectively, *reversed* S*-shaped*) if $\mathcal{C}_\varepsilon$ turns exactly twice and eventually to the right (respectively, the left); call $\mathcal{C}_\varepsilon$ to be S*-like shaped* (respectively, *reversed* S*-like shaped*) if $\mathcal{C}_\varepsilon$ turns exactly positive even times and eventually to the right (respectively, the left). S*-like shaped* is also referred to as *roughly* S*-shaped* in [37, 38]. Note that Lemma 2.7 in [8] implies that for each value r, (1.2) admits at most one λ and at most one positive solution u so that $\|u\|_\infty = r$. This means that $\mathcal{C}_\varepsilon$ cannot bend down or up, but can turn to the left and the right. So the above definitions are not ambiguous. Here, we do not require that the curve $\mathcal{C}_\varepsilon$ must be connected.

Our main tools are time-map method, total positivity theory of integral operator, and Sturm's Theorem for polynomials. Different from the recent works [4, 5], the nonlinearity $f(u) = \frac{1}{(1-u)^2} - \frac{\varepsilon^2}{(1-u)^4}$ here is a convex–concave but non-monotonic function, which brings new difficulties to analyze the time-map. We develop some new time-map techniques to deal with the complicated situation in which the derivative of the time-map changes sign at the boundary of its domain as λ varies. With the aid of total positivity theory and Sturm's Theorem, we overcome this difficulty and prove global bifurcation results for different ranges of ε and L; see Theorems 2.1–2.8. These results show that the patterns of bifurcation curves are very complex and diverse, significantly different from those in the semilinear case (1.3) and in the quasilinear case (1.6) (i.e., (1.2) with $\varepsilon = 0$). We also give a conjecture on the complete classification and evolution of bifurcation curves $\mathcal{C}_\varepsilon$ for (1.2) in the $(\lambda, \|u\|_\infty)$-plane with varying ε and L; see Fig.15. For the convenience of the reader, some constants appearing in the present paper are collected in Table 1.

We organize the paper as follows. In Section 2, we state our main results. In Section 3, we give various lemmas about the properties of the time-map. The main theorems will be proved in Section 4. Finally, we give two conjectures on the classification and evolution of bifurcation curve $\mathcal{C}_\varepsilon$ in Section 5. The proofs of two inequalities required in the previous arguments are placed in the Appendix.

2. Main results

The following theorems are our main results.

Table 1: Definitions and numerical evaluations of some constants appearing in the present paper.

Symbol of Constant	Numerical Evaluation	Definition and Source
L_0	≈ 0.34997	By Theorem 1.1 or [1, 32]
$\breve{\varepsilon}$ $(=\frac{4\sqrt{30}}{75})$	≈ 0.29212	By (3.10) or [27]
$\tilde{\varepsilon}$	≈ 0.26262	By Lemma 3.5 or [27]
$\hat{\varepsilon}$	≈ 0.25458	By Lemma 3.5 or [27]
ε_*	≈ 0.13123	By Lemma 3.15

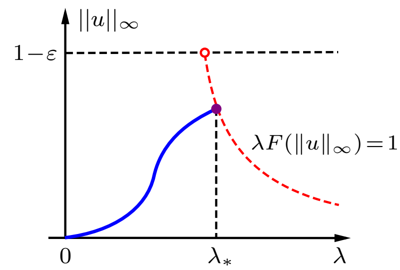

Figure 3: Monotone increasing bifurcation curves $\mathcal{C}_\varepsilon$ for (1.2) with $\frac{\sqrt{2}}{2} \leqslant \varepsilon < 1$.

Theorem 2.1. *For any $\varepsilon \in (0,1)$ and $L > 0$, the bifurcation curve $\mathcal{C}_\varepsilon$ of (1.1) is bounded in the $(\lambda, \|u\|_\infty)$-plane, always starts from the origin $(0,0)$ and eventually stops at some point on the derivative blow-up curve $\{(\lambda, r) \mid \lambda > \lambda_{1-\varepsilon}$ and $r \in (0, 1-\varepsilon)$ satisfy $\lambda F(r) = 1\}$, where $\lambda_{1-\varepsilon} := \frac{1}{F(1-\varepsilon)}$, $r := \|u\|_\infty$, and $F(r) := \frac{1}{1-r} - \frac{\varepsilon^2}{3(1-r)^3} + \frac{\varepsilon^2}{3} - 1$. Furthermore, $\mathcal{C}_\varepsilon$ is either a C^1 curve or composed of two separate C^1 curves (one of them may be a singleton).*

Remark 2.1. *The solutions u corresponding to the stopping points on the blow-up curve $\lambda F(\|u\|_\infty) = 1$ have distinct boundary regularity from the other solutions. Precisely, these solutions belong to $C^2(-L, L) \bigcap C[-L, L]$ satisfying $u'(\pm L) = \mp\infty$, while the others belong to $C^2[-L, L]$ and hence have higher regularity to the boundary; see Section 3 for more details.*

Theorem 2.2. *If $\frac{\sqrt{2}}{2} \leqslant \varepsilon < 1$, then the bifurcation curve $\mathcal{C}_\varepsilon$ for (1.2) is strictly increasing. Precisely, for any $L > 0$, there exists $\lambda_* > 0$ such that (1.2) has no positive solution for $\lambda \in (\lambda_*, \infty)$ and exactly one positive solution for $\lambda \in (0, \lambda_*]$.*

In view of Theorem 2.1, the bifurcation diagram for $\frac{\sqrt{2}}{2} \leqslant \varepsilon < 1$ is depicted in Fig.3. In what follows, we will focus on the different shapes of $\mathcal{C}_\varepsilon$ for varying ε and no longer discuss the exact position of the stopping point for simplicity.

Theorem 2.3. *If $\frac{\sqrt{30}}{10} \leqslant \varepsilon < \frac{\sqrt{2}}{2}$, then the bifurcation curve $\mathcal{C}_\varepsilon$ for (1.2) is $\supset$-like shaped or S-like shaped or strictly increasing. Precisely, there exist positive numbers $\bar{L} < \bar{\bar{L}}$ such that the following assertions hold:*
(i) (See Fig.4(ii)) *If $0 < L < \bar{L}$, then $\mathcal{C}_\varepsilon$ is $\supset$-like shaped. Moreover, there exist positive numbers $\lambda_* < \lambda^*$ such that (1.2) has no positive solution for $\lambda \in (\lambda^*, +\infty)$, at least one positive solution for $\lambda \in (0, \lambda_*) \cup \{\lambda^*\}$,*

at least two positive solutions for $\lambda \in [\lambda_*, \lambda^*)$.

(ii)(See Fig.4(iii) or (iv)) *If* $\bar{L} < L < \bar{\bar{L}}$, *then* $\mathcal{C}_\varepsilon$ *is S-like shaped or strictly increasing. If* $\mathcal{C}_\varepsilon$ *is strictly increasing, then the results in Theorem 2.2 hold, while if* $\mathcal{C}_\varepsilon$ *is S-like shaped, then* (1.2) *has at least three positive solutions for some* $\lambda > 0$.

(iii)(See Fig.4(iv)) *If* $L \geqslant \bar{\bar{L}}$, *then* $\mathcal{C}_\varepsilon$ *is strictly increasing, and the results in Theorem 2.2 hold.*

(iv)(See Fig.4(ii) or (iii)) *If* $L = \bar{L}$, *then* $\mathcal{C}_\varepsilon$ *is* $\supset$*-like shaped or S-like shaped.*

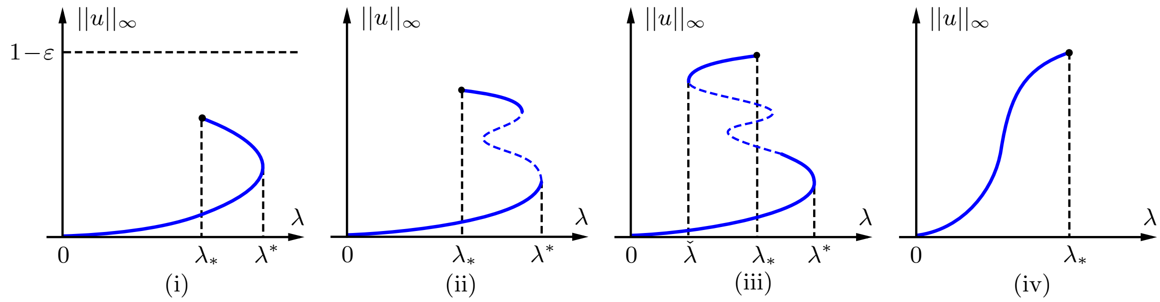

Figure 4: $\supset$-shaped, $\supset$-like shaped, S-like shaped, and strictly increasing bifurcation curves $\mathcal{C}_\varepsilon$ for (1.2) with different ranges of ε and L. Case (A) $\frac{\sqrt{30}}{10} \leqslant \varepsilon < \frac{\sqrt{2}}{2}$: (ii) $0 < L < \bar{L}$, (ii)(iii) $L = \bar{L}$, (iii)(iv) $\bar{L} < L < \bar{\bar{L}}$, (iv) $L \geqslant \bar{\bar{L}}$; Case (B) $\frac{4\sqrt{30}}{75} \leqslant \varepsilon < \frac{\sqrt{30}}{10}$: (i) $0 < L < \bar{L}$, (ii)(iii) $L = \bar{L}$, (ii) $L_\gamma < L \leqslant \bar{L}$, (iii)(iv) $\bar{L} < L < \bar{\bar{L}}$, (iv) $L \geqslant \bar{\bar{L}}$; Case (C) $\tilde{\varepsilon} \leqslant \varepsilon < \frac{4\sqrt{30}}{75}(\approx 0.29212)$: (i) $0 < L \leqslant L_\gamma$, (ii) $L_\gamma < L < \bar{L}$, (ii)(iii) $L = \bar{L}$, (iii)(iv) $L > \bar{L}$; Case (D) $\frac{19}{100} \leqslant \varepsilon < \tilde{\varepsilon}(\approx 0.26262)$: (i) $0 < L \leqslant L_\gamma$, (ii) $L_\gamma < L < \bar{L}$, (ii)(iii) $L = \bar{L}$, (iii) $L > \bar{L}$.

Theorem 2.4. *If* $\frac{4\sqrt{30}}{75} \leqslant \varepsilon < \frac{\sqrt{30}}{10}$, *then the bifurcation curve* $\mathcal{C}_\varepsilon$ *for* (1.2) *is* $\supset$*-like shaped or S-like shaped or strictly increasing. Precisely, there exist positive numbers* $L_\gamma < \bar{L} < \bar{\bar{L}}$ *such that the following assertions hold:*

(i) (See Fig.4(i)) *If* $0 < L \leqslant L_\gamma$, *then* $\mathcal{C}_\varepsilon$ *is* $\supset$*-shaped. Precisely, there exist positive numbers* $\lambda_* < \lambda^*$ *such that* (1.2) *has no positive solution for* $\lambda \in (\lambda^*, \infty)$, *exactly one positive solution for* $\lambda \in (0, \lambda_*) \cup \{\lambda^*\}$, *exactly two positive solutions for* $\lambda \in [\lambda_*, \lambda^*)$.

(ii) (See Fig.4(ii)) *If* $L_\gamma < L < \bar{L}$, *then* $\mathcal{C}_\varepsilon$ *is* $\supset$*-like shaped, and the results in Theorem 2.3(i) hold.*

(iii) (See Fig.4(iii) or (iv)) *If* $\bar{L} < L < \bar{\bar{L}}$, *then* $\mathcal{C}_\varepsilon$ *is S-like shaped or strictly increasing, and the results in Theorem 2.3(ii) hold.*

(iv) (See Fig.4(iv)) *If* $L \geqslant \bar{\bar{L}}$, *then* $\mathcal{C}_\varepsilon$ *is strictly increasing, and the results in Theorem 2.2 hold.*

(v) (See Fig.4(ii) or (iii)) *If* $L = \bar{L}$, *then* $\mathcal{C}_\varepsilon$ *is* $\supset$*-like shaped or S-like shaped.*

Let $\tilde{\varepsilon}$ and ε_* be the constants given in Table 1.

Theorem 2.5. *If* $\tilde{\varepsilon} \leqslant \varepsilon < \frac{4\sqrt{30}}{75}$, *then the bifurcation curve* $\mathcal{C}_\varepsilon$ *for* (1.2) *is* $\supset$*–like shaped, S-like shaped or strictly increasing. Precisely, there exist positive numbers* $L_\gamma < \bar{L}$ *such that the following assertions hold:*

(i) (See Fig.4(i)) *If* $0 < L \leqslant L_\gamma$, *then* $\mathcal{C}_\varepsilon$ *is* $\supset$*-shaped, and the results in Theorem 2.4(i) hold.*

(ii) (See Fig.4(ii)) *If* $L_\gamma < L < \bar{L}$, *then* $\mathcal{C}_\varepsilon$ *is* $\supset$*-like shaped, and the results in Theorem 2.3(i) hold.*

(iii) (See Fig.4(iii) or (iv)) *If* $L > \bar{L}$, *then* $\mathcal{C}_\varepsilon$ *is S-like shaped or strictly increasing, and the results in Theorem 2.3(ii) hold.*

(iv) (See Fig.4(ii) or (iii)) *If* $L = \bar{L}$, *then* $\mathcal{C}_\varepsilon$ *is* $\supset$*-like shaped or S-like shaped.*

Theorem 2.6. *If $\frac{19}{100} \leqslant \varepsilon < \tilde{\varepsilon}$, then the bifurcation curve $\mathcal{C}_\varepsilon$ for (1.2) is $\supset$-like shaped or S-like shaped. Precisely, there exist positive numbers $L_\gamma < \bar{L}$ such that the following assertions hold:*

(i) (See Fig.4(i)) *If $0 < L \leqslant L_\gamma$, then $\mathcal{C}_\varepsilon$ is $\supset$-shaped, and all the results in Theorem 2.4(i) hold.*

(ii) (See Fig.4(ii)) *If $L_\gamma < L < \bar{L}$, then $\mathcal{C}_\varepsilon$ is $\supset$-like shaped, and all the results in Theorem 2.3(i) hold.*

(iii) (See Fig.4(iii)) *If $L > \bar{L}$, then $\mathcal{C}_\varepsilon$ is S-like shaped and (1.2) has at least three positive solutions for some $\lambda > 0$.*

(iv) (See Fig.4(ii) or (iii)) *If $L = \bar{L}$, then $\mathcal{C}_\varepsilon$ is $\supset$-like shaped or S-like shaped.*

Theorem 2.7. *If $\varepsilon_* < \varepsilon < \frac{19}{100}$, then the bifurcation curve $\mathcal{C}_\varepsilon$ for (1.2) is $\supset$-like shaped or S-like shaped, and it possibly consists of two disjoint C^1 components. Precisely, there exists $\bar{L} > 0$ such that the following assertions hold:*

(i) *If $0 < L < \bar{L}$, then one of the following assertions holds:*

(i-1) (See Fig.5(i-1)) *$\mathcal{C}_\varepsilon$ consists of two disjoint C^1 components, where the lower component is $\supset$-shaped and the upper component is reversed S-like shaped or strictly decreasing. Moreover, there exist positive numbers $\lambda_* < \hat{\lambda} < \check{\lambda} < \lambda^*$ such that (1.2) has no positive solutions for $\lambda \in (\lambda_*, +\infty)$, exactly one positive solution for $\lambda \in (\hat{\lambda}, \check{\lambda}) \cup \{\lambda^*\}$, at least one positive solution for $\lambda \in (0, \lambda_*)$, exactly two positive solutions for $\lambda \in [\check{\lambda}, \lambda^*)$, at least two positive solutions for $\lambda \in [\lambda_*, \hat{\lambda}]$.*

(i-2) (See Fig.5(i-2)) *$\mathcal{C}_\varepsilon$ is C^1 and $\supset$-like shaped, and the results in Theorem 2.3(i) hold.*

(ii) *If $L > \bar{L}$, then one of the following assertions holds:*

(ii-1)(See Fig.5(ii-1)) *$\mathcal{C}_\varepsilon$ consists of two disjoint C^1 components, where the lower component is $\supset$-shaped and the upper component is $\subset$-like shaped. Moreover, there exist positive numbers $\tilde{\lambda} < \lambda_* < \hat{\lambda} < \check{\lambda} < \lambda^*$ such that (1.2) has no positive solution for $\lambda \in (\lambda^*, +\infty)$, exactly one positive solution for $\lambda \in (\hat{\lambda}, \check{\lambda}) \cup \{\lambda^*\}$, at least one positive solution for $\lambda \in (0, \tilde{\lambda})$, exactly two positive solutions for $[\check{\lambda}, \lambda^*)$, at least two positive solutions for $\lambda \in (\lambda_*, \hat{\lambda}] \cup \{\tilde{\lambda}\}$, at least three positive solutions for $\lambda \in (\tilde{\lambda}, \lambda_*]$.*

(ii-2)(See Fig.5(ii-2)) *$\mathcal{C}_\varepsilon$ is C^1 and S-like shaped, and (1.2) has at least three positive solutions for some $\lambda > 0$.*

(iii) *If $L = \bar{L}$, then either (i) or (ii) occurs.*

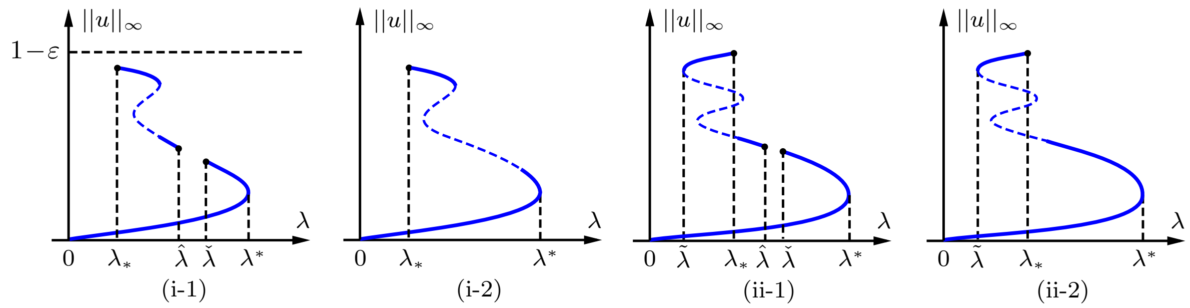

Figure 5: $\supset$-like shaped and S-like shaped bifurcation curves $\mathcal{C}_\varepsilon$ (possibly split into two disjoint C^1 components) of (1.2) with $\varepsilon_*(\approx 0.13123) < \varepsilon < \frac{19}{100}$: (i) $0 < L < \bar{L}$; (ii) $L > \bar{L}$.

Theorem 2.8. *If $0 < \varepsilon \leqslant \varepsilon_*$, then the bifurcation curve $\mathcal{C}_\varepsilon$ for (1.2) is $\supset$-like shaped or S-like shaped, and possibly consists of two disjoint C^1 components. Precisely, there exist positive numbers L_*, L^*, and $\bar{L}$, with*

$L_* < \min\{\bar{L}, L^*\}$, *such that the following assertions hold:*

(i) (See Fig.6(i)) *If* $0 < L < L_*$, *then* $\mathcal{C}_\varepsilon$ *is* $\supset$*-shaped, and the results in Theorem 2.4(i) hold.*

(ii) (See Fig.6(ii)) *If* $L_* \leqslant L < L^*$ *and* $L < \bar{L}$, *then* $\mathcal{C}_\varepsilon$ *consists of two disjoint* C^1 *components, where the lower component is* $\supset$*-shaped and the upper component is reversed* S*-like shaped or strictly decreasing, and the results in Theorem 2.7(i-1) hold.*

(iii) *If* L *is between* $\bar{L}$ *and* L^*, *then one of the following assertions holds:*

(iii-1) (See Fig.6(iii-1)) *If* $L^* > \bar{L}$ *and* $\bar{L} < L < L^*$, *then* $\mathcal{C}_\varepsilon$ *consists of two disjoint* C^1 *components, where the lower component is* $\supset$*-shaped and the upper component is* $\subset$*-like shaped, and the results in Theorem 2.7(ii-1) hold.*

(iii-2) (See Fig.6(iii-2)) *If* $L^* < \bar{L}$ *and* $L^* \leqslant L < \bar{L}$, *then* $\mathcal{C}_\varepsilon$ *is* $\supset$*-like shaped, and all results in Theorem 2.3(i) hold.*

(iv) (See Fig.6(iv)) *If* $L > \bar{L}$ *and* $L \geqslant L^*$, *then* $\mathcal{C}_\varepsilon$ *is* C^1 *and* S*-like shaped, and* (1.2) *has at least three positive solutions for some* $\lambda > 0$.

(v) *If* $L = \bar{L}$, *then either (ii) or (iii) occurs.*

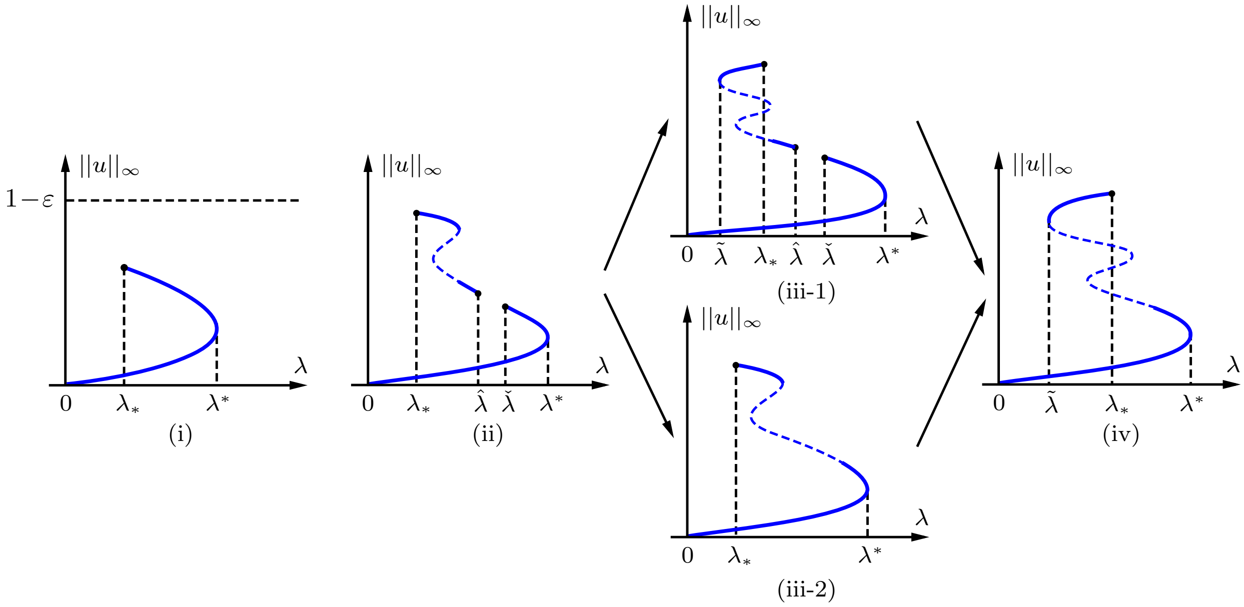

Figure 6: $\supset$-shaped, $\supset$-like shaped, and S-like shaped bifurcation curves $\mathcal{C}_\varepsilon$ of (1.2) with $0 < \varepsilon \leqslant \varepsilon_*(\approx 0.13123)$. (i) $0 < L < L_*$; (ii) $L_* \leqslant L < L^*$ and $L < \bar{L}$; (iii-1) $L^* > \bar{L}$ and $\bar{L} < L < L^*$; (iii-2) $L^* < \bar{L}$ and $L^* \leqslant L < \bar{L}$; (iv) $L > \bar{L}$ and $L \geqslant L^*$.

The proofs will be given in Section 4.

3. Lemmas

Let $\varepsilon \in (0, 1)$. Set $f(u) := \frac{1}{(1-u)^2} - \frac{\varepsilon^2}{(1-u)^4}$, $a := a(\varepsilon) = 1 - \sqrt{2}\varepsilon$, and $\gamma := \gamma(\varepsilon) = 1 - \frac{\sqrt{30}}{3}\varepsilon$. Then $f(u)$ satisfies the following properties:

(a) $f(u) > 0$ on $[0, 1-\varepsilon)$ and $f(1-\varepsilon) = 0$; $f(u) \in C^\infty[0, 1-\varepsilon]$.

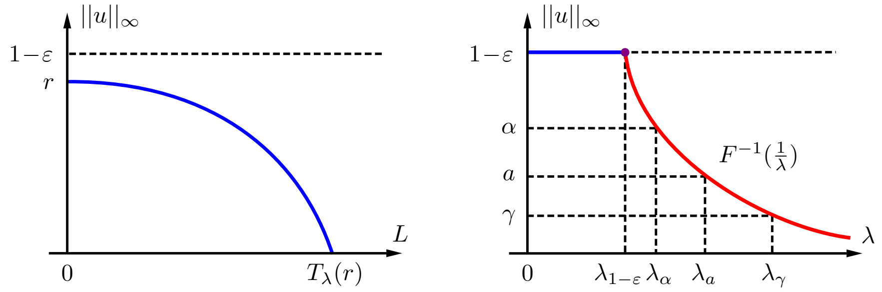

Figure 7: Time-map $T_\lambda(r)$ and the domain $\mathcal{D}_{\varepsilon,\lambda}$. The horizontal line $\|u\|_\infty = 1-\varepsilon$, the blow-up curve $\|u\|_\infty = F^{-1}(\frac{1}{\lambda})$, and λ_α for any $\alpha \in (0, 1-\varepsilon]$, the special values $\lambda_{1-\varepsilon} := \frac{1}{F(1-\varepsilon)}$ for $\varepsilon \in (0,1)$, $\lambda_a := \lambda_{a(\varepsilon)}$ for $\varepsilon \in (0, \frac{\sqrt{2}}{2})$, and $\lambda_\gamma := \lambda_{\gamma(\varepsilon)}$ for $\varepsilon \in (0, \frac{\sqrt{30}}{10})$.

(b) If $\varepsilon \in (0, \frac{\sqrt{2}}{2})$, then $a > 0$, $f'(u) > 0$ on $[0, a)$, $f'(a) = 0$, and $f'(u) < 0$ on $(a, 1-\varepsilon]$. If $\varepsilon \in [\frac{\sqrt{2}}{2}, 1)$, then $f'(u) < 0$ on $(0, 1-\varepsilon]$.
(c) If $\varepsilon \in (0, \frac{\sqrt{30}}{10})$, then $\gamma > 0$, $f''(u) > 0$ on $[0, \gamma)$, $f''(\gamma) = 0$, and $f''(u) < 0$ on $(\gamma, 1-\varepsilon]$. If $\varepsilon \in [\frac{\sqrt{30}}{10}, 1)$, then $f''(u) < 0$ on $(0, 1-\varepsilon]$.

Set $F(u) = \int_0^u f(t)\,\mathrm{d}t$. Then $F(u)$ is strictly increasing when $f(u)$ is nonnegative. Define

$$\lambda_\alpha := \frac{1}{F(\alpha)} \qquad \text{for } \alpha \in (0, 1-\varepsilon];$$

see the right of Fig.7. Clearly, λ_α is strictly decreasing with respect to α. Since all positive solutions of (1.2) are symmetric and even, the time-map formula which we apply to study (1.2) takes the form ([6, 15]):

$$T_\lambda(r) = \int_0^r \frac{1 + \lambda F(u) - \lambda F(r)}{\sqrt{1 - [1 + \lambda F(u) - \lambda F(r)]^2}}\,\mathrm{d}u \tag{3.1}$$

$$= r\int_0^1 \frac{1 + \lambda F(rs) - \lambda F(r)}{\sqrt{1 - [1 + \lambda F(rs) - \lambda F(r)]^2}}\,\mathrm{d}s, \tag{3.2}$$

and it is well-defined for $r = u(0) = \|u\|_\infty \in \mathcal{D}_{\varepsilon,\lambda}$, where

$$\mathcal{D}_{\varepsilon,\lambda} = \begin{cases} \left(0, F^{-1}(\frac{1}{\lambda})\right], & \text{if } \lambda > \lambda_{1-\varepsilon}, \\ (0, 1-\varepsilon), & \text{if } \lambda \leqslant \lambda_{1-\varepsilon}. \end{cases} \tag{3.3}$$

That is, the domain $\mathcal{D}_{\varepsilon,\lambda}$ of $T_\lambda(r)$ depends on the values of ε and λ; see Fig.7. Moreover, by [6, Lemma 3.1] and [15, Lemmas 3.6 and 3.7], the smoothness of f implies that $T_\lambda(r)$ is smooth with respect to $r \in \mathcal{D}_{\varepsilon,\lambda}$ as well as to $\lambda \in (0, \lambda_r]$ (the derivative at the endpoint means the left derivative).

From the uniqueness of solution for the associated initial value problem, it follows that each positive solution u of (1.2) just corresponds to a solution of the equation

$$T_\lambda(r) = L \quad \text{satisfying } r = \|u\|_\infty. \tag{3.4}$$

Thus ones can derive global bifurcation curves and exact numbers of positive solutions for (1.2) by analyzing shapes of $T_\lambda(r)$ on the domain $\mathcal{D}_{\varepsilon,\lambda}$. Moreover, since the explicit formula (3.1) of the time-map comes from the first integral (the energy identity)

$$1 - \frac{1}{\sqrt{1 + (u')^2}} = \lambda F(r) - \lambda F(u),$$

it follows that when $r = F^{-1}\big(\frac{1}{\lambda}\big)$, the associated solution u must satisfy $|u'(\pm L)| = \infty$, and hence this solution only belongs to $C^2(-L,L)\bigcap C[-L,L]$, not $C^2[-L,L]$. Other solutions corresponding to $r \neq F^{-1}\big(\frac{1}{\lambda}\big)$ actually belong to $C^2[-L,L]$ and have higher regularity due to the boundedness of the derivatives.

For any given $\varepsilon \in (0,1)$, denote by h the supremum of T_λ on $\mathcal{D}_{\varepsilon,\lambda}$, i.e.,

$$h(\lambda) := \sup\big\{T_\lambda(r) \mid r \in \big(0, F^{-1}(\frac{1}{\lambda})\big]\big\} \quad \text{ for } \lambda > \lambda_{1-\varepsilon}.$$

The following lemmas give some properties of $T_\lambda(r)$ and $h(\lambda)$.

Lemma 3.1. *Let $T_\lambda(r)$ and $h(\lambda)$ be defined as above. Then the following assertions hold:*

(i) *For any $r \in (0, 1-\varepsilon)$, $T_\lambda(r)$ is strictly decreasing with respect to $\lambda \in (0, \lambda_r]$ and $\lim_{\lambda\to 0} T_\lambda(r) = +\infty$.*

(ii) *$\lim_{r\to 0} T_\lambda(r) = 0$ and $\lim_{r\to 0} T'_\lambda(r) = +\infty$ for all $\lambda > 0$.*

(iii) *$h(\lambda)$ is well-defined, continuous, and strictly decreasing with respect to $\lambda > \lambda_{1-\varepsilon}$. Moreover, $\lim_{\lambda\to+\infty} h(\lambda) = 0$.*

Proof. (i) Notice that for any $r \in (0, 1-\varepsilon)$, $(0, r] \subset (0, F^{-1}(\frac{1}{\lambda})]$ for all $\lambda \in (0, \lambda_r]$. Since $0 < F(r) - F(u) < \frac{1}{\lambda}$ for all $0 < u < r < 1-\varepsilon$ and $t \mapsto \frac{1-t}{\sqrt{1-(1-t)^2}}$ is strictly decreasing with respect to $t \in (0,1)$, it follows from (3.1) that T is strictly decreasing with respect to $\lambda \in (0, \lambda_r]$. Moreover, since $\sqrt{1-[1+\lambda F(u) - \lambda F(1-\varepsilon)]^2} \to 0^+$ and $1 + \lambda F(u) - \lambda F(1-\varepsilon) \to 1^-$ as $\lambda \to 0$, it follows from (3.1) that $\lim_{\lambda\to 0} T_\lambda(r) = +\infty$.

(ii) Since $f(0) = 1 - \varepsilon^2 > 0$, by [12, Propositions 2.6 and 2.7], we obtain the results of limits.

(iii) First, since $T_\lambda(r)$ is continuous with respect to λ and r and $\lim_{r\to 0} T_\lambda(r) = 0$, it follows that $h(\lambda) < +\infty$ is well-defined and continuous. Second, since $\lim_{r\to 0} T_\lambda(r) = 0$, by the proof of [12, Proposition 2.13(2)], we immediately obtain that $h(\lambda)$ is strictly decreasing with respect to $\lambda > \lambda_{1-\varepsilon}$. At final, since $\lim_{\lambda\to+\infty} F^{-1}(\frac{1}{\lambda}) = 0$ and $r \in (0, F^{-1}(\frac{1}{\lambda})]$ as λ is enough large, it follows from (ii) that $\lim_{\lambda\to+\infty} h(\lambda) = 0$. $\square$

In what follows, we denote $\frac{\partial T_\lambda(r)}{\partial r}$ by $T'_\lambda(r)$ for simplicity.

Lemma 3.2. *For any $\varepsilon \in [\frac{\sqrt{2}}{2}, 1)$ and $\lambda > 0$, $T'_\lambda(r) > 0$ on $\mathcal{D}_{\varepsilon,\lambda}$.*

Proof. If $\varepsilon \geqslant \frac{\sqrt{2}}{2}$, then $f'(u) < 0$ on $(0, 1-\varepsilon)$. For any $r \in \mathcal{D}_{\varepsilon,\lambda}$, by setting $s = 1 - \lambda F(r)$ and $y = 1 + \lambda F(u) - \lambda F(r)$, we obtain from (3.1) that

$$T_\lambda(r) = \widetilde{T}_\lambda(s) = \int_s^1 \frac{y}{\sqrt{1-y^2}} \frac{1}{\lambda f\big(F^{-1}\big(\frac{y-s}{\lambda}\big)\big)}\, \mathrm{d}y.$$

Differentiating yields

$$\begin{aligned} T'_\lambda(r) &= \widetilde{T}'_\lambda(s) \cdot \frac{\mathrm{d}s}{\mathrm{d}r} \\ &= \left(-\frac{s}{\sqrt{1-s^2}} \frac{1}{\lambda f(F^{-1}(0))} + \int_s^1 \frac{y}{\sqrt{1-y^2}} \frac{f'\big(F^{-1}\big(\frac{y-s}{\lambda}\big)\big)}{\lambda^2 \big[f\big(F^{-1}\big(\frac{y-s}{\lambda}\big)\big)\big]^3}\, \mathrm{d}y \right) \cdot (-\lambda f(r)), \end{aligned} \tag{3.5}$$

where the first term of the right-hand side is well-defined due to $f(0) > 0$. Since $f'(u) < 0$ on $(0, 1-\varepsilon)$, it follows from (3.5) that $T'_\lambda(r) > 0$ for all $r \in \mathcal{D}_{\varepsilon,\lambda}$. $\square$

Lemma 3.3. *For the time-map $T_\lambda(r)$, the following assertions hold:*

(i) *For $\varepsilon \in (0, \frac{\sqrt{2}}{2})$ and $\lambda \geqslant \lambda_a$, $T_\lambda(r)$ has at least one critical point, a local maximum, on $\big(0, F^{-1}\big(\frac{1}{\lambda}\big)\big]$.*

(ii) *For $\varepsilon \in (0, \frac{\sqrt{30}}{10})$ and $\lambda \geqslant \lambda_\gamma$, $T_\lambda(r)$ has exactly one critical point, a local maximum, on $\big(0, F^{-1}\big(\frac{1}{\lambda}\big)\big]$.*

Proof. (i) If $\varepsilon < \frac{\sqrt{2}}{2}$, then $a > 0$. For any $y \in (0,1)$ and $\lambda \geqslant \lambda_a$, $F^{-1}\left(\frac{y}{\lambda}\right) < F^{-1}(\frac{1}{\lambda}) \leqslant F^{-1}(\frac{1}{\lambda_a}) = a$. Thus $f'(F^{-1}(\frac{y}{\lambda})) > 0$ for all $y \in (0,1)$.

Let $r = F^{-1}\big(\frac{1}{\lambda}\big)$ in (3.5). Then $s = 0$ and

$$T_\lambda'(F^{-1}(\frac{1}{\lambda})) = -\int_0^1 \frac{y}{\sqrt{1-y^2}} \frac{f'\big(F^{-1}\big(\frac{y}{\lambda}\big)\big)}{\lambda^2 \left[f\big(F^{-1}\big(\frac{y}{\lambda}\big)\big)\right]^3} \, \mathrm{d}y \cdot \lambda f(F^{-1}(\frac{1}{\lambda})) < 0,$$

since $f'(F^{-1}(\frac{y}{\lambda})) > 0$ for all $y \in (0,1)$. Combining Lemma 3.1(ii), we obtain that $T_\lambda'(r)$ has at least one critical point, a local maximum, on $\big(0, F^{-1}\big(\frac{1}{\lambda}\big)\big]$.

(ii) If $\varepsilon < \frac{\sqrt{30}}{10}$, then $\gamma > 0$. For any $y \in (0,1)$ and $\lambda \geqslant \lambda_\gamma$, $F^{-1}(\frac{y}{\lambda}) < F^{-1}(\frac{1}{\lambda}) \leqslant F^{-1}(\frac{1}{\lambda_\gamma}) = \gamma < a$. Thus $f'(F^{-1}(\frac{y}{\lambda})) > 0$ for all $y \in (0,1)$. The same argument as in (i) gives that $T_\lambda'(F^{-1}(\frac{1}{\lambda})) < 0$ for all $\varepsilon \in (0, \frac{\sqrt{10}}{3})$ and $\lambda \geqslant \lambda_\gamma$. Thus, $T_\lambda(r)$ has at least one critical point on $\big(0, F^{-1}\big(\frac{1}{\lambda}\big)\big]$.

If $\lambda \geqslant \lambda_\gamma$, then $\big(0, F^{-1}\big(\frac{1}{\lambda}\big)\big) \subset (0, \gamma)$. Since f is convex on $\big(0, F^{-1}\big(\frac{1}{\lambda}\big)\big]$ for $\lambda \geqslant \lambda_\gamma$, by the proof of [8, Theorem 3.4] or [15, Theorem 2.1], we obtain that $T_\lambda(r)$ has at most one critical point on $\big(0, F^{-1}\big(\frac{1}{\lambda}\big)\big]$. Therefore, $T_\lambda(r)$ has exactly one critical point, a local maximum, on $\big(0, F^{-1}\big(\frac{1}{\lambda}\big)\big]$. $\square$

Next we give some auxiliary functions to investigate the properties of $T_\lambda'(r)$.

By (3.1), we obtain that

$$\begin{aligned} T_\lambda'(r) &= \int_0^1 \frac{1 + \lambda F(rs) - \lambda F(r)}{\sqrt{1 - [1 + \lambda F(rs) - \lambda F(r)]^2}} \, \mathrm{d}s + \int_0^1 \frac{\lambda r [f(rs)s - f(r)]}{\{1 - [1 + \lambda F(rs) - \lambda F(r)]^2\}^{3/2}} \, \mathrm{d}s \\ &= \int_0^1 \frac{[1 + \lambda F(rs) - \lambda F(r)]\{1 - [1 + \lambda F(rs) - \lambda F(r)]^2\} + \lambda r [f(rs)s - f(r)]}{\{1 - [1 + \lambda F(rs) - \lambda F(r)]^2\}^{3/2}} \, \mathrm{d}s. \end{aligned} \tag{3.6}$$

Set

$$\Delta F = F(r) - F(rs) \quad \text{and} \quad \Delta f = rf(r) - rsf(rs) \qquad \text{for } s \in (0,1).$$

Notice that $0 < \lambda \Delta F < 1$ for all $\lambda > 0$, $r \in \mathcal{D}_{\varepsilon,\lambda}$ and $s \in (0,1)$. Then differentiating yields that

$$T_\lambda'(r) = \int_0^1 \frac{(1 - \lambda\Delta F)(1 - (1 - \lambda\Delta F)^2) - \lambda\Delta f}{\{1 - (1 - \lambda\Delta F)^2\}^{3/2}} \, \mathrm{d}s = \int_0^1 \frac{\lambda^3 \Delta F^3 - 3\lambda^2 \Delta F^2 + 2\lambda\Delta F - \lambda\Delta f}{\{1 - (1 - \lambda\Delta F)^2\}^{3/2}} \, \mathrm{d}s \tag{3.7}$$

$$\begin{aligned} &< \int_0^1 \frac{\lambda^2 \Delta F^2 - 3\lambda^2 \Delta F^2 + 2\lambda\Delta F - \lambda\Delta f}{\{1 - (1 - \lambda\Delta F)^2\}^{3/2}} \, \mathrm{d}s < \int_0^1 \frac{\lambda(2\Delta F - \Delta f)}{\{1 - (1 - \lambda\Delta F)^2\}^{3/2}} \, \mathrm{d}s \\ &= \frac{\lambda}{r} \int_0^r \frac{\theta_\varepsilon(r) - \theta_\varepsilon(u)}{\{1 - [1 + \lambda F(u) - \lambda F(r)]^2\}^{3/2}} \, \mathrm{d}u, \end{aligned} \tag{3.8}$$

where

$$\theta_\varepsilon(u) := 2F(u) - uf(u) = 2\int_0^u f(t) \, \mathrm{d}t - uf(u). \tag{3.9}$$

Then

$$\theta_\varepsilon'(u) = f(u) - uf'(u) = \frac{(1-u)^2(1-3u) - \varepsilon^2(1-5u)}{(1-u)^5}.$$

By [27, Lemma 3], $\theta_\varepsilon(u)$ is strictly increasing on $[\check{\varepsilon}, 1)$, where

$$\check{\varepsilon} := \max_{u\in\left(\frac{1}{3},1\right)} \sqrt{\frac{(1-u)^2(1-3u)}{1-5u}} = \frac{4\sqrt{30}}{75}(\approx 0.29212). \tag{3.10}$$

The following lemma gives more properties of $\theta_\varepsilon(u)$.

Lemma 3.4 ([27]). *The function $\theta_\varepsilon(u)$ defined by (3.9) satisfies the following properties:*
(i) *$\theta_\varepsilon(0) = 0$ and $\theta'_\varepsilon(0) = 1 - \varepsilon^2 > 0$ for all $\varepsilon \in (0, 1)$.*
(ii) *There exists a positive constant $\bar{\varepsilon}(\approx 0.22793)$ such that $\theta_\varepsilon(\gamma(\varepsilon)) \leqslant 0$ for all $\varepsilon \in (0, \bar{\varepsilon}]$.*
(iii) *If $\varepsilon \leqslant \check{\varepsilon}$, then there exist two numbers $p_1 := p_1(\varepsilon)$ and $p_2 := p_2(\varepsilon)$, with $0 < p_1 \leqslant \gamma \leqslant p_2 < 1-\varepsilon$, such that*

$$\theta'_\varepsilon(u) \begin{cases} < 0 & \text{on } (p_1, p_2); \\ = 0 & \text{if } u = p_1, p_2; \\ > 0 & \text{on } [0, p_1) \cup (p_2, 1-\varepsilon). \end{cases} \tag{3.11}$$

Moreover, $p_1(\varepsilon) < \gamma(\varepsilon) < p_2(\varepsilon)$ for all $\varepsilon \in (0, \check{\varepsilon})$ and $p_1(\check{\varepsilon}) = \gamma(\check{\varepsilon}) = p_2(\check{\varepsilon}) = \frac{7}{15}$.

We also need another auxiliary function $H_\varepsilon(u)$.

Lemma 3.5 ([27]). *Let $H_\varepsilon(u) := \int_0^u t\theta'_\varepsilon(t)\,\mathrm{d}t$ and $\bar{\varepsilon}, \check{\varepsilon}$ be given as above. Then both $H_\varepsilon(p_2(\varepsilon))$ and $H_\varepsilon(\gamma(\varepsilon))$ are strictly increasing functions of $\varepsilon \in (0, \tilde{\varepsilon})$, and there exist two constants $\hat{\varepsilon}, \tilde{\varepsilon}$ with $\bar{\varepsilon} < \hat{\varepsilon} < \tilde{\varepsilon} < \check{\varepsilon}$, such that*

$$H_{\hat{\varepsilon}}(\gamma(\hat{\varepsilon})) = 0 \quad \text{and} \quad H_\varepsilon(\gamma(\varepsilon)) < 0 \quad \text{for all } \varepsilon < \hat{\varepsilon},$$
$$H_{\tilde{\varepsilon}}(p_2(\tilde{\varepsilon})) = 0 \quad \text{and} \quad H_\varepsilon(p_2(\varepsilon)) < 0 \quad \text{for all } \varepsilon < \tilde{\varepsilon},$$

where $\hat{\varepsilon} = \sup\{\varepsilon < \check{\varepsilon} \mid H_\varepsilon(\gamma(\varepsilon)) < 0\}$ and $\tilde{\varepsilon} = \sup\{\varepsilon < \check{\varepsilon} \mid H_\varepsilon(p_2(\varepsilon)) < 0\}$ (numerical evaluation shows that $\hat{\varepsilon} \approx 0.25458$ and $\tilde{\varepsilon} \approx 0.26262$).

Lemma 3.6. *Let $T_\lambda(r)$, $\gamma(\varepsilon)$, and $p_2(\varepsilon)$ be given as above. Then the following assertions hold:*
(i) *If $0 < \varepsilon < \hat{\varepsilon}$ and $\lambda < \lambda_\gamma$, then $T_\lambda(r)$ has exactly one critical point on $(0, \gamma)$ and $T'_\lambda(\gamma) < 0$.*
(ii) *If $0 < \varepsilon < \tilde{\varepsilon}$ and $\lambda < \lambda_{p_2}$, then $T_\lambda(r)$ has at least one critical point on $(0, p_2)$ and $T'_\lambda(p_2) < 0$.*

Proof. With the aids of Lemmas 3.4 and 3.5, we obtain the results from inequality (3.8), following the same line as in the proof of [4, Lemma 3.10(i)(ii)]. So we omit the details. □

Next we investigate the behaviors of $T'_\lambda(r)$ at the right endpoint of $\mathcal{D}_{\varepsilon,\lambda}$.

Lemma 3.7. *For any $\varepsilon \in (0, 1)$, $\lim_{\lambda\to(\lambda_{1-\varepsilon})^+} T'_\lambda\big(F^{-1}\big(\frac{1}{\lambda}\big)\big) = +\infty$.*

The proof of Lemma 3.7 will be provided after that of Lemma 3.12 below.

Define

$$G(\lambda) := T'_\lambda\big(F^{-1}\big(\frac{1}{\lambda}\big)\big) \quad \text{for } \lambda > \lambda_{1-\varepsilon}. \tag{3.12}$$

Then $G(\lambda)$ is continuous with respect to $\lambda \in (\lambda_{1-\varepsilon}, +\infty)$. In the earlier works on exact multiplicity of positive solutions for (1.1), the nonlinearity f is usually chosen to be a monotonic function. According to

[12, Proposition 2.10], $G(\lambda)$ has the opposite monotonicity to f provided that f is a monotonic function. In the present paper, f is not monotonic, which brings new difficulties and more complexity. On the one hand, according to the proof of Lemma 3.3(i), $G(\lambda) < 0$ for all $\lambda \geqslant \lambda_a$ when $\varepsilon \in (0, \frac{\sqrt{2}}{2})$. On the other hand, $\lim_{\lambda \to (\lambda_{1-\varepsilon})^+} G(\lambda) > 0$ by Lemma 3.7(ii). So $G(\lambda)$ changes sign at least once on $(\lambda_{1-\varepsilon}, +\infty)$ for $\varepsilon \in (0, \frac{\sqrt{2}}{2})$.

To give the exact number of sign changes of $G(\lambda)$ on $(\lambda_{1-\varepsilon}, +\infty)$, we need the following crucial lemma, which was proved by using the theory of total positivity. About this theory and the relevant applications, we refer the reader to [39, 40].

Lemma 3.8 ([14]). *Let n be a positive integer. Then the integral operator*

$$(\mathcal{L}_n \varphi)(\mu) = \int_0^\mu \left(\mu^2 - z^2\right)^{n-\frac{1}{2}} z\varphi(z) dz$$

is variation diminishing provided that φ changes sign no more than n times. Precisely, if φ has no more than n changes of sign on $\mathbb{R}^+$, then $\mathcal{L}_n\varphi$ has at most as many sign changes on $\mathbb{R}^+$ as φ has.

Note that zeros do not count as sign changes, and $\mathbb{R}^+$ can be replaced with a finite interval $(0, \rho)$.

The following result can be applied to more general problem (1.1) than (1.2) and hence has independent interest.

Lemma 3.9. *Consider problem (1.1). Suppose $f(u) \in C^2([0, \rho))$ and $f(u) > 0$ on $[0, \rho)$ for some $\rho \in (0, +\infty]$. Let $G(\lambda)$ be defined on $(\frac{1}{F(\rho)}, +\infty)$ by (3.12) (here, $\frac{1}{F(\rho)} = 0$ if $F(\rho) = +\infty$). Set*

$$\omega_1(t) := 3f'(t)f^2(t) + F(t)f''(t)f(t) - 3F(t)f'^2(t). \tag{3.13}$$

If $\omega_1(t)$ changes sign no more than once on $(0, \rho)$, then $G(\lambda)$ changes sign no more than once on $(\frac{1}{F(\rho)}, +\infty)$.

Proof. By (3.5) and (3.12), we have

$$G(\lambda) = T'_\lambda\big(F^{-1}(\frac{1}{\lambda})\big) = \widetilde{T}'(0)\frac{\mathrm{d}s}{\mathrm{d}r}(F^{-1}(\frac{1}{\lambda})) = -\lambda f(F^{-1}(\frac{1}{\lambda}))\int_0^1 \frac{y}{\sqrt{1-y^2}} \frac{f'\big(F^{-1}(\frac{y}{\lambda})\big)}{\lambda^2\left[f\big(F^{-1}(\frac{y}{\lambda})\big)\right]^3} \mathrm{d}y,$$

where $s = 1 - \lambda F(r)$ and $y = 1 + \lambda F(u) - \lambda F(r)$. To apply Lemma 3.8, we take $z = \frac{y}{\lambda}$ and change $G(\lambda)$ into the form

$$\frac{-1}{\lambda f(F^{-1}(\frac{1}{\lambda}))} G(\lambda) = \int_0^{\frac{1}{\lambda}} \frac{\lambda z}{\sqrt{1-(\lambda z)^2}} \frac{f'(F^{-1}(z))}{\lambda^2[f(F^{-1}(z))]^3} \lambda \,\mathrm{d}z = \int_0^{\frac{1}{\lambda}} \frac{z}{\sqrt{1-(\lambda z)^2}} \frac{f'(F^{-1}(z))}{f^3(F^{-1}(z))} \mathrm{d}z.$$

Taking $\mu = \frac{1}{\lambda}$, we further have

$$\frac{-1}{f(F^{-1}(\frac{1}{\lambda}))} G(\lambda) = \frac{-1}{f(F^{-1}(\mu))} G(\frac{1}{\mu}) = \int_0^\mu \frac{z}{\sqrt{\mu^2 - z^2}} \frac{f'(F^{-1}(z))}{f^3(F^{-1}(z))} \mathrm{d}z = \int_0^\mu \frac{1}{\sqrt{\mu^2 - z^2}} z\mathcal{H}(z) \,\mathrm{d}z, \tag{3.14}$$

where

$$\mathcal{H}(z) := \frac{f'(F^{-1}(z))}{f^3(F^{-1}(z))}. \tag{3.15}$$

Multiplying (3.14) by μ^2 and integrating by parts, similar to [14, (2.6)], we have

$$\frac{-1}{\lambda^2 f(F^{-1}(\frac{1}{\lambda}))} G(\lambda) = \frac{-\mu^2}{f(F^{-1}(\mu))} G(\frac{1}{\mu}) = \int_0^\mu \frac{\mu^2}{\sqrt{\mu^2 - z^2}} z\mathcal{H}(z) \,\mathrm{d}z$$

$$= \int_0^\mu \sqrt{\mu^2 - z^2} z\mathcal{H}(z) \, \mathrm{d}z + \int_0^\mu \frac{z^2}{\sqrt{\mu^2 - z^2}} z\mathcal{H}(z) \, \mathrm{d}z$$
$$= \int_0^\mu \sqrt{\mu^2 - z^2} z \left[3\mathcal{H}(z) + z\mathcal{H}'(z) \right] \mathrm{d}z.$$

From (3.15), we obtain that

$$\mathcal{H}'(z) = \left(\frac{f'(F^{-1}(z))}{f^3(F^{-1}(z))} \right)' = \frac{f''(F^{-1}(z))f(F^{-1}(z)) - 3f'^2(F^{-1}(z))}{f^5(F^{-1}(z))}.$$

Set

$$\varphi(z) := 3\mathcal{H}(z) + z\mathcal{H}'(z) = \frac{3f'(F^{-1}(z))f^2(F^{-1}(z)) + zf''(F^{-1}(z))f(F^{-1}(z)) - 3zf'^2(F^{-1}(z))}{f^5(F^{-1}(z))}.$$

Since $f(u)$ is positive on $[0, \rho)$, it follows that F is increasing and F^{-1} is well defined on $[0, F(\rho))$. Letting $t = F^{-1}(z)$, we obtain that $t \in [0, \rho)$ and

$$\varphi(t) = \frac{3f'(t)f^2(t) + F(t)f''(t)f(t) - 3F(t)f'^2(t)}{f^5(t)} =: \frac{\omega_1(t)}{f^5(t)},$$

where ω_1 is defined in (3.13). So it is clear that $\varphi(t)$ and $\omega_1(t)$ have the same number of sign changes on $(0, \rho)$. If $\omega_1(t)$ changes sign no more than once on $(0, \rho)$, then so does $\varphi(t)$. Applying Lemma 3.8 to the case of $n = 1$, we immediately obtain that $G(\lambda)$ changes sign no more than once on $(\frac{1}{F(\rho)}, +\infty)$. $\square$

With the aid of the previous lemma, we next determine the number of the sign changes of $G(\lambda)$.

Lemma 3.10. *Consider problem* (1.2). *For any* $\varepsilon \in (0, \frac{\sqrt{2}}{2})$, $G(\lambda)$ *changes sign exactly once on* $(\lambda_{1-\varepsilon}, +\infty)$. *That is, there exists a unique* $\bar{\lambda} := \sup\{\lambda > 0 \mid G(\lambda) > 0\} > \lambda_{1-\varepsilon}$ *such that* $G(\lambda) \begin{cases} > 0, & \text{if } \lambda_{1-\varepsilon} < \lambda < \bar{\lambda}, \\ = 0, & \text{if } \lambda = \bar{\lambda}, \\ < 0, & \text{if } \lambda > \bar{\lambda}. \end{cases}$

Proof. By Lemma 3.9 and the analysis below (3.12), it suffices to prove that $\omega_1(u)$ changes sign at most once on $(0, 1 - \varepsilon)$.

Substituting $f(u) = \frac{1}{(1-u)^2} - \frac{\varepsilon^2}{(1-u)^4}$ into (3.13) yields that $\omega_1(u) = \frac{1}{(1-u)^{13}}\psi(u)$, where ψ is a polynomial of degree 7 with a parameter ε and is defined by

$$\psi(u) := (2\varepsilon^2 - 6)u^7 + (-14\varepsilon^2 + 42)u^6 + (-\tfrac{22}{3}\varepsilon^4 + 64\varepsilon^2 - 126)u^5 + (\tfrac{110}{3}\varepsilon^4 - 180\varepsilon^2 + 210)u^4 + (\tfrac{28}{3}\varepsilon^6 - \tfrac{304}{3}\varepsilon^4 + 290\varepsilon^2 - 210)u^3 + (-28\varepsilon^6 + 152\varepsilon^4 - 262\varepsilon^2 + 126)u^2 + (28\varepsilon^6 - 110\varepsilon^4 + 124\varepsilon^2 - 42)u - 12\varepsilon^6 + 30\varepsilon^4 - 24\varepsilon^2 + 6.$$

Then $\omega_1(u)$ and $\psi(u)$ have the same number of sign changes.

Set $\psi_i(u) := \frac{\partial^i}{\partial u^i}(\psi(u))$, $i = 1, 2, \ldots, 7$. Successive differentiation yields

$$\psi_1(u) = 7(2\varepsilon^2 - 6)u^6 + 6(-14\varepsilon^2 + 42)u^5 + 5(\frac{-22}{3}\varepsilon^4 + 64\varepsilon^2 - 126)u^4 + 4(\frac{110}{3}\varepsilon^4 - 180\varepsilon^2 + 210)u^3$$
$$+3(\frac{28}{3}\varepsilon^6 - \frac{304}{3}\varepsilon^4 + 290\varepsilon^2 - 210)u^2 + 2(-28\varepsilon^6 + 152\varepsilon^4 - 262\varepsilon^2 + 126)u + 28\varepsilon^6 - 110\varepsilon^4 + 124\varepsilon^2 - 42,$$
$$\psi_2(u) = 42(2\varepsilon^2 - 6)u^5 + 30(-14\varepsilon^2 + 42)u^4 + 20(\frac{-22}{3}\varepsilon^4 + 64\varepsilon^2 - 126)u^3 + 12(\frac{110}{3}\varepsilon^4 - 180\varepsilon^2 + 210)u^2$$

$$
\begin{aligned}
&\quad + 6(\frac{28}{3}\varepsilon^6 - \frac{304}{3}\varepsilon^4 + 290\varepsilon^2 - 210)u + 2(-28\varepsilon^6 + 152\varepsilon^4 - 262\varepsilon^2 + 126),\\
\psi_3(u) &= 210(2\varepsilon^2 - 6)u^4 + 120(-14\varepsilon^2 + 42)u^3 + 60(\frac{-22}{3}\varepsilon^4 + 64\varepsilon^2 - 126)u^2\\
&\quad + 24(\frac{110}{3}\varepsilon^4 - 180\varepsilon^2 + 210)u + 6(\frac{28}{3}\varepsilon^6 - \frac{304}{3}\varepsilon^4 + 290\varepsilon^2 - 210),\\
\psi_4(u) &= 840(2\varepsilon^2 - 6)u^3 + 360(-14\varepsilon^2 + 42)u^2 + 120(\frac{-22}{3}\varepsilon^4 + 64\varepsilon^2 - 126)u + 24(\frac{110}{3}\varepsilon^4 - 180\varepsilon^2 + 210),\\
\psi_5(u) &= 2520(2\varepsilon^2 - 6)u^2 + 720(-14\varepsilon^2 + 42)u + 120(\frac{-22}{3}\varepsilon^4 + 64\varepsilon^2 - 126),\\
\psi_6(u) &= 5040(2\varepsilon^2 - 6)u + 720(-14\varepsilon^2 + 42),\\
\psi_7(u) &= 5040(2\varepsilon^2 - 6).
\end{aligned}
$$

Then for any $\varepsilon \in (0, \frac{\sqrt{2}}{2})$, further computation yields:
(i) Since $\psi_7(u) < 0$ on $(0, 1-\varepsilon)$, and $\psi_6(1-\varepsilon) = -10080\varepsilon^3 + 30240\varepsilon > 0$, then $\psi_6(u) > 0$ on $(0, 1-\varepsilon)$.
(ii) Since $\psi_6(u) > 0$ on $(0, 1-\varepsilon)$, and $\psi_5(1-\varepsilon) = 4160\varepsilon^2(\varepsilon^2 - 3) < 0$, then $\psi_5(u) < 0$ on $(0, 1-\varepsilon)$.
(iii) Since $\psi_5(u) < 0$ on $(0, 1-\varepsilon)$, and $\psi_4(1-\varepsilon) = -800\varepsilon^3(\varepsilon^2 - 3) > 0$, then $\psi_4(u) > 0$ on $(0, 1-\varepsilon)$.
(iv) Since $\psi_4(u) > 0$ on $(0, 1-\varepsilon)$, and $\psi_3(1-\varepsilon) = 36\varepsilon^4(\varepsilon^2 - 3) < 0$, then $\psi_3(u) < 0$ on $(0, 1-\varepsilon)$.
(v) Since $\psi_2(0) = -56\varepsilon^6 + 304\varepsilon^4 - 524\varepsilon^2 + 252 > -56(\frac{\sqrt{2}}{2})^6 + 304\varepsilon^4 - 524\varepsilon^2 + 252 = 304\varepsilon^4 - 524\varepsilon^2 + 245 = 304(\varepsilon^2 - \frac{131}{152})^2 + \frac{1459}{76} > 0$ and $\psi_2(1-\varepsilon) = \frac{4\varepsilon^4(5\varepsilon^3 - 15\varepsilon - 8)}{3} < 0$, combining with (iv), we obtain that $\psi_2(u)$ has exactly one root on $(0, 1-\varepsilon)$. That is, $\psi_1(u)$ has exactly one local maximum on $(0, 1-\varepsilon)$.
(vi) Since $\psi_1(0) = 2(\varepsilon^2 - 1)(14\varepsilon^4 - 41\varepsilon^2 + 21) = 2(\varepsilon^2 - 1)[14(\varepsilon^2 - \frac{41}{28})^2 - \frac{505}{56}] < 0$ and $\psi_1(1-\varepsilon) = \frac{16\varepsilon^5(\varepsilon+2)(\varepsilon-1)^2}{3} > 0$, combining with (v), we obtain that $\psi_1(u)$ has exactly one root on $(0, 1-\varepsilon)$. That is, $\psi(u)$ has exactly one local minimum on $(0, 1-\varepsilon)$.
(vii) Since $\psi(0) = -6(2\varepsilon^2 - 1)(\varepsilon - 1)^2(\varepsilon + 1)^2 > 0$ and $\psi(1-\varepsilon) = -4\varepsilon^6(\varepsilon + 2)(\varepsilon - 1)^2 < 0$, combining with (vi), we obtain that $\psi(u)$ has exactly one root on $(0, 1-\varepsilon)$. That is, $\psi(u)$ changes sign exactly once on $(0, 1-\varepsilon)$ for all $\varepsilon \in (0, \frac{\sqrt{2}}{2})$, and so does $\omega_1(u)$. □

We next determine the sign of $T'_\lambda(r)$ for small λ.

Lemma 3.11. *Let $\breve{\varepsilon}$ be the constant given in (3.10). For any $\varepsilon \in [\breve{\varepsilon}, 1)$, there exists a number $\Lambda_\varepsilon \in (0, \lambda_{1-\varepsilon})$ such that $T'_\lambda(r) > 0$ on $(0, 1-\varepsilon)$ for all $\lambda < \Lambda_\varepsilon$.*

Proof. From (3.7), we obtain that

$$T'_\lambda(r) = \int_0^1 \frac{\lambda^3 \Delta F^3 - 3\lambda^2 \Delta F^2 + 2\lambda \Delta F - \lambda \Delta f}{\{1 - (1 - \lambda \Delta F)^2\}^{3/2}} \mathrm{d}s \geqslant \int_0^1 \frac{-3\lambda^2 \Delta F^2 + 2\lambda \Delta F - \lambda \Delta f}{\{1 - (1 - \lambda \Delta F)^2\}^{3/2}} \mathrm{d}s.$$

Take a constant $c > 3$. If $\lambda < \frac{1}{cF(1-\varepsilon)}$, then

$$\lambda \Delta F < \lambda F(r) \leqslant \lambda F(1-\varepsilon) < \frac{1}{c} \quad \text{for all } r \in (0, 1-\varepsilon],\ s \in (0, 1).$$

It follows that

$$
\begin{aligned}
-3\lambda^2 \Delta F^2 + 2\lambda \Delta F - \lambda \Delta f > \frac{-3}{c}\lambda \Delta F + 2\lambda \Delta F - \lambda \Delta f &= \lambda[(2 - \frac{3}{c})\Delta F - \Delta f]\\
&= \lambda(b\Delta F - \Delta f) = \lambda[A_{\varepsilon,b}(r) - A_{\varepsilon,b}(rs)],
\end{aligned}
$$

where $b = 2 - \frac{3}{c} \in (1,2)$ and the auxiliary function $A_{\varepsilon,b}$ is defined by

$$A_{\varepsilon,b}(u) := bF(u) - uf(u) = \frac{b}{1-u} - \frac{u}{(1-u)^2} - \frac{\varepsilon^2 b}{3(1-u)^3} + \frac{\varepsilon^2 u}{(1-u)^4} + \frac{\varepsilon^2 b}{3} - b. \tag{3.16}$$

Then

$$T'_\lambda(r) > \int_0^1 \frac{\lambda[A_{\varepsilon,b}(r) - A_{\varepsilon,b}(rs)]}{\{1-(1-\lambda\Delta F)^2\}^{3/2}}\, \mathrm{d}s.$$

We next prove that for any given $\varepsilon \in [\breve{\varepsilon}, 1)$, there exists $b_\varepsilon \in (1,2)$ such that $A_{\varepsilon,b}(u)$ is strictly increasing on $(0, 1-\varepsilon]$. Differentiating yields that $\frac{\partial}{\partial u} A_{\varepsilon,b}(u) = \frac{1}{(1-u)^5} k_{\varepsilon,b}(u)$, where

$$k_{\varepsilon,b}(u) := (-b-1)u^3 + (3b+1)u^2 + (\varepsilon^2 b + 3\varepsilon^2 - 3b + 1)u - \varepsilon^2 b + \varepsilon^2 + b - 1.$$

Since

$$k_{\varepsilon,b}(0) = (1-\varepsilon^2)(b-1) > 0,\ k_{\varepsilon,b}(1-\varepsilon) = 2\varepsilon^2(1-\varepsilon) > 0,\ k_{\varepsilon,b}(-\infty) = +\infty,\ k_{\varepsilon,b}(+\infty) = -\infty,$$

then the Mean Value Theorem implies that $k_{\varepsilon,b}(u)$ has at most two zeros on $(0, 1-\varepsilon)$. Furthermore,

$$\begin{aligned}
&\frac{\partial k_{\varepsilon,b}(u)}{\partial u} = (-3b-3)u^2 + (6b+2)u + \varepsilon^2 b + 3\varepsilon^2 - 3b + 1,\\
&\frac{\partial k_{\varepsilon,b}}{\partial u}(1-\varepsilon) = -2\varepsilon(\varepsilon b - 2) > 0,\ \frac{\partial k_{\varepsilon,b}}{\partial u}(-\infty) = -\infty,\ \frac{\partial k_{\varepsilon,b}}{\partial u}(+\infty) = -\infty.
\end{aligned}$$

Then $\frac{\partial}{\partial u} k_{\varepsilon,b}(u)$ has exactly one zero $z_{\varepsilon,b} := \frac{3b+1-\sqrt{3\varepsilon^2 b^2 + 12\varepsilon^2 b + 9\varepsilon^2 + 4}}{3b+3}$ on $(-\infty, 1-\varepsilon)$. This implies that $k_{\varepsilon,b}(u)$ has at most one critical point, a local minimum, on $(0, 1-\varepsilon)$. In addition, $A_{\varepsilon,b}(0) = 0$ and $A_{\varepsilon,b}(1-\varepsilon) = \frac{b(2+\varepsilon)(1-\varepsilon)^2}{3\varepsilon} > 0$. So we can sketch possible graphs of $k_{\varepsilon,b}(u)$ and $A_{\varepsilon,b}(u)$, see Fig.8.

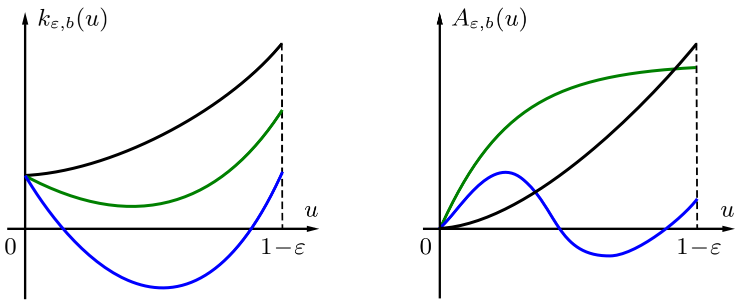

Figure 8: Left: possible graphs of $k_{\varepsilon,b}(u)$. Right: possible graphs of $A_{\varepsilon,b}(u)$.

Case (i). Suppose $z_{\varepsilon,b} \leqslant 0$. Then $k_{\varepsilon,b}(u) > 0$ on $(0, 1-\varepsilon]$ and hence $A_{\varepsilon,b}(u)$ is strictly increasing on $(0, 1-\varepsilon]$, which implies that $A_{\varepsilon,b}(r) - A_{\varepsilon,b}(rs) > 0$ for all $r \in (0, 1-\varepsilon]$ and $s \in (0,1)$. So $T'_\lambda(r) > 0$ on $(0, 1-\varepsilon)$.

Case (ii). Suppose $z_{\varepsilon,b} > 0$. Then $k_{\varepsilon,b}(u)$ has exactly one critical point, a local minimum $z_{\varepsilon,b}$ on $(0, 1-\varepsilon)$. We next investigate the sign of $k_{\varepsilon,b}(z_{\varepsilon,b})$.

We first show that $\frac{\partial}{\partial b} k_{\varepsilon,b}(z_{\varepsilon,b}) > 0$ for all $\varepsilon \in (0,1)$. Notice that $\frac{\partial k_{\varepsilon,b}}{\partial u}(z_{\varepsilon,b}) = 0$. Then

$$\frac{\partial}{\partial b} k_{\varepsilon,b}(z_{\varepsilon,b}) = \frac{\partial k_{\varepsilon,b}}{\partial b}(z_{\varepsilon,b}) + \frac{\partial k_{\varepsilon,b}}{\partial u}(z_{\varepsilon,b}) \frac{\partial z_{\varepsilon,b}}{\partial b} = \frac{\partial k_{\varepsilon,b}}{\partial b}(z_{\varepsilon,b}),$$

and computing yields that

$$\frac{\partial k_{\varepsilon,b}}{\partial b}(u) = -u^3 + 3u^2 + (\varepsilon^2 - 3)u + 1 - \varepsilon^2,$$

and the three roots of $\frac{\partial k_{\varepsilon,b}}{\partial b}(u)$ are $1, 1-\varepsilon$ and $1+\varepsilon$, which implies that $\frac{\partial k_{\varepsilon,b}}{\partial b}(u) > 0$ on $(0, 1-\varepsilon)$. So $\frac{\partial k_{\varepsilon,b}}{\partial b}(z_{\varepsilon,b}) > 0$. Thus $k_{\varepsilon,b}(z_{\varepsilon,b})$ is a strictly increasing function of $b \in (1,2)$.

If $b = 2$, there is then $A_{\varepsilon,2}(u) = 2F(u) - uf(u) = \theta_\varepsilon(u)$, by (3.10), we have $\frac{\partial}{\partial u} A_{\varepsilon,2}(u) = \frac{1}{(1-u)^5} k_{\varepsilon,2}(u) > 0$ on $(0, 1-\varepsilon]$ for all $\varepsilon \in [\check{\varepsilon}, 1)$, then

$$k_{\varepsilon,2}(z_{\varepsilon,2}) = \min_{u \in (0,1-\varepsilon)} k_{\varepsilon,2}(u) > 0 \text{ on } (0, 1-\varepsilon] \text{ for all } \varepsilon \in [\check{\varepsilon}, 1). \tag{3.17}$$

If $b = 1$, then

$$k_{\varepsilon,1}(z_{\varepsilon,1}) = \frac{4}{27}(\sqrt{6\varepsilon^2+1} - 2)(\sqrt{6\varepsilon^2+1} + 1 - 6\varepsilon^2) \leqslant 0 \text{ for all } \varepsilon \in (0,1).$$

So for any given $\varepsilon \in [\check{\varepsilon}, 1)$, there exists a unique $b_\varepsilon \in (1,2)$ such that $k_{\varepsilon,b}(z_{\varepsilon,b_\varepsilon}) = 0$. Then for any fixed $b \in (b_\varepsilon, 2)$, $k_{\varepsilon,b}(u) \geqslant k_{\varepsilon,b}(z_{\varepsilon,b}) > 0$. So $A_{\varepsilon,b}(u)$ is strictly increasing on $(0, 1-\varepsilon)$, which implies that $T'_\lambda(r) > 0$ on $(0, 1-\varepsilon)$. Notice that $c = \frac{3}{2-b}$ and $\lambda < \frac{1}{cF(1-\varepsilon)}$. Then $b \in (b_\varepsilon, 2)$ is equivalent to $\lambda \in (0, \frac{2-b_\varepsilon}{3F(1-\varepsilon)})$. We denote $\Lambda_\varepsilon := \frac{2-b_\varepsilon}{3F(1-\varepsilon)}$, then $T'_\lambda(r) > 0$ on $(0, 1-\varepsilon)$ for all $\lambda < \Lambda_\varepsilon$. □

For all $\varepsilon \in (0,1)$, denote by $g(\lambda)$ the value of $T_\lambda(r)$ at the right endpoint of the domain $\mathcal{D}_{\varepsilon,\lambda}$, i.e.,

$$g(\lambda) := T_\lambda(F^{-1}(\frac{1}{\lambda})) = \int_0^{F^{-1}(\frac{1}{\lambda})} \frac{\lambda F(u)}{\sqrt{1 - [\lambda F(u)]^2}} \, \mathrm{d}u \quad \text{ for } \lambda > \lambda_{1-\varepsilon}. \tag{3.18}$$

It is clear that $g(\lambda)$ is continuous on $(\lambda_{1-\varepsilon}, +\infty)$. Furthermore, the smoothness of T_λ with respect to r and λ, together with the smoothness of f, implies that g is smooth with respect to $\lambda \in (\lambda_{1-\varepsilon}, \infty)$.

We next give some properties of $g(\lambda)$ for (1.2).

Lemma 3.12. *For any $\varepsilon \in (0,1)$, the following assertions hold:*
(i) $\lim_{\lambda \to +\infty} g(\lambda) = 0$. (ii) $\lim_{\lambda \to (\lambda_{1-\varepsilon})^+} g(\lambda) = +\infty$.

Proof. (i) $\lim_{\lambda \to 0} g(\lambda) = \lim_{\lambda \to 0} T_\lambda(1-\varepsilon) = +\infty$ directly follows from Lemma 3.1(i). If $\lambda \to +\infty$, then $\lambda > \lambda_{1-\varepsilon}$ and $g(\lambda) = T_\lambda(F^{-1}(\frac{1}{\lambda}))$. Since

$$\lim_{u \to 0} \frac{F(u)}{f(u)} = \lim_{u \to 0} \frac{(1-u)^3 - \frac{\varepsilon^2}{3}(1-u) + (\frac{\varepsilon^2}{3} - 1)(1-u)^4}{(1-u)^2 - \varepsilon^2} = 0,$$

by [12, Proposition 2.9], we obtain that $\lim_{\lambda \to +\infty} g(\lambda) = 0$.

(ii) For $\varepsilon \in (0,1)$ and $f(u) = \frac{1}{(1-u)^2} - \frac{\varepsilon^2}{(1-u)^4}$, it is clear that $0 < \lim_{u \to (1-\varepsilon)^-} \frac{f(u)}{1-\varepsilon-u} = \frac{2}{\varepsilon^3} < +\infty$. So there exist positive values $R < 1-\varepsilon$ and k such that $f(u) \leqslant k(1-\varepsilon-u)$ for $R < u < 1-\varepsilon$. It follows that

$$F(r) - F(u) = \int_u^r f(v) \, \mathrm{d}v \leqslant \int_u^r k(1-\varepsilon-v) \, \mathrm{d}v = k(1-\varepsilon)(r-u) - \frac{1}{2}k(r^2 - u^2). \tag{3.19}$$

By writing $\lambda = \frac{1}{F(r)}$ in (3.19), we have

$$\tilde{g}(r) := g(\lambda)|_{\lambda = \frac{1}{F(r)}} = \int_0^r \frac{F(u)}{\sqrt{F^2(r) - F^2(u)}} \, \mathrm{d}u$$

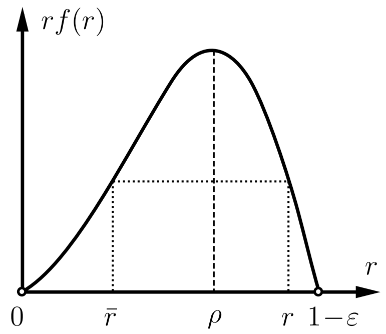

Figure 9: Graph of $rf(r)$ on $(0, 1-\varepsilon)$ with fixed $\varepsilon \in (0,1)$ and definition of $\bar{r}$.

and hence

$$\begin{aligned}\lim_{\lambda \to (\lambda_{1-\varepsilon})^+} g(\lambda) &= \lim_{r \to (1-\varepsilon)^-} \tilde{g}(r) = \lim_{r \to (1-\varepsilon)^-} \int_0^r \frac{F(u)}{\sqrt{F^2(r) - F^2(u)}} \mathrm{d}u \\ &\geqslant \lim_{r \to (1-\varepsilon)^-} \int_R^r \frac{F(u)}{\sqrt{F(r) + F(u)}} \cdot \frac{1}{\sqrt{F(r) - F(u)}} \mathrm{d}u \\ &\geqslant \lim_{r \to (1-\varepsilon)^-} \frac{F(R)}{\sqrt{2F(r)}} \int_R^r \frac{1}{\sqrt{F(r) - F(u)}} \mathrm{d}u.\end{aligned}$$

Then it follows from (3.19) that

$$\begin{aligned}\lim_{r \to (1-\varepsilon)^-} \tilde{g}(r) &\geqslant -\frac{F(R)}{\sqrt{kF(1-\varepsilon)}} \cdot \lim_{r \to (1-\varepsilon)^-} \int_R^r \frac{1}{\sqrt{(1-\varepsilon-u)^2 - (1-\varepsilon-r)^2}} \mathrm{d}(1-\varepsilon-u) \\ &= -\frac{F(R)}{\sqrt{kF(1-\varepsilon)}} \cdot \lim_{r \to (1-\varepsilon)^-} \ln(1-\varepsilon-u+\sqrt{(1-\varepsilon-u)^2 - (1-\varepsilon-r)^2})\big|_R^r \\ &= +\infty, \end{aligned} \tag{3.20}$$

which completes the proof. $\square$

Proof of Lemma 3.7. For $f(r) = \frac{1}{(1-r)^2} - \frac{\varepsilon^2}{(1-r)^4}$ with $\varepsilon \in (0,1)$, differentiating $rf(r)$ yields

$$(rf(r))' = \frac{r^3 - r^2 - (3\varepsilon^2+1)r - \varepsilon^2 + 1}{(1-r)^5} =: \frac{w(r)}{(1-r)^5}.$$

Since $w(0) = 1 - \varepsilon^2 > 0$, $w(1-\varepsilon) = 2\varepsilon^2(\varepsilon - 1) < 0$ and $\lim_{r \to \pm\infty} w(r) = \pm\infty$, we have that w has a unique zero $\rho := \rho(\varepsilon) \in (0, 1-\varepsilon)$. So $rf(r)$ is strictly increasing on $(0,\rho)$ and decreasing on $(\rho, 1-\varepsilon)$. Then for each $r > \rho$, there exists a unique value $\bar{r} := \bar{r}(\varepsilon, r) < \rho$ such that

$$uf(u) - rf(r) = \begin{cases} < 0 & \text{if } u \in (0, \bar{r}) \\ = 0 & \text{if } u = \bar{r} \\ > 0 & \text{if } u \in (\bar{r}, r) \end{cases}; \tag{3.21}$$

see Fig.9. Moreover, $\bar{r} \to 0$ as $r \to 1-\varepsilon$.

From (3.19), we have

$$T'_\lambda(r) = I_1(r) + I_2(r) + I_3(r),$$

where

$$I_1(r) := \frac{1}{r}\int_0^r \frac{1+\lambda F(u)-\lambda F(r)}{\sqrt{1-[1+\lambda F(u)-\lambda F(r)]^2}}\,\mathrm{d}u,$$
$$I_2(r) := \frac{1}{r}\int_0^{\bar r} \frac{\lambda[uf(u)-rf(r)]}{\{1-[1+\lambda F(u)-\lambda F(r)]^2\}^{3/2}}\,\mathrm{d}u,$$
$$I_3(r) := \frac{1}{r}\int_{\bar r}^r \frac{\lambda[uf(u)-rf(r)]}{\{1-[1+\lambda F(u)-\lambda F(r)]^2\}^{3/2}}\,\mathrm{d}u.$$

In view of (3.21), it is clear that $I_2(r)<0$ and $I_3(r)>0$ for $r\in(\rho,1-\varepsilon)$. Then

$$\liminf_{\lambda\to(\lambda_{1-\varepsilon})^+} T'_\lambda\big(F^{-1}(\frac{1}{\lambda})\big) \geqslant \liminf_{\lambda\to(\lambda_{1-\varepsilon})^+}\Big[I_1(F^{-1}(\frac{1}{\lambda}))+I_2(F^{-1}(\frac{1}{\lambda}))\Big]. \tag{3.22}$$

It follows from (3.20) that

$$\begin{aligned}\lim_{\lambda\to(\lambda_{1-\varepsilon})^+} I_1(F^{-1}(\frac{1}{\lambda})) &= \lim_{\lambda\to(\lambda_{1-\varepsilon})^+}\frac{1}{F^{-1}(\frac{1}{\lambda})}g(\lambda) \overset{r=F^{-1}(\frac{1}{\lambda})}{=\!=\!=\!=} \lim_{r\to(1-\varepsilon)^-}\frac{1}{r}\tilde g(r)\\ &= \frac{1}{1-\varepsilon}\lim_{r\to(1-\varepsilon)^-}\tilde g(r)=+\infty.\end{aligned} \tag{3.23}$$

Computing yields

$$\begin{aligned}I_2(F^{-1}(\frac{1}{\lambda})) &= \frac{1}{F^{-1}(\frac{1}{\lambda})}\int_0^{F^{-1}(\frac{1}{\lambda})}\frac{\lambda[uf(u)-F^{-1}(\frac{1}{\lambda})f(F^{-1}(\frac{1}{\lambda}))]}{\{1-[\lambda F(u)]^2\}^{3/2}}\,\mathrm{d}u - I_3(F^{-1}(\frac{1}{\lambda}))\\ &\overset{r=F^{-1}(\frac{1}{\lambda})}{=\!=\!=\!=} \frac{1}{r}\Big(\int_0^r-\int_{\bar r}^r\Big)\frac{uf(u)-rf(r)}{F(r)\Big\{1-[\frac{F(u)}{F(r)}]^2\Big\}^{3/2}}\,\mathrm{d}u\\ &= \frac{F^2(r)}{r}\int_0^{\bar r}\frac{uf(u)-rf(r)}{[F^2(r)-F^2(u)]^{3/2}}\,\mathrm{d}u.\end{aligned}$$

Since $f(1-\varepsilon)=0$ and $F(1-\varepsilon)>0$, and by (3.21), we have that

$$0>\int_0^{\bar r}\frac{uf(u)-rf(r)}{[F^2(r)-F^2(u)]^{3/2}}\,\mathrm{d}u > \int_0^{\bar r}\frac{-rf(r)}{[F^2(r)-F^2(\bar r)]^{3/2}}\,\mathrm{d}u = \frac{-\bar r rf(r)}{[F^2(r)-F^2(\bar r)]^{3/2}}\to 0,$$

as $r\to(1-\varepsilon)^-$. Hence,

$$\lim_{\lambda\to(\lambda_{1-\varepsilon})^+} I_2(F^{-1}(\frac{1}{\lambda})) = \frac{F^2(1-\varepsilon)}{1-\varepsilon}\lim_{r\to(1-\varepsilon)^-}\int_0^{\bar r}\frac{uf(u)-rf(r)}{[F^2(r)-F^2(u)]^{3/2}}\,\mathrm{d}u=0. \tag{3.24}$$

Consequently, (3.22)–(3.24) imply that $\lim_{\lambda\to(\lambda_{1-\varepsilon})^+} T'_\lambda\big(F^{-1}(\frac{1}{\lambda})\big)=+\infty$. □

Lemma 3.13. *For any $\varepsilon\in[\frac{19}{100},1)$, $g(\lambda)$ is strictly decreasing on $(\lambda_{1-\varepsilon},+\infty)$.*

Proof. For $\lambda>\lambda_{1-\varepsilon}$, according to [12, Proposition 2.8], differentiating (3.18) yields

$$g'(\lambda)=\frac{\mathrm{d}}{\mathrm{d}\lambda}\big[T_\lambda\big(F^{-1}(\frac{1}{\lambda})\big)\big] = -\frac{1}{\lambda^2}\int_0^1\frac{y}{\sqrt{1-y^2}}\frac{f^2(t)-f'(t)F(t)}{f^3(t)}\Big|_{t=F^{-1}(\frac{y}{\lambda})}\,\mathrm{d}y. \tag{3.25}$$

Set $q(u):=(1-u)^8[f^2(u)-f'(u)F(u)]$. Inserting f into q yields

$$\begin{aligned}q(u) &= (\frac{2\varepsilon^2}{3}-2)u^5+(9-\frac{10\varepsilon^2}{3})u^4+(-16-\frac{4}{3}\varepsilon^4+\frac{32}{3}\varepsilon^2)u^3\\ &\quad+(4\varepsilon^4-16\varepsilon^2+14)u^2+(-4\varepsilon^4+10\varepsilon^2-6)u+\varepsilon^4-2\varepsilon^2+1.\end{aligned} \tag{3.26}$$

So it suffices to prove that

$$q(u) > 0, \quad \text{on } (0, 1-\varepsilon) \text{ for all } \varepsilon \in [\frac{19}{100}, 1). \tag{3.27}$$

By Sturm's Theorem, after a long calculation it can be shown that (3.27) holds, which implies $g'(\lambda) < 0$ on $(\lambda_{1-\varepsilon}, +\infty)$. Since the proof of (3.27) is lengthy and tedious, we place it in Appendix. □

Lemma 3.14. *For any $\varepsilon \in (0, \frac{19}{100})$, $g'(\lambda) < 0$ on $[\lambda_{1/2}, +\infty)$.*

Proof. In view of (3.25) and (3.26), we first prove that $q(u) > 0$ on $(0, \frac{1}{2})$ for all $\varepsilon \in (0, \frac{19}{100})$. Set $q_i(u) := \frac{\partial^i}{\partial u^i}(q(u))$. For $\varepsilon \in (0, \frac{19}{100})$, we obtain from (3.26) that
(i) $q_5(u) = 80\varepsilon^2 - 240 < 0$ on $(0, 1-\varepsilon)$ and $q_4(0) = -80\varepsilon^2 + 216 > 0$, then $q_4(u)$ has at most one zero on $(0, 1-\varepsilon)$.
(ii) Since $q_4(u)$ has at most one zero, $q_3(0) = -8(\varepsilon^2 - 6)(\varepsilon^2 - 2) < 0$ and $q_3(1-\varepsilon) = 8\varepsilon(4\varepsilon^3 - 12\varepsilon + 3) > 0$, then $q_3(u)$ has exactly one zero on $(0, 1-\varepsilon)$.
(iii) Since $q_3(u)$ has exactly one zero, $q_2(0) = 8\varepsilon^4 - 32\varepsilon^2 + 28 > 0$ and $q_2(1-\varepsilon) = \frac{-4\varepsilon^2}{3}(4\varepsilon^3 - 12\varepsilon + 5) < 0$, then $q_2(u)$ has exactly one zero on $(0, 1-\varepsilon)$.
(iv) Since $q_2(u)$ has exactly one zero, $q_1(0) = 2(1-\varepsilon^2)(2\varepsilon^2 - 3) < 0$ and $q_1(1-\varepsilon) = \frac{-2\varepsilon^3}{3}(\varepsilon + 2)(\varepsilon - 1)^2 < 0$, then $q_1(u)$ has at most two zero on $(0, 1-\varepsilon)$.

Also, computing yields that

$$q_1(\frac{1}{2}) = -\varepsilon^4 + \frac{13}{24}\varepsilon^2 - \frac{1}{8} < 0 \quad \text{and} \quad q_1(\frac{13}{20}) = \frac{7}{100}(-7\varepsilon^4 - \frac{2377}{480}\varepsilon^2 + \frac{49}{160}) > 0 \quad \text{for } \varepsilon \in (0, \frac{19}{100}),$$

which, together with (iv), imply that $q_1(u) < 0$ on $(0, \frac{1}{2})$. Since

$$q(\frac{1}{2}) = -\frac{1}{6}\varepsilon^4 + \frac{7}{48}\varepsilon^2 = \frac{1}{6}\varepsilon^2(\frac{7}{8} - \varepsilon^2) > 0 \quad \text{for } \varepsilon \in (0, \frac{19}{100}),$$

it follows that $q(u) > 0$ on $(0, \frac{1}{2})$ for all $\varepsilon \in (0, \frac{19}{100})$, which implies that

$$f^2(u) - f'(u)F(u) > 0, \quad \text{on } (0, \frac{1}{2}). \tag{3.28}$$

Since

$$t = F^{-1}(\frac{y}{\lambda}) \in \Big(0, F^{-1}(\frac{1}{\lambda})\Big) \subset \Big(0, F^{-1}\big(F(\frac{1}{2})\big)\Big) = \big(0, \frac{1}{2}\big) \quad \text{for all } \lambda \in [\lambda_{1/2}, +\infty) \text{ and } y \in (0, 1),$$

it follows from (3.25) and (3.28) that $g'(\lambda) < 0$ for all $\lambda \in [\lambda_{1/2}, +\infty)$. □

Lemma 3.15. *There exists a positive constant $\varepsilon_*(\approx 0.13123)$ such that $g'(\lambda_\gamma) > 0$ for all $\varepsilon \in (0, \varepsilon_*]$.*

Proof. Substituting $\gamma(\varepsilon) = 1 - \frac{\sqrt{30}}{3}\varepsilon$ into (3.26) yields

$$q(\gamma) = -\frac{80}{81}\varepsilon^4\Big(\sqrt{30}(\varepsilon^3 - 3\varepsilon) + \frac{207}{80}\Big) < 0 \quad \text{for } \varepsilon \in (0, \varepsilon_0),$$

where $\varepsilon_0(\approx 0.158805)$ is the first positive root of polynomial $\sqrt{30}(\varepsilon^3 - 3\varepsilon) + \frac{207}{80}$. The proof of Lemma 3.14 shows that for $\varepsilon \in (0, \varepsilon_0)$, $q(u)$ has at most two critical points on $(0, 1-\varepsilon)$. Notice that $q(0) = (\varepsilon^2 - 1)^2 > 0$,

$q(1-\varepsilon)=\frac{2\varepsilon^4(\varepsilon+2)(\varepsilon-1)^2}{3}>0$. Since $\gamma(\varepsilon)>\frac{1}{2}$ and $q(\gamma)<0$, it follows that there exists a unique $\xi:=\xi(\varepsilon)\in\left(\frac{1}{2},\gamma(\varepsilon)\right)$ such that

$$f^2(u)-f(u)F(u)=\frac{1}{(1-u)^8}q(u)\begin{cases}>0 & \text{for } u\in(0,\xi),\\ <0 & \text{for } u\in(\xi,\gamma).\end{cases}$$

Substituting $\lambda=\lambda_\gamma$ into (3.25), we obtain that

$$\begin{aligned}-\frac{1}{\lambda_\gamma}g'(\lambda_\gamma)&=-F(\gamma)g'(\frac{1}{F(\gamma)})=\int_0^\gamma\frac{F(t)}{\sqrt{F^2(\gamma)-F^2(t)}}\frac{f^2(t)-f'(t)F(t)}{f^2(t)}dt\\ &<\int_0^\xi\frac{F(t)}{\sqrt{F^2(\gamma)-F^2(\xi)}}\frac{f^2(t)-f'(t)F(t)}{f^2(t)}dt+\int_\xi^\gamma\frac{F(t)}{\sqrt{F^2(\gamma)-F^2(\xi)}}\frac{f^2(t)-f'(t)F(t)}{f^2(t)}dt\\ &=\frac{1}{\sqrt{F^2(\gamma)-F^2(\xi)}}\left(\int_0^\gamma F(t)\frac{f^2(t)-f'(t)F(t)}{f^2(t)}dt\right)\\ &=\frac{1}{\sqrt{F^2(\gamma)-F^2(\xi)}}\frac{F^2(y)-f(y)\int_0^y F(t)dt}{f(y)}\Big|_0^\gamma.\end{aligned}\tag{3.29}$$

Further computation yields

$$\frac{F^2(\gamma)-f(\gamma)\int_0^\gamma F(t)dt}{f(\gamma)}=\frac{\phi(\varepsilon)}{18000\varepsilon^2 f(\gamma)},$$

where

$$\phi(\varepsilon):=2000\varepsilon^6+1500\sqrt{30}\varepsilon^3-12000\varepsilon^4-4500\sqrt{30}\varepsilon+16110\varepsilon^2+3780\ln(\frac{\sqrt{30}\varepsilon}{3})+8343.$$

Since $\lim_{\varepsilon\to0^+}\phi(\varepsilon)=-\infty$ and $\phi(\varepsilon_0)(\approx 180.45)>0$, then $\phi(\varepsilon)$ has at least one zero on $(0,\varepsilon_0)$. Denote by ε_* the first zero of $\phi(\varepsilon)$ on $(0,\varepsilon_0)$. Thus $\frac{F^2(y)-f(y)\int_0^y F(t)dt}{f(y)}\big|_{y=\gamma}\leqslant 0$ for all $\varepsilon\in(0,\varepsilon_*]$. Numerical evaluation shows that $\varepsilon_*\approx 0.13123$. Since $F(0)=0$, we obtain from (3.29) that $g'(\lambda_\gamma)>0$ for all $\varepsilon\in(0,\varepsilon_*]$. $\square$

Lemma 3.16. *For any $\varepsilon\in(0,\frac{19}{100})$, $g'(\lambda)$ changes sign at most twice on $(\lambda_{1-\varepsilon},+\infty)$. Moreover, for any $\varepsilon\in(0,\varepsilon_*]$, $g'(\lambda)$ changes sign exactly twice on $(\lambda_{1-\varepsilon},+\infty)$.*

Proof. By Lemmas 3.12, 3.14 and 3.15, we immediately obtain that $g'(\lambda)$ changes sign at least twice on $(\lambda_{1-\varepsilon},+\infty)$ for $\varepsilon\in(0,\varepsilon_*]$. So it suffices to prove that $g'(\lambda)$ changes sign at most twice on $(\lambda_{1-\varepsilon},+\infty)$ for $\varepsilon\in(0,0.19)$. To this end, consider the function

$$\begin{aligned}\omega_2(u):=&\,15f^6(u)-33F(u)f^4(u)f'(u)+36F^2(u)f^2(u)f'^2(u)-12F^2(u)f^3(u)f''(u)\\ &-15F^3(u)f'^3(u)+10F^3(u)f(u)f'(u)f''(u)-F^3(u)f^2(u)f'''(u).\end{aligned}\tag{3.30}$$

According to [14, Corollary 2.8], it suffices to prove that $\omega_2(u)$ changes sign at most twice on $(0,1-\varepsilon)$ for all $\varepsilon\in(0,0.19)$.

Substituting f into (3.30) yields that $\omega_2(u)=\frac{1}{9(1-u)^{24}}p(u)$, where p is a polynomial of degree 15 with a parameter ε and is defined by

$$p(u):=(8\,\varepsilon^6-72\,\varepsilon^4+216\,\varepsilon^2-216)u^{15}+(-120\,\varepsilon^6+1080\,\varepsilon^4-3240\,\varepsilon^2+3240)u^{14}+(1224\,\varepsilon^6-8712\,\varepsilon^4+23814\,\varepsilon^2-\frac{128\,\varepsilon^8}{3}-22626)u^{13}+(-8632\,\varepsilon^6+47736\,\varepsilon^4-113022\,\varepsilon^2+\frac{1664\,\varepsilon^8}{3}+97551)u^{12}+$$

$(88\,\varepsilon^{10}-4120\,\varepsilon^{8}+43248\,\varepsilon^{6}-190404\,\varepsilon^{4}+382968\,\varepsilon^{2}-290304)u^{11}+(\frac{62792\,\varepsilon^{8}}{3}-160080\,\varepsilon^{6}+570780\,\varepsilon^{4}-969192\,\varepsilon^{2}-968\,\varepsilon^{10}+631422)u^{10}+(-\frac{230240\,\varepsilon^{8}}{3}+448268\,\varepsilon^{6}-1308744\,\varepsilon^{4}+1871910\,\varepsilon^{2}-\frac{280\,\varepsilon^{12}}{3}+5680\,\varepsilon^{10}-1036530)u^{9}+(840\,\varepsilon^{12}-21912\,a^{10}+207984\,\varepsilon^{8}-958788\,\varepsilon^{6}+2313225\,\varepsilon^{4}-2786238\,\varepsilon^{2}+1307097)u^{8}+(-3360\,\varepsilon^{12}+57936\,\varepsilon^{10}-419064\,\varepsilon^{8}+1566672\,\varepsilon^{6}-3151656\,\varepsilon^{4}+3203064\,\varepsilon^{2}-1275912)u^{7}+(7896\,\varepsilon^{12}-106848\,\varepsilon^{10}+625824\,\varepsilon^{8}-1942608\,\varepsilon^{6}+3287484\,\varepsilon^{4}-2831976\,\varepsilon^{2}+963468)u^{6}+(-12096\,\varepsilon^{12}+138942\,\varepsilon^{10}-686610\,\varepsilon^{8}+1802952\,\varepsilon^{6}-2589408\,\varepsilon^{4}+1903986\,\varepsilon^{2}-557766)u^{5}+(12600\,\varepsilon^{12}-127350\,\varepsilon^{10}+543969\,\varepsilon^{8}-1223928\,\varepsilon^{6}+1505214\,\varepsilon^{4}-953370\,\varepsilon^{2}+242865)u^{4}+(-8964\,\varepsilon^{12}+80784\,\varepsilon^{10}-301104\,\varepsilon^{8}+585144\,\varepsilon^{6}-622188\,\varepsilon^{4}+343224\,\varepsilon^{2}-76896)u^{3}+(4212\,\varepsilon^{12}-33696\,\varepsilon^{10}+109350\,\varepsilon^{8}-184680\,\varepsilon^{6}+171720\,\varepsilon^{4}-83592\,\varepsilon^{2}+16686)u^{2}+(-1188\,\varepsilon^{12}+8154\,\varepsilon^{10}-22950\,\varepsilon^{8}+34020\,\varepsilon^{6}-28080\,\varepsilon^{4}+12258\,\varepsilon^{2}-2214)u+135\,\varepsilon^{12}-810\,\varepsilon^{10}+2025\,\varepsilon^{8}-2700\,\varepsilon^{6}+2025\,\varepsilon^{4}-810\,\varepsilon^{2}+135.$

Clearly, $\omega_2(u)$ and $p(u)$ have the same number of sign changes. By Sturm's Theorem, after a long calculation it can be shown that

$$p(u) \text{ changes sign at most twice on } (0,1-\varepsilon) \text{ for all } \varepsilon\in(0,0.19), \tag{3.31}$$

which completes the proof. Since the proof of (3.31) is lengthy and tedious, we place it in Appendix. □

The next lemma shows that $1-\varepsilon$ does not belong to the domain of T_λ for all $\lambda\leqslant\lambda_{1-\varepsilon}$.

Lemma 3.17. *For any $\varepsilon\in(0,1)$, the following assertions hold:*
(i) $\lim_{r\to(1-\varepsilon)^-}T_\lambda(r)=+\infty$ for all $\lambda\leqslant\lambda_{1-\varepsilon}$. (ii) $\lim_{\lambda\to(\lambda_{1-\varepsilon})^+}h(\lambda)=+\infty$.

Proof. For fixed $r\in(0,1-\varepsilon)$, since $T_\lambda(r)$ is strictly decreasing with respect to $\lambda\in(0,\lambda_r]$, it follows that

$$T_\lambda(r)>T_{\lambda_{1-\varepsilon}}(r) \qquad \text{for all } \lambda\in(0,\lambda_{1-\varepsilon}).$$

So it suffices to prove that $\lim_{r\to(1-\varepsilon)^-}T_{\lambda_{1-\varepsilon}}(r)=+\infty$.

Take the transformation $r=F^{-1}(\frac{1}{\lambda})$ for $\lambda>\lambda_{1-\varepsilon}$; see Fig.7. Since $r\to(1-\varepsilon)^-$ as $\lambda\to(\lambda_{1-\varepsilon})^+$, it follows that

$$\lim_{r\to(1-\varepsilon)^-}T_{\lambda_{1-\varepsilon}}(r)=\lim_{\lambda\to(\lambda_{1-\varepsilon})^+}T_{\lambda_{1-\varepsilon}}(F^{-1}(\frac{1}{\lambda})).$$

Since $T_\lambda(r)$ is strictly decreasing with respect to $\lambda\in(0,\lambda_r]$, it follows that

$$T_{\lambda_{1-\varepsilon}}(F^{-1}(\frac{1}{\lambda}))>T_\lambda(F^{-1}(\frac{1}{\lambda})).$$

So

$$\lim_{\lambda\to(\lambda_{1-\varepsilon})^+}T_{\lambda_{1-\varepsilon}}(F^{-1}(\frac{1}{\lambda}))\geqslant\lim_{\lambda\to(\lambda_{1-\varepsilon})^+}T_\lambda(F^{-1}(\frac{1}{\lambda})).$$

For $\lambda>\lambda_{1-\varepsilon}$, it follows from Lemma 3.12 that

$$\lim_{\lambda\to(\lambda_{1-\varepsilon})^+}T_\lambda(F^{-1}(\frac{1}{\lambda}))\equiv\lim_{\lambda\to(\lambda_{1-\varepsilon})^+}g(\lambda)=+\infty,$$

which proves (i). (ii) is a direct result of (i). □

Lemma 3.18. *For any $\varepsilon\in(0,1)$, there exists a number $\sigma_\varepsilon\in(0,1-\varepsilon)$, such that, for $\lambda\leqslant\lambda_{\sigma_\varepsilon}$, $T''_\lambda(r)>0$ on $[\sigma_\varepsilon,1-\varepsilon)\cap\mathcal{D}_{\varepsilon,\lambda}$.*

Proof. Similar to (3.7), by a direct computation, we have

$$\begin{aligned}
T_\lambda''(r) =& \int_0^1 \frac{\lambda\left\{2[f(rs)s-f(r)]+r\left[f'(rs)s^2-f'(r)\right]\right\}\left\{1-[1+\lambda F(rs)-\lambda F(r)]^2\right\}}{\left\{1-[1+\lambda F(rs)-\lambda F(r)]^2\right\}^{5/2}}\,\mathrm{d}s \\
&+\int_0^1 \frac{3\lambda^2 r[1+\lambda F(rs)-\lambda F(r)][f(rs)s-f(r)]^2}{\left\{1-[1+\lambda F(rs)-\lambda F(r)]^2\right\}^{5/2}}\,\mathrm{d}s \\
=&\frac{\lambda^2}{r}\int_0^1 \frac{-(\Delta f'+2\Delta f)(2-\lambda\Delta F)\Delta F+3(1-\lambda\Delta F)(\Delta f)^2}{[1-(1-\lambda\Delta F)^2]^{5/2}}\,\mathrm{d}s, \qquad (3.32)
\end{aligned}$$

where $\Delta f := rf(r)-rsf(rs)$, $\Delta f' := r^2 f'(r)-r^2s^2f'(rs)$, and $\Delta F := F(r)-F(rs)$. Since

$$\Delta F > 0 \ \text{ for } u\in(0,r) \quad \text{and} \quad 1-\lambda\Delta F>0 \ \text{ on } \mathcal{D}_{\varepsilon,\lambda},$$

we have that $T_\lambda''(r)>0$ holds if $\Delta f'+2\Delta f<0$ for all $s\in(0,1)$.

For any $\varepsilon\in(0,1)$, set

$$\eta(r) := r^2 f'(r)+2rf(r) = \frac{2r}{(1-r)^5}\left[r^2-(\varepsilon^2+2)r-\varepsilon^2+1\right].$$

Differentiating η yields

$$\eta'(r) = \frac{2}{(1-r)^6}\left[2r^3-(3\varepsilon^2+3)r^2-6\varepsilon^2 r-\varepsilon^2+1\right].$$

Then $\ell(r) := 2r^3-(3\varepsilon^2+3)r^2-6\varepsilon^2r-\varepsilon^2+1$ has at most three real zeros. Furthermore,

$$\ell(\pm\infty)=\pm\infty, \quad \ell(0)=1-\varepsilon^2>0, \quad \text{and} \quad \ell(1-\varepsilon)=\varepsilon^2(1-\varepsilon)(3\varepsilon-7)<0.$$

So $\ell(r)$ has exactly one zero on $(0,1-\varepsilon)$, denoted by σ_ε. Since $\ell(r)<0$ on $(\sigma_\varepsilon,1-\varepsilon)$, it implies that $\eta(r)$ is strictly decreasing on $(\sigma_\varepsilon,1-\varepsilon)$. Hence, for any $s\in(0,1)$, $\lambda>0$ and $r\in(\sigma_\varepsilon,1-\varepsilon)\cap\mathcal{D}_{\varepsilon,\lambda}$, we have

$$\Delta f'+2\Delta f = r^2f'(r)-r^2s^2f'(rs)+2rf(r)-2rsf(rs) = \eta(r)-\eta(rs)<0.$$

It follows from (3.32) that $T_\lambda''(r)>0$ on $[\sigma_\varepsilon,1-\varepsilon)\cap\mathcal{D}_{\varepsilon,\lambda}$, which is not empty for any $\lambda\leqslant\lambda_{\sigma_\varepsilon}$. □

4. Proofs of main results

From (3.4) and uniqueness of solutions of the initial value problem, we obtain that for any fixed $\lambda>0$ and $\varepsilon\in(0,1)$, the exact number of positive solutions of (1.2) is equal to that of the root of the equation $T_\lambda(r)=L$ in $\mathcal{D}_{\varepsilon,\lambda}$.

By the definitions and the lemmas, we have actually established the following properties of the time-map:

(1) For any $\varepsilon\in(0,1),\lambda>0$, the domain of $T_\lambda(r)$ is $\mathcal{D}_{\varepsilon,\lambda}=\begin{cases}\left(0,F^{-1}\left(\frac{1}{\lambda}\right)\right] & \text{if } \lambda>\lambda_{1-\varepsilon}\\ (0,1-\varepsilon) & \text{if } \lambda\leqslant\lambda_{1-\varepsilon}\end{cases}$. Moreover, $T_\lambda(r)$ is twice continuously differentiable with respect to $(r,\lambda)\in\mathcal{D}_{\varepsilon,\lambda}\times(0,\lambda_r]$.

(2) For any given $r\in(0,1-\varepsilon)$, $T_\lambda(r)$ is a strictly decreasing function of $\lambda\in(0,\lambda_r]$ and $\lim_{\lambda\to0}T_\lambda(r)=+\infty$.

(3) $\lim_{r\to0^+}T_\lambda(r)=0$ and $\lim_{r\to0^+}T_\lambda'(r)=+\infty$ for any $\lambda>0$; $\lim_{r\to(1-\varepsilon)^-}T_\lambda(r)=+\infty$ for all $\lambda\in(0,\lambda_{1-\varepsilon}]$.

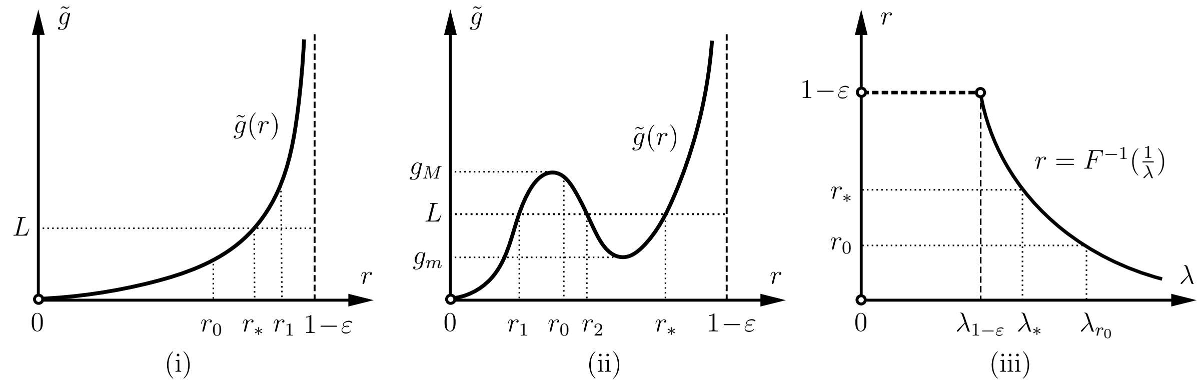

Figure 10: (i) Case A. $\tilde{g}(r)$ is strictly increasing on $(0, 1-\varepsilon)$. (ii) Case B. $\tilde{g}(r)$ has exactly two critical points, a pair of maximum and minimum points, on $(0, 1-\varepsilon)$. (iii) The domain $\mathcal{D}_{\varepsilon,\lambda}$ and the transformation $r = F^{-1}(\frac{1}{\lambda})$.

(4) $g(\lambda) := T_\lambda(F^{-1}(\frac{1}{\lambda}))$ is a continuous function of $\lambda \in (\lambda_{1-\varepsilon}, \infty)$, satisfying $\lim_{\lambda \to (\lambda_{1-\varepsilon})^+} g(\lambda) = +\infty$ and $\lim_{\lambda \to +\infty} g(\lambda) = 0$.
(5) $h(\lambda) := \sup_{r \in (0, F^{-1}(1/\lambda)]} T_\lambda(r)$ is a strictly decreasing, continuous function of $\lambda \in (\lambda_{1-\varepsilon}, \infty)$, satisfying that $\lim_{\lambda \to (\lambda_{1-\varepsilon})^+} h(\lambda) = +\infty$ and $\lim_{\lambda \to +\infty} h(\lambda) = 0$.
(6) If $\varepsilon \in (0, \frac{\sqrt{2}}{2})$, then there exists a unique $\bar{\lambda} > \lambda_{1-\varepsilon}$ such that

$$G(\lambda) := T'(F^{-1}(\frac{1}{\lambda})) \begin{cases} > 0 & \text{if } \lambda_{1-\varepsilon} < \lambda < \bar{\lambda} \\ = 0 & \text{if } \lambda = \bar{\lambda} \\ < 0 & \text{if } \lambda > \bar{\lambda} \end{cases}.$$

If $\varepsilon \in [\frac{\sqrt{2}}{2}, 1)$, then $T'_\lambda(r) > 0$ on $\mathcal{D}_{\varepsilon,\lambda}$ for all $\lambda > 0$.
(7) For any $\varepsilon \in (0,1)$, $g(\lambda)$ either is strictly decreasing or has exactly two critical points, a pair of maximum and minimum points, on $(\lambda_{1-\varepsilon}, +\infty)$. In particular, if $\varepsilon \in [\frac{19}{100}, 1)$, then the former occurs; if $\varepsilon \in (0, \frac{13}{100}]$, then the latter happens.

Proof of Theorem 2.1. From **(4)** and **(7)** above, it follows that there are only two possible shapes for g on $(\lambda_{1-\varepsilon}, \infty)$. By the transformation $r = F^{-1}(\frac{1}{\lambda})$, the same situation holds for $\tilde{g}(r) := g(\frac{1}{F(r)})$ on $(0, 1-\varepsilon)$; see Fig.10(i) and (ii). So we divide the proof into two cases.

Case A. Assume that $\tilde{g}(r)$ is strictly increasing on $(0, 1-\varepsilon)$; see Fig.10(i) and (iii). Then for any given $L > 0$, **(4)** implies that there exists a unique $r_* := F^{-1}(\frac{1}{\lambda_*})$, with $\lambda_* > \lambda_{1-\varepsilon}$, such that $\tilde{g}(r_*) = g(\lambda_*) = T_{\lambda_*}(r_*) = L$.

For given $L > 0$ and any $r_0 \in (0, r_*)$, it follows from **(2)** that there exists a unique $\lambda \in (0, \lambda_*)$, denoted by $\lambda_L(r_0)$ or λ_{r_0} for short, such that $T_{\lambda_L(r_0)}(r_0) = L$. In fact, if $T_\lambda(r_0) > L$ for all $\lambda \in (0, \lambda_{r_0})$, then it contradicts the following relation:

$$\lim_{\lambda \to \lambda_{r_0}} T_\lambda(r_0) = T_{\lambda_{r_0}}(r_0) = \tilde{g}(r_0) < \tilde{g}(r_*) = L.$$

Thus, we define a function $\lambda_L(r)$ of $r \in (0, r_*]$.

We claim that for any $r_1 \in (r_*, 1-\varepsilon)$, the equation $T_\lambda(r_1) = L$ has no solution. Indeed, since $(0, r_1] \subset (0, F^{-1}(\frac{1}{\lambda})]$ for all $\lambda < \lambda_{r_1}$, $T_\lambda(r_1)$ makes sense only for $\lambda < \lambda_{r_1}$. However, for any $\lambda < \lambda_{r_1}$, we have

$$T_\lambda(r_1) > T_{\lambda_{r_1}}(r_1) = \tilde{g}(r_1) > \tilde{g}(r_*) = L,$$

which implies that there does not exist $\lambda > 0$ such that $T_\lambda(r_1) = L$. The claim is true.

By the smoothness and monotonicity of T_λ in **(1)** and **(2)**, using the implicit function theorem at each $r_0 \in (0, r_*)$ yields that $\lambda_L(r) \in C^1(0, r_*)$. By the definition of $\mathcal{C}_\varepsilon$, we have

$$\begin{aligned}\mathcal{C}_\varepsilon &= \{(\lambda, r) \mid T_\lambda(r) = L \text{ for } r \in \mathcal{D}_{\varepsilon,\lambda}\} \\ &= \{(\lambda_L(r), r) \mid r \in (0, r_*], \text{ where } r_* \text{ satisfies that } \tilde{g}(r_*) = L\},\end{aligned}$$

then $\mathcal{C}_\varepsilon$ is continuously differentiable in the $(\lambda, \|u\|_\infty)$-plane. Since $r_* = F^{-1}(\frac{1}{\lambda_*}) < 1-\varepsilon$, $\mathcal{C}_\varepsilon$ will not end at the horizontal line $\|u\|_\infty = 1-\varepsilon$ but rather at some point (λ_*, r_*) on the derivative blow-up curve defined by $\{(\lambda, r) \mid \lambda F(r) = 1 \text{ and } \lambda > \lambda_{1-\varepsilon}\}$ in the $(\lambda, \|u\|_\infty)$-plane.

Case B. Assume that $\tilde{g}(r)$ possesses precisely two critical points, a pair of maximum and minimum points, on $(\lambda_{1-\varepsilon}, +\infty)$. Denote by g_m and g_M the local minimum and maximum values, respectively; see Fig.10(ii) and (iii).

Since the proofs dealing with the monotone segments of $\tilde{g}$ resemble that presented in Case A, we next only focus on analyzing the part where $L \in [g_m, g_M]$. Then for such L, there exist three positive numbers $\lambda_{r_1} \geqslant \lambda_{r_2} \geqslant \lambda_* > \lambda_{1-\varepsilon}$, with at least one of '$\geqslant$' being strict, such that

$$g(\lambda_{r_1}) = g(\lambda_{r_2}) = g(\lambda_*) = L, \text{ i.e., } \tilde{g}(r_1) = \tilde{g}(r_2) = \tilde{g}(r_*) = L.$$

According to the transformation $r = F^{-1}(\frac{1}{\lambda})$, it is clear that $r_1 \leqslant r_2 \leqslant r_* < 1-\varepsilon$ and at least one of '$\leqslant$' is strict.

By the same argument as in Case A, we obtain that for any $r \in (0, r_1] \cup [r_2, r_*]$, there exists a unique $\lambda_L(r) > 0$ such that $T_{\lambda_L(r)}(r) = L$; for $r \in (r_*, 1-\varepsilon)$, the equation $T_{\lambda_L(r)}(r) = L$ has no solution. Furthermore, for any $r_0 \in (r_1, r_2)$, the equation $T_\lambda(r_0) = L$ also has no solution for $\lambda \in (0, \lambda_{r_0})$. Indeed, it follows from the inequality relation:

$$T_\lambda(r_0) > T_{\lambda_{r_0}}(r_0) = \tilde{g}(r_0) > \tilde{g}(r_1) = \tilde{g}(r_2) = L.$$

Similarly, by the implicit function theorem, we obtain that the function $\lambda_L(r) \in C^1\{(0, r_1) \cup (r_2, r_*)\}$ and

$$\mathcal{C}_\varepsilon = \{(\lambda_L(r), r) \mid r \in (0, r_1] \cup [r_2, r_*] \text{ with } r_1, r_2 \text{ and } r_* \text{ satisfying } \tilde{g}(r_1) = \tilde{g}(r_2) = \tilde{g}(r_*) = L\}.$$

Since at least one of '$\leqslant$' is strict in the expression $r_1 \leqslant r_2 \leqslant r_* < 1-\varepsilon$, it is clear that $\mathcal{C}_\varepsilon$ consists of two disjoint path-connected components. Clearly, $\mathcal{C}_\varepsilon$ ends at the points $(\lambda_1, r_1), (\lambda_2, r_2)$, and (λ_*, r_*) on the derivative blow-up curve.

Next, we prove that $\lim_{r\to 0} \lambda_L(r) = 0$. It suffices to show that for any given $L > 0$, $\limsup_{r\to 0} \lambda_L(r) \leqslant 0$. Otherwise, there exists a number $s > 0$ such that $\limsup_{r\to 0} \lambda_L(r) > s$. Furthermore, there exists a sequence $\{r_n\} \to 0$ satisfying that $\lambda_L(r_n) > s$ and $T_{\lambda_L(r_n)}(r_n) = L$. Since $T_\lambda(r)$ is strictly decreasing with respect to $\lambda \in (0, \lambda_r)$, it follows from **(3)** that

$$L = \lim_{n\to\infty} T_{\lambda_L(r_n)}(r_n) \leqslant \lim_{n\to\infty} T_s(r_n) = 0,$$

which is a contradiction. So $0 \leqslant \liminf_{r\to 0} \lambda_L(r) \leqslant \limsup_{r\to 0} \lambda_L(r) \leqslant 0$.

Finally, the curve $\mathcal{C}_\varepsilon$ is bounded. Indeed, for any $\{(\lambda_n, r_n)\} \subset \mathcal{C}_\varepsilon \subset (0, \lambda_r] \times \mathcal{D}_{\varepsilon,\lambda}$, if $\lambda_n \to +\infty$, then $r_n \to 0$, contradicting the fact $\lim_{n\to\infty} \lambda_n = \lim_{r_n \to 0} \lambda_L(r_n) = 0$. $\square$

Proof of Theorem 2.2. For $\frac{\sqrt{2}}{2} \leqslant \varepsilon < 1$, by Lemmas 3.1, 3.2, 3.7, 3.12, 3.13 and 3.17, we obtain the properties **(1)**–**(5)(7)** mentioned above and the following

(6)′ For any fixed $\lambda > 0$, $T_\lambda(r)$ is strictly increasing with respect to $r \in \mathcal{D}_{\varepsilon,\lambda}$.

See Fig.11(i). Then for any $L > 0$, by the above properties, there exists a unique $\lambda_* > 0$ such that $g(\lambda_*) = L$. Thus by (3.4) and properties **(1)**–**(5)(6)′(7)**, we immediately obtain the results in Theorem 2.2. The bifurcation curve $\mathcal{C}_\varepsilon$ is depicted in Fig.3. $\square$

Proof of Theorem 2.3. For $\frac{\sqrt{30}}{10} \leqslant \varepsilon < \frac{\sqrt{2}}{2}$, in addition to **(1)**–**(7)** above, the following properties are given by Lemmas 3.3 and 3.11:

(8) $G(\lambda) < 0$ and $T_\lambda(r)$ has at least one critical point, a maximum point, in $\left(0, F^{-1}\left(\frac{1}{\lambda}\right)\right)$ for all $\lambda \geqslant \lambda_a$, with $a = 1 - \sqrt{2}\varepsilon > 0$.

(9) $T'_\lambda(r) > 0$ on $\mathcal{D}_{\varepsilon,\lambda}$ for all $\lambda < \Lambda_\varepsilon$ for some $\Lambda_\varepsilon > 0$.

Set $\xi := \sup\{\varrho \,|\, T_\lambda(r)$ is strictly increasing in $\mathcal{D}_{\varepsilon,\lambda}$ for all $\lambda < \varrho\}$. By **(6)**, **(8)** and **(9)**, it is clear that $\Lambda_\varepsilon \leqslant \xi \leqslant \bar{\lambda} < \lambda_a$. Take σ_ε as in Lemma 3.18 and define

$$\bar{L} := g(\bar{\lambda}), \quad \bar{\bar{\lambda}} := \min\{\xi, \lambda_{\sigma_\varepsilon}\}, \quad \text{and} \quad \bar{\bar{L}} := T_{\bar{\bar{\lambda}}}(\sigma_\varepsilon); \tag{4.1}$$

see Fig.11(ii). Then $\sigma_\varepsilon \in (0, F^{-1}(1/\bar{\bar{\lambda}})]$ and $\bar{L} \leqslant \bar{\bar{L}}$.

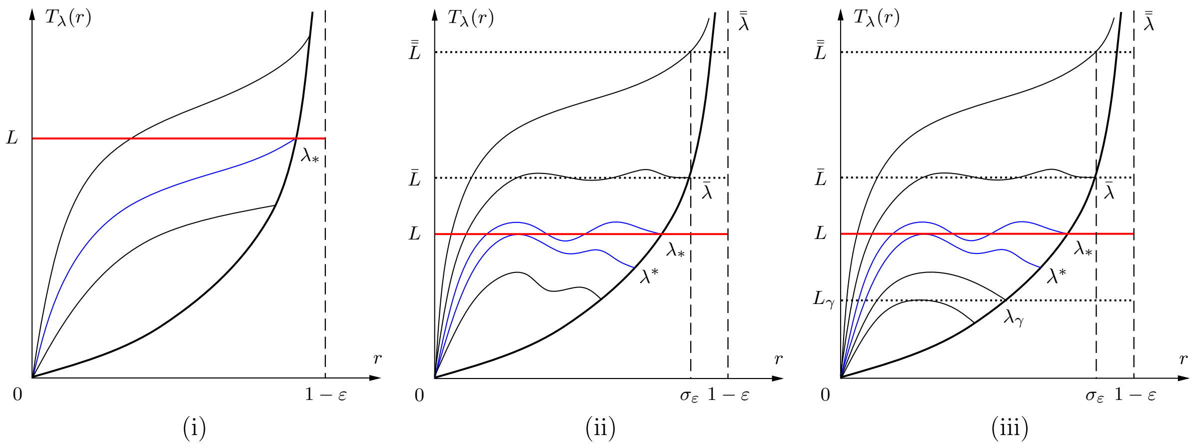

Figure 11: Possible graphs of the time-map $T_\lambda(r)$ with varying $\lambda > 0$. (i) $\frac{\sqrt{2}}{2} \leqslant \varepsilon < 1$. (ii) $\frac{\sqrt{30}}{10} \leqslant \varepsilon < \frac{\sqrt{2}}{2}$. (iii) $\check{\varepsilon} \leqslant \varepsilon < \frac{\sqrt{30}}{10}$. The numbers $\lambda_* \leqslant \lambda^*$ and $\bar{\lambda}$ satisfy that $g(\lambda_*) = L = h(\lambda^*)$ and $G(\bar{\lambda}) = 0$, respectively. σ_ε and $\bar{\lambda}$ are given in Lemma 3.18 and **(6)**, respectively. $\bar{\bar{\lambda}}$, $\bar{L}$ and $\bar{\bar{L}}$ are defined in (4.1).

For any $L > 0$, by **(4)** and **(7)** for g, there exists a unique $\lambda_* > 0$ such that $g(\lambda_*) = L$. Furthermore, if $L \in (0, \bar{L}]$, by **(3)** or **(5)**, there exists a unique $\lambda^*(\geqslant \lambda_*)$ such that $h(\lambda^*) = L$. If $L \in [\bar{\bar{L}}, +\infty)$, the equation $T_\lambda(r) = L$ has no solution when $\lambda > \lambda_*$ and exactly one solution when $\lambda < \bar{\bar{\lambda}}$ because of monotonicity of T_λ on $\mathcal{D}_{\varepsilon,\lambda}$.

For $L \in [\bar{\bar{L}}, +\infty)$ and $\bar{\bar{\lambda}} \leqslant \lambda \leqslant \lambda_*$, we claim that the equation $T_\lambda(r) = L$ has exactly one solution on $(\sigma_\varepsilon, 1-\varepsilon) \cap \mathcal{D}_{\varepsilon,\lambda}$. Indeed, if there exist $\lambda_0 \geqslant \bar{\bar{\lambda}}$ and $r_0 > \sigma_\varepsilon$ such that $T_{\lambda_0}(r_0) = L$ and $T'_{\lambda_0}(r_0) \leqslant 0$, since $T''_\lambda(r) > 0$ on $[\sigma_\varepsilon, 1-\varepsilon) \cap \mathcal{D}_{\varepsilon,\lambda}$, then $T'_{\lambda_0}(r) < 0$ on $(\sigma_\varepsilon, r_0)$. It indicates that

$$T_{\lambda_0}(\sigma_\varepsilon) > T_{\lambda_0}(r_0) = L \geqslant \bar{\bar{L}} = T_{\bar{\bar{\lambda}}}(\sigma_\varepsilon),$$

which contradicts the strict monotonicity of T_λ with respect to λ in **(2)**. The claim is true. Consequently, we obtain the assertions (i)–(iv). The curve $\mathcal{C}_\varepsilon$ is depicted in Fig.4. $\square$

Proof of Theorem 2.4. For $\breve{\varepsilon} \leqslant \varepsilon < \frac{\sqrt{30}}{10}$, by Lemmas 3.1, 3.3, 3.7, 3.10–3.13, 3.17 and 3.18, we obtain the properties **(1)**–**(7)(9)** mentioned above and the following

(8)$'$ $T_\lambda(r)$ has exactly one critical point, a local maximum, on $\left(0, F^{-1}\left(\frac{1}{\lambda}\right)\right]$ for all $\lambda \geqslant \lambda_\gamma$.

See Fig.11(iii). Let $\bar{L}$, $\bar{\bar{\lambda}}$, and $\bar{\bar{L}}$ be defined as in (4.1) and $L_\gamma = g(\lambda_\gamma)$. Notice that $L_\gamma < \bar{L} < \bar{\bar{L}}$. Then for any $L > 0$, by the above properties, there exists a unique $\lambda_* > 0$ such that $g(\lambda_*) = L$. Moreover, for any $L \in (0, L_\gamma]$, there exists a unique $\lambda^*(> \lambda_\gamma)$ such that $T_{\lambda^*}(r)$ has a unique local maximum $\max_{r \in \mathcal{D}_{\varepsilon,\lambda}} T_{\lambda^*}(r) = h(\lambda^*) = L$. For any $L \in (L_\gamma, \bar{L}]$, there exists a unique $\lambda^*(> \lambda_*)$ such that $h(\lambda^*) = L$. For any $L \in [\bar{\bar{L}}, +\infty)$, $T_\lambda(r)$ is strictly increasing on $\mathcal{D}_{\varepsilon,\lambda}$. Thus by (3.4) and properties **(1)**–**(7)(8)$'$(9)**, we immediately obtain the results in Theorem 2.4(i)–(iv). The bifurcation curve $\mathcal{C}_\varepsilon$ is depicted in Fig.4(B). $\square$

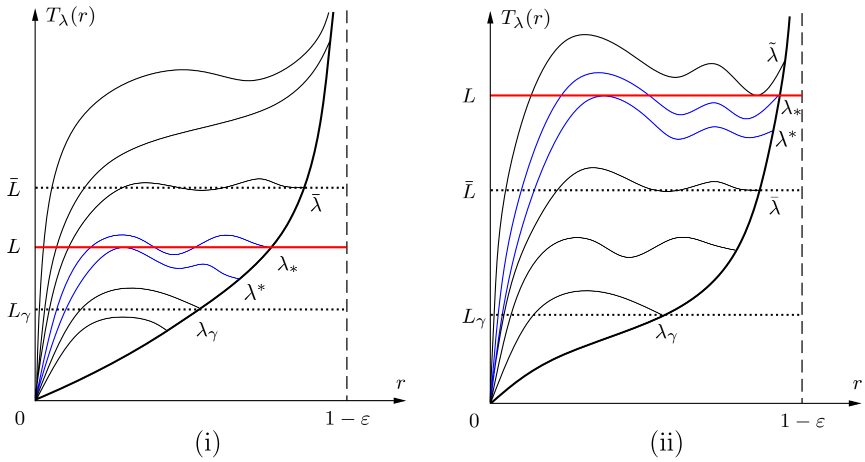

Figure 12: Possible graphs of $T_\lambda(r)$ with varying $\lambda > 0$. (i) $\tilde{\varepsilon} \leqslant \varepsilon < \check{\varepsilon}(= \frac{4\sqrt{30}}{75})$. (ii) $\frac{19}{100} \leqslant \varepsilon < \tilde{\varepsilon}(\approx 0.26262)$.

Proof of Theorem 2.5. For $\tilde{\varepsilon} \leqslant \varepsilon < \check{\varepsilon}(= \frac{4\sqrt{30}}{75})$, by Lemmas 3.1, 3.3, 3.7, 3.10, 3.12, 3.13 and 3.17, we obtain the properties **(1)**–**(7)(8)$'$** mentioned above. See Fig.12(i). Let $L_\gamma = g(\lambda_\gamma)$ and $\bar{L} = g(\bar{\lambda})$. Then for any $L > 0$, by the above properties, there exists a unique $\lambda_* > 0$ such that $g(\lambda_*) = L$. Moreover, for any $L \in (0, L_\gamma]$, there exists a unique $\lambda^*(> \lambda_\gamma)$ such that $T_{\lambda^*}(r)$ has a unique local maximum $\max_{r \in \mathcal{D}_{\varepsilon,\lambda}} T_{\lambda^*}(r) = h(\lambda^*) = L$. For any $L \in (L_\gamma, \bar{L}]$, there exists a unique $\lambda^*(> \lambda_*)$ such that $h(\lambda^*) = L$. Thus by (3.4) and properties **(1)**–**(7)(8)$'$**, we immediately obtain the results in Theorem 2.5(i)–(iv). The bifurcation curve $\mathcal{C}_\varepsilon$ is depicted in Fig.4(C). $\square$

Proof of Theorem 2.6. For $\frac{19}{100} \leqslant \varepsilon < \tilde{\varepsilon}(\approx 0.26262)$, by Lemmas 3.1, 3.3, 3.6, 3.7, 3.10, 3.12, 3.13 and 3.17, we obtain the properties **(1)–(7)(8)′** mentioned above and the following

(9)′ $T_\lambda(r)$ has at least one critical point on $(0, p_2)$ for $\lambda < \lambda_{p_2}$, where p_2 is defined in Lemma 3.4.

See Fig.12(ii). Let $L_\gamma := g(\lambda_\gamma)$ and $\bar{L} = g(\bar{\lambda})$. By the above properties, for any $L > 0$, there exists a unique $\lambda_* > 0$ such that $g(\lambda_*) = L$. Moreover, we have:

(i) For any $L \in (0, L_\gamma]$, there exists a unique $\lambda^*(> \lambda_\gamma)$ such that $T_{\lambda^*}(r)$ has a unique local maximum $\max_{r\in\mathcal{D}_{\varepsilon,\lambda}} T_{\lambda^*}(r) = h(\lambda^*) = L$.

(ii) For any $L \in (L_\gamma, \bar{L}]$, there exists a unique $\lambda^*(> \lambda_*)$ such that $h(\lambda^*) = L$.

(iii) For any $L \in (\bar{L}, +\infty)$, there exists a unique λ^* such that $h(\lambda^*) = L$ and there exists $0 < \tilde{\lambda} < \max\{\lambda^*, \lambda_*\}$ satisfying $\tilde{\lambda} = \inf\{\lambda \mid T_\lambda(r)$ has a local minimum at some $r_0 \in (0, 1-\varepsilon)$ such that $T_\lambda(r_0) = L\}$.

Thus by (3.4) and properties **(1)–(7)(8)′(9)′**, we immediately obtain the results in Theorem 2.6(i)–(iv). The bifurcation curve $\mathcal{C}_\varepsilon$ is depicted in Fig.4(D). □

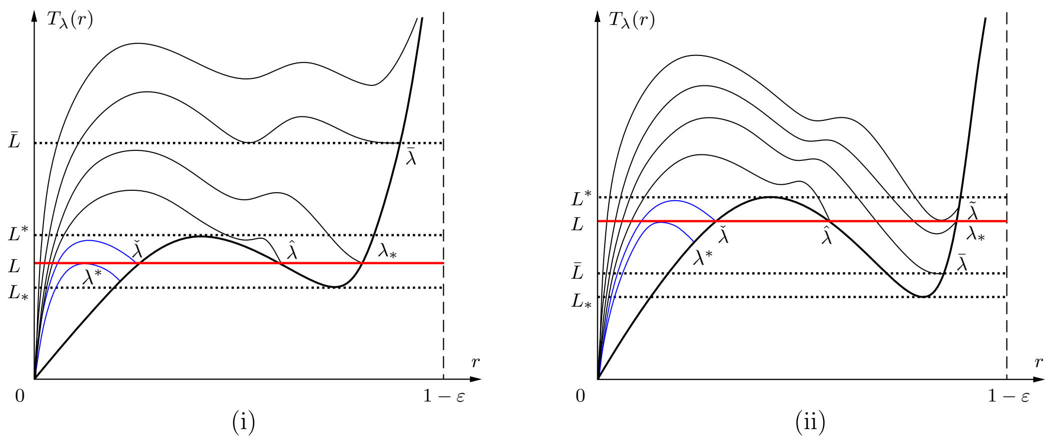

Figure 13: Possible graphs of the time-map $T_\lambda(r)$ for $\varepsilon \leqslant \varepsilon_*(\approx 0.13123)$ with varying $\lambda > 0$. (i) $L^* < \bar{L}$. (ii) $L^* > \bar{L}$.

Proof of Theorem 2.8. For $0 < \varepsilon \leqslant \varepsilon_*(\approx 0.13123)$, by Lemmas 3.1, 3.3, 3.6, 3.7, 3.10, 3.12 and 3.14–3.17, we obtain the properties **(1)–(6)(8)′** mentioned above and the following

(7)′ $g'(\lambda_\gamma) > 0$.

(9)″ $T_\lambda(r)$ has exactly one critical point, a local maximum, on $(0, \gamma)$ for all $\lambda < \lambda_\gamma$.

(10) $g(\lambda)$ has exactly two critical points on $(\lambda_{1-\varepsilon}, +\infty)$, a local minimum λ_m and a local maximum λ_M satisfying $\lambda_{1-\varepsilon} < \lambda_m < \lambda_\gamma < \lambda_M$.

See Fig.13. Let $L_* = g(\lambda_m)$, $L^* = g(\lambda_M)$, and $\bar{L} = g(\bar{\lambda})$. Then $L_* < \min\{\bar{L}, L^*\}$, and for any $L > 0$, there exist positive numbers λ^*, λ_* such that $h(\lambda^*) = g(\lambda_*) = L$. Moreover, for any $L_* \leqslant L < L^*$ and $L \leqslant \bar{L}$, there exist positive numbers $\check{\lambda}$ and $\hat{\lambda}$ with $\lambda_* < \hat{\lambda} < \check{\lambda} < \lambda^*$ such that $g(\check{\lambda}) = g(\hat{\lambda}) = L$. If $\bar{L} < L^*$, then for any $\bar{L} < L < L^*$, there exist positive numbers $\check{\lambda}$, $\hat{\lambda}$ and $\tilde{\lambda}$ with $\tilde{\lambda} \leqslant \lambda_* < \hat{\lambda} < \check{\lambda} < \lambda^*$ satisfying $g(\hat{\lambda}) = g(\check{\lambda}) = L$ and $\tilde{\lambda} = \inf\{\lambda \mid T_\lambda(r)$ has a local minimum at some $r_0 \in (0, 1-\varepsilon)$ such that $T_\lambda(r_0) = L\}$. Thus by (3.4) and properties **(1)–(6)(7)′(8)′(9)″(10)**, we immediately obtain the results in Theorem 2.8(i)–(v). The bifurcation curve $\mathcal{C}_\varepsilon$ is depicted in Fig.6. □

Proof of Theorem 2.7. For $\varepsilon_* < \varepsilon < \frac{19}{100}$, by Lemmas 3.1, 3.3, 3.6, 3.7, 3.10, 3.12 and 3.14–3.16, we obtain the properties **(1)–(6)(8)′(9)″** mentioned above and the following

(10)′ $g(\lambda)$ is strictly increasing or has exactly two critical points on $(\lambda_{1-\varepsilon}, +\infty)$.

See Fig.12(ii) and Fig.13(i)-(ii). Let $\bar{L} = g(\bar{\lambda})$. If $g(\lambda)$ is strictly increasing on $(\lambda_{1-\varepsilon}, +\infty)$, then we can get the same results (i)–(iii) in the proof of Theorem 2.6 by the same arguments; If $g(\lambda)$ has two critical points, then we can get the same results in the proof of Theorem 2.8 by the same arguments. Hence, we immediately obtain the results in Theorem 2.7(i)–(iii). The bifurcation curve $\mathcal{C}_\varepsilon$ is depicted in Fig.5. $\square$

5. Conjectures

Basing on the main theorems and lemmas in previous sections, we propose two conjectures on the global bifurcation curve $\mathcal{C}_\varepsilon$ of (1.2). The first one is about the shape for the key function $g(\lambda)$ — the value of time-map $T_\lambda(r)$ at the right endpoint of the domain $\mathcal{D}_{\varepsilon,\lambda}$. Recall that L_0, ε_*, $\hat{\varepsilon}$, and $\check{\varepsilon}$ are the known constants; see Table 1 in Section 1.

Conjecture 5.1 (See Fig.14)**.** *Let $g(\lambda)$ be given in (3.18). There exists a constant $\varepsilon_3 \in (0.13, 0.19)$ such that if $0 < \varepsilon < \varepsilon_3$, then $g(\lambda)$ has exactly two critical points, a local minimum and a local maximum, on $(\lambda_{1-\varepsilon}, +\infty)$; while if $\varepsilon_3 \leqslant \varepsilon < 1$, then $g(\lambda)$ is strictly decreasing on $(\lambda_{1-\varepsilon}, +\infty)$.*

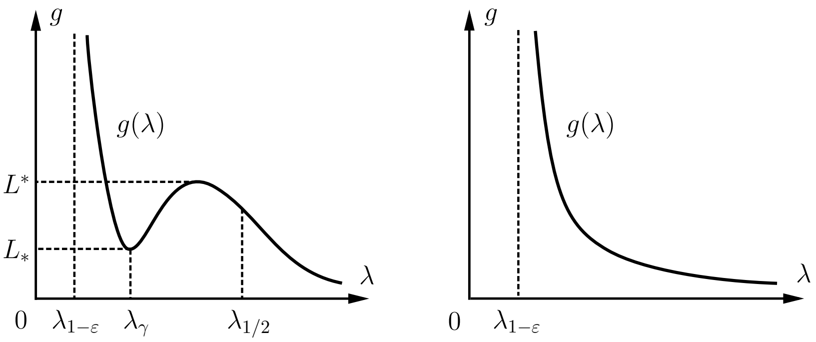

Figure 14: A conjecture on the shape of $g(\lambda)$: there exists a threshold value $\varepsilon_3 \in (0.13, 0.19)$ such that either g has exactly two critical points, a pair of maximum and a minimum points, or it is strictly decreasing on $(\lambda_{1-\varepsilon}, +\infty)$, depending on $\varepsilon < \varepsilon_3$ (Left) or $\varepsilon \geqslant \varepsilon_3$ (Right).

Remark 5.1. *In view of Lemmas 3.12–3.16, it remains to prove that the conjecture is true on $(\lambda_{1-\varepsilon}, \lambda_{1/2})$ for $\varepsilon \in (\varepsilon_*, \frac{19}{100})$. Denote by λ_3^* the unique critical point of $g(\lambda)$ provided that $\varepsilon = \varepsilon_3$, and define $L_1 := g(\lambda_3^*)$. Numerical simulation shows that the threshold values $\varepsilon_3 \approx 0.14622 > \varepsilon_*$ and $L_1 \approx 0.36983 > L_0$.*

The second conjecture is about the complete classification and evolution of bifurcation curve $\mathcal{C}_\varepsilon$ of (1.2).

Conjecture 5.2 (See Fig.15)**.** *For any $\varepsilon \in (0,1)$ and $L > 0$, the global bifurcation curve $\mathcal{C}_\varepsilon$ of (1.2) is strictly increasing or $\supset$-shaped or S-shaped in the $(\lambda, \|u\|_\infty)$-plane. Precisely, there exist eight positive constants $L_i (i = 1, 2, 3)$ and $\varepsilon_j (j = 1, \cdots, 5)$, with $L_0 < L_1 < L_2 < L_3$ and $\varepsilon_1 < \varepsilon_2 < \varepsilon_* < \varepsilon_3 < \frac{19}{100} < \hat{\varepsilon} < \varepsilon_4 < \check{\varepsilon} < \varepsilon_5 < \frac{\sqrt{2}}{2}$, and six continuous curves $\bar{L}_1(\varepsilon), \bar{L}_2(\varepsilon), \bar{\bar{L}}(\varepsilon), L_*(\varepsilon), L^*(\varepsilon)$, and $\check{L}(\varepsilon)$ in the (L, ε)-plane such that*

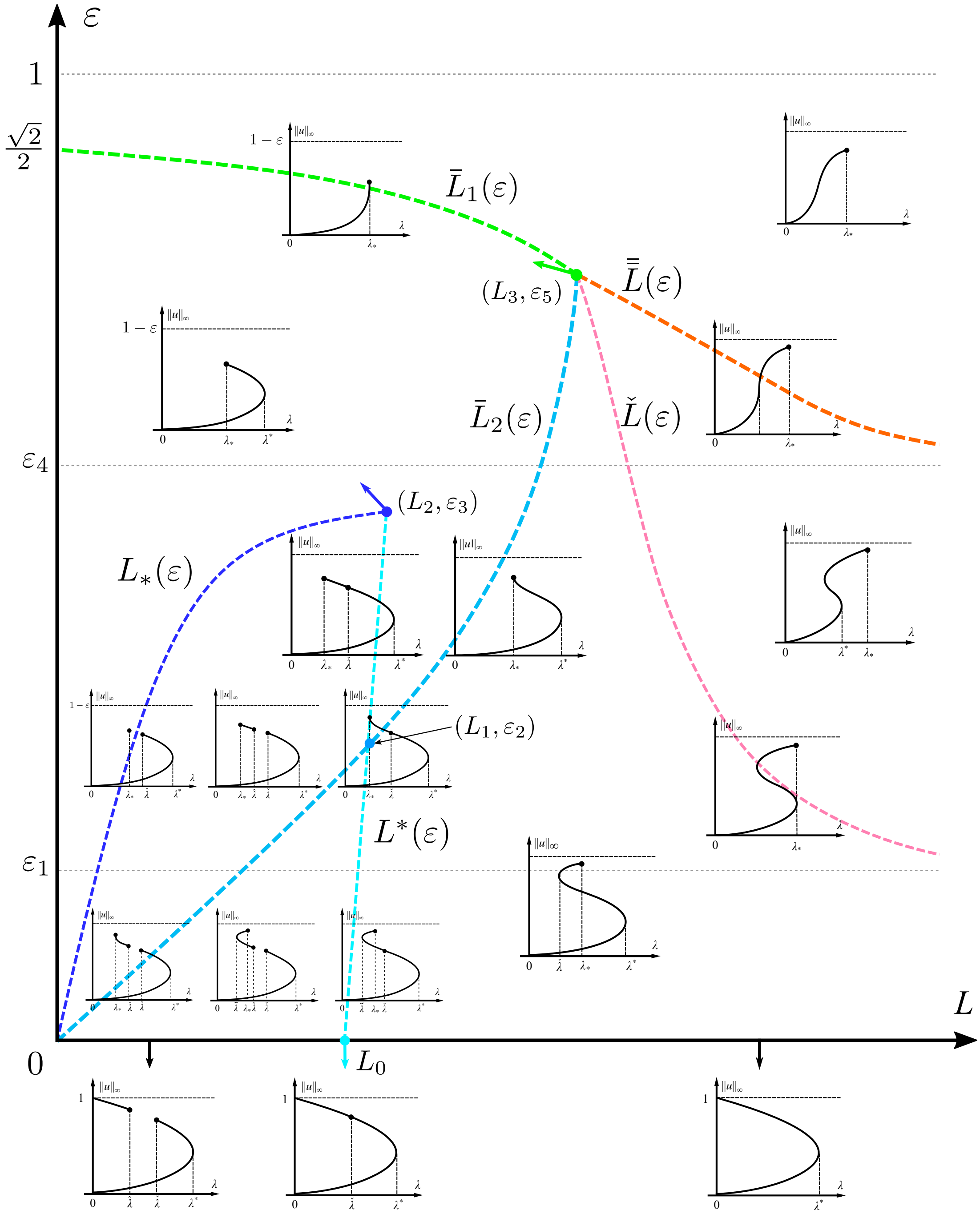

Figure 15: A conjecture on the complete classification and evolution of bifurcation curves $\mathcal{C}_\varepsilon$ of (1.2) in the $(\lambda, \|u\|_\infty)$-plane with varying evolution parameters $\varepsilon \in [0, 1)$ and $L > 0$: there exist eight constants $L_1, L_2, L_3, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5$, and six continuous curves $\bar{L}_1(\varepsilon), \bar{L}_2(\varepsilon), \bar{\bar{L}}(\varepsilon), L_*(\varepsilon), L^*(\varepsilon)$, and $\check{L}(\varepsilon)$ in the (L, ε)-plane such that in the separated regions or on these curves, $\mathcal{C}_\varepsilon$ is one of three alternatives: increasing, $\supset$-shaped or S-shaped, and it is path-connected except for the region $\{(L, \varepsilon) \mid 0 \leqslant \varepsilon < \varepsilon_3 \text{ and } L_*(\varepsilon) \leqslant L < L^*(\varepsilon)\}$, where $\mathcal{C}_\varepsilon$ consists of two disjoint path-connected components: the lower component is a $\supset$-shaped curve, and the upper component is either a decreasing curve, a $\subset$-shaped curve, or a singleton. The case of $\varepsilon = 0$ has been known in Theorem 1.1; some partial results for the conjecture have been proven in Theorems 2.1–2.8.

the following assertions hold:
(a) $\bar{L}_1(\varepsilon)$ *and* $\bar{L}_2(\varepsilon)$ *emanate from the point* $(0, \frac{\sqrt{2}}{2})$ *and the origin respectively and both stop at the point* (L_3, ε_5)*; both* $\bar{\bar{L}}(\varepsilon)$ *and* $\check{L}(\varepsilon)$ *emanate from the point* (L_3, ε_5) *and take* $\varepsilon = \varepsilon_4$ *and* $\varepsilon = \varepsilon_1$ *as asymptotes, respectively;* $L_*(\varepsilon)$ *and* $L^*(\varepsilon)$ *emanate from the origin and the point* $(L_0, 0)$ *respectively and both stop at the point* (L_2, ε_3)*;* $L^*(\varepsilon)$ *and* $\bar{L}_2(\varepsilon)$ *intersect at* (L_1, ε_2)*.*
(b) $\mathcal{C}_\varepsilon$ *is strictly increasing as* (L, ε) *is in the region above* $\bar{L}_1(\varepsilon)$ *and* $\bar{\bar{L}}(\varepsilon)$*, containing the curves* $\bar{L}_1(\varepsilon)$ *and* $\bar{\bar{L}}(\varepsilon)$*.*
(c) $\mathcal{C}_\varepsilon$ *is* $\supset$*-shaped as* (L, ε) *is in the region below* $\bar{L}_1(\varepsilon)$ *and to the left of* $\bar{L}_2(\varepsilon)$*, containing the curve* $\bar{L}_2(\varepsilon)$*.*
(d) $\mathcal{C}_\varepsilon$ *is* S*-shaped as* (L, ε) *is in the region below* $\bar{\bar{L}}(\varepsilon)$ *and to the right of* $\bar{L}_2(\varepsilon)$*. Furthermore, the stopping point of* $\mathcal{C}_\varepsilon$ *is on the left (respectively, the right) of the first turning point as* (L, ε) *is in the subregion below and to the left (respectively, above and to the right) of* $\check{L}(\varepsilon)$*.*
(e) $\mathcal{C}_\varepsilon$ *is split into two disjoint* C^1 *components as* (L, ε) *is in the region enclosed by* $L_*(\varepsilon)$*,* $L^*(\varepsilon)$ *and* $\varepsilon = 0$*, containing the curve* $L_*(\varepsilon)$*.*
(f) $\mathcal{C}_\varepsilon$ *has a vertical tangent (respectively, two vertical tangents) as* (L, ε) *is on the curves* $\bar{L}_1(\varepsilon)$ *and* $\bar{\bar{L}}(\varepsilon)$ *(respectively, on the curve* $\bar{L}_2(\varepsilon)$*).*

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Appendix

In this appendix, we give rigorous proofs of inequalities (3.27) and (3.31) appearing in Lemmas 3.13 and 3.16, respectively. We use an idea from [41, Appendix A] and apply the following Sturm's Theorem to determine the precise number of different real zeros of relevant univariate polynomials on the given intervals. We have used the symbolic manipulator *Maple* to check all the computations.

Theorem 5.3 (Sturm's Theorem)**.** *Assume that* $\tau_1, \tau_2 \in \mathbb{R}$ *and* $Q(x)$ *is a polynomial such that* $Q(\tau_1) = Q(\tau_2) \neq 0$*. Let the Sturm sequence* $\{Q_0, Q_1, \ldots, Q_k\}$ *of* Q *be defined by*

$$\begin{aligned} &Q_0 = Q,\ Q_1 = Q',\ Q_2 = -\operatorname{rem}(Q_0, Q_1), \\ &Q_3 = -\operatorname{rem}(Q_1, Q_2), \ldots, 0 = -\operatorname{rem}(Q_{k-1}, Q_k), \end{aligned}$$

where $\operatorname{rem}(Q_i, Q_j)$ *is the remainder of the polynomial long division of* Q_i *by* Q_j*,* $'$ *means* $\frac{\mathrm{d}}{\mathrm{d}x}$*, and* k *is the minimal number of polynomial divisions needed to obtain a zero remainder. Let* $\sigma_Q(\zeta)$ *denote the number of sign changes (ignoring zeros) in the sequence*

$$\{Q_0(\zeta), Q_1(\zeta), \ldots, Q_k(\zeta)\}.$$

Then Q *has* $\sigma_Q(\tau_1) - \sigma_Q(\tau_2)$ *distinct real zeros on the interval* (τ_1, τ_2)*.*

This theorem can be found in standard textbooks of algebra or polynomials, e.g., [42, 43]. Denote by $\mathrm{sgn}(u)$ the sign function

$$\mathrm{sgn}(u) = \begin{cases} 1, & \text{if } u > 0, \\ 0, & \text{if } u = 0, \\ -1, & \text{if } u < 0. \end{cases}$$

For given finite real sequence $s = \{s_i\}_{i=1}^{m}$, we also denote by $\mathrm{sgn}(s)$ the sign sequence

$$\mathrm{sgn}(s) = \{\mathrm{sgn}(s_1), \mathrm{sgn}(s_2), \ldots, \mathrm{sgn}(s_m)\}.$$

Proof of inequality (3.27)**.** We divide the proof into three steps:

Step 1. We prove that $q(u) > 0$ on $(0, 1-\varepsilon)$ for all $\varepsilon \in (\frac{9}{20}, 1)$.

Set $q_i(u) := \frac{\partial^i}{\partial u^i}(q(u))$, $i = 1, 2, \ldots, 5$. Successive differentiation yields

$$\begin{aligned} q_1(u) &= 5(\frac{2\varepsilon^2}{3} - 2)u^4 + 4(9 - \frac{10\varepsilon^2}{3})u^3 + 3(-16 - \frac{4\varepsilon^4}{3} + \frac{32\varepsilon^2}{3})u^2 + 2(4\varepsilon^4 - 16\varepsilon^2 + 14)u \\ &\quad + (-4\varepsilon^4 + 10\varepsilon^2 - 6), \\ q_2(u) &= 20(\frac{2\varepsilon^2}{3} - 2)u^3 + 12(9 - \frac{10\varepsilon^2}{3})u^2 + 6(-16 - \frac{4\varepsilon^4}{3} + \frac{32\varepsilon^2}{3})u + 2(4\varepsilon^4 - 16\varepsilon^2 + 14), \\ q_3(u) &= 60(\frac{2\varepsilon^2}{3} - 2)u^2 + 24(9 - \frac{10\varepsilon^2}{3})u + 6(-16 - \frac{4\varepsilon^4}{3} + \frac{32\varepsilon^2}{3}), \\ q_4(u) &= 120(\frac{2\varepsilon^2}{3} - 2)u + 24(9 - \frac{10\varepsilon^2}{3}), \\ q_5(u) &= 80\varepsilon^2 - 240. \end{aligned}$$

Then for any $\varepsilon \in (\frac{9}{20}, 1)$, we proceed orderly to complete Step 1 as follows:

(1) Since $q_5(u) < 0$ on $(0, 1-\varepsilon)$ and $q_4(1-\varepsilon) = -80\varepsilon^3 + 240\varepsilon - 24 > 0$, it follows that $q_4(u) > 0$ on $(0, 1-\varepsilon)$;

(2) Since $q_4(u) > 0$ on $(0, 1-\varepsilon)$ and $q_3(1-\varepsilon) = 8\varepsilon(4\varepsilon^3 - 12\varepsilon + 3) < 0$, it follows that $q_3(u) < 0$ on $(0, 1-\varepsilon)$;

(3) Since $q_3(u) < 0$ on $(0, 1-\varepsilon)$ and $q_2(1-\varepsilon) = -\frac{4\varepsilon^2}{3}(4\varepsilon^3 - 12\varepsilon + 5) > 0$, it follows that $q_2(u) > 0$ on $(0, 1-\varepsilon)$;

(4) Since $q_2(u) > 0$ on $(0, 1-\varepsilon)$ and $q_1(1-\varepsilon) = -\frac{2\varepsilon^3}{3}(\varepsilon + 2)(\varepsilon - 1)^2 < 0$, it follows that $q_1(u) < 0$ on $(0, 1-\varepsilon)$;

(5) Since $q_1(u) < 0$ on $(0, 1-\varepsilon)$ and $q(1-\varepsilon) = \frac{2\varepsilon^4}{3}(\varepsilon + 2)(\varepsilon - 1)^2 > 0$, it follows that $q(u) > 0$ on $(0, 1-\varepsilon)$.

Step 2. We prove that $q_1(u) < 0$ on $(0, 1-\varepsilon)$ for all $\varepsilon \in (\frac{3}{10}, \frac{9}{20}]$. Similar to [41, Appendix A], we prove it by applying Sturm's Theorem (Theorem 5.3).

Set $q_1(u) = \frac{\tilde{q_1}(u,x,x)}{3}$, where

$$\begin{aligned} \tilde{q_1}(u, a, b) :=& (10a^2 - 30)u^4 + (-40b^2 + 108)u^3 + (-12b^4 + 96a^2 - 144)u^2 \\ &+ (24a^4 - 96b^2 + 84)u - 12b^4 + 30a^2 - 18. \end{aligned}$$

Let $\{k_i\}_{i=0}^{301}$ be a uniform partition of $(\frac{3}{10}, \frac{9}{20}]$. Then for $u \in (0, \frac{7}{10})$ and $\varepsilon \in (\frac{3}{10}, \frac{9}{20}]$ (notice that $(0, 1-\varepsilon) \subset (0, \frac{7}{10})$ for all $\varepsilon \in (\frac{3}{10}, \frac{9}{20}]$), we have $q_1(u) < \frac{q_1^i(u)}{3}$ $(i = 0, 1, \ldots, 300)$, where $q_1^i(u) := \tilde{q_1}(u, k_{i+1}, k_i)$ since for $\tilde{q_1}(u, a, b)$ all coefficients of terms $a^n (n = 2, 4)$ are positive and all coefficients of terms $b^n (n = 2, 4)$ are negative.

Next we use the Sturm sequence in Theorem 5.3 to prove that $q_1^i(u) < 0$ for $u \in (0, \frac{7}{10})$. Let $s_{q_1^i}(u)$ be the Sturm sequence of $q_1^i(u)$ for $i = 0, 1, \dots 300$. Direct computations yield that for $i = 0, 1, \dots 300$,

$$\operatorname{sgn}(s_{q_1^i}(0)) = \{-1, 1, -1, 1, 1\}, \quad \operatorname{sgn}(s_{q_1^i}(\frac{7}{10})) = \{-1, -1, 1, -1, 1\}.$$

Denote by $\sigma_{q_1^i}(u)$ the number of sign changes in the Sturm sequence $q_1^i(u)$. Notice that $q_1^i(0)$ and $q_1^i(\frac{7}{10})$ are the first terms of Sturm sequences $s_{q_1^i}(0)$ and $s_{q_1^i}(\frac{7}{10})$ respectively. Then for $i = 0, 1, \dots 300$, we have

$$q_1^i(0) < 0,\ q_1^i(\frac{7}{10}) < 0,\ \sigma_{q_1^i}(0) - \sigma_{q_1^i}(\frac{7}{10}) = 3 - 3 = 0.$$

This, together with Sturm's Theorem, implies that $q_1^i(u) < 0$ on $(0, \frac{7}{10})$ for $i = 0, 1, \dots 300$.

Therefore, $q_1(u) < 0$ on $(0, \frac{7}{10})$ for $\varepsilon \in (\frac{3}{10}, \frac{9}{20}]$. Since $q(0) > 0$ and $q(1-\varepsilon) > 0$ for all $\varepsilon \in (0, 1)$, it follows that $q(u) > 0$ on $(0, 1-\varepsilon)$ for all $\varepsilon \in (\frac{3}{10}, \frac{9}{20}]$.

Step 3. We prove that $q(u) > 0$ on $(0, 1-\varepsilon)$ for all $\varepsilon \in [\frac{19}{100}, \frac{3}{10}]$.

Set $q(u) = \frac{\tilde{q}(u,x,x)}{3}$, where

$$\begin{aligned}\tilde{q}(u, a, b) :=& (2a^2 - 6)u^5 + (-10b^2 + 27)u^4 + (-4b^4 + 32a^2 - 48)u^3 + (12a^4 - 48b^2 + 42)u^2 \\ &+ (-12b^4 + 30a^2 - 18)u + 3a^4 - 6b^2 + 3.\end{aligned}$$

Let $\{k_i\}_{i=0}^{1501}$ be a uniform partition of $[\frac{19}{100}, \frac{3}{10}]$. Then for $u \in (0, \frac{81}{100})$, $\varepsilon \in [\frac{19}{100}, \frac{3}{10}]$ and $i = 0, 1, \dots, 1500$, we have $q(u) > \frac{q^i(u)}{3}$, where $q^i(u) := \tilde{q}(u, k_i, k_{i+1})$ since for $\tilde{q}(u, a, b)$ all coefficients of terms $a^n (n = 2, 4)$ are positive and all coefficients of terms $b^n (n = 2, 4)$ are negative.

Similar to Step 2, we use the Sturm sequence to prove that $q^i(u) < 0$ for $u \in (0, \frac{81}{100})$. Let $s_{q^i}(u)$ be the Sturm sequence of $q^i(u)$ for $i = 0, 1, \dots 1500$. We compute that for $i = 0, 1, \dots 1500$,

$$\operatorname{sgn}(s_{q^i}(0)) = \begin{cases} \{1, -1, 1, -1, -1, 1\} & \text{for } i = 0, 1, \dots, 350, \\ \{1, -1, 1, 1, -1, 1\} & \text{for } i = 351, 352, \dots, 1500, \end{cases}$$
$$\operatorname{sgn}(s_{q^i}(\frac{81}{100})) = \{1, -1, -1, 1, -1, 1\}.$$

Denote by $\sigma_{q^i}(u)$ the number of sign changes in the Sturm sequence $q^i(u)$. Notice that $q^i(0)$ and $q^i(\frac{81}{100})$ are the first terms of Sturm sequences $s_{q^i}(0)$ and $s_{q^i}(\frac{81}{100})$ respectively. Then for $i = 0, 1, \dots 1500$,

$$q^i(0) > 0,\ q^i(\frac{81}{100}) > 0,\ \sigma_{q^i}(0) - \sigma_{q^i}(\frac{81}{100}) = 4 - 4 = 0.$$

This, together with Sturm's Theorem, implies that $q^i(u) > 0$ on $(0, \frac{81}{100})$ for $i = 0, 1, \dots 1500$. Thus $q(u) > 0$ on $(1-\varepsilon) \subset (0, \frac{81}{100})$ for all $\varepsilon \in [\frac{19}{100}, \frac{3}{10}]$. □

Proof of (3.31)**.** The proof is similar to the previous one, so we here only show the key steps and omit some tedious calculations.

Set $p_i(u) := \frac{\partial^i}{\partial u^i}(p(u))$, $i = 1, 2, \dots, 15$. For any $\varepsilon \in (0, \frac{19}{100})$, successive differentiation and computation yield:

(1) $p_{15}(u) = 10461394944000(\varepsilon^2 - 3)^3 < 0$ and $p_{14}(1) = 0$. Then $p_{14}(u) > 0$ for all $u \in (0, 1-\varepsilon)$.

(2) $p_{13}(0) = 4151347200(3-\varepsilon^2)(64\varepsilon^6 - 1644\varepsilon^4 + 8136\varepsilon^2 - 11313) < 4151347200(3-\varepsilon^2)\cdot(-11019.28739) < 0$ and $p_{13}(1-\varepsilon) = 4151347200(\varepsilon^2 - 3)(1196\varepsilon^6 - 7176\varepsilon^4 + 10764\varepsilon^2 - 27)$. Then $p_{13}(1-\varepsilon)$ has only one zero on

$(0, \frac{19}{100})$, denoting by $x_1(\approx 0.05126)$. It follows from (1) that $p_{13}(u) < 0$ on $(0, 1-\varepsilon)$ for $\varepsilon \in (x_1, \frac{19}{100})$ and p_{13} has only one zero on $(0, 1-\varepsilon)$ for $\varepsilon \in (0, x_1)$.

(3) $p_{12}(0) = 265686220800\varepsilon^8 - 4134741811200\varepsilon^6 + 22865620377600\varepsilon^4 - 54137718835200\varepsilon^2 + 46727085081600 > 4.47725 \cdot 10^{13} > 0$ and $p_{12}(1-\varepsilon) = (-14778796032\varepsilon^9 + 133009164288\varepsilon^7 - 399027492864\varepsilon^5 + 400148356608\varepsilon^3 - 3362591232\varepsilon - 129330432) \cdot 10^2$. By applying Sturm's Theorem we obtain that $p_{12}(1-\varepsilon)$ has one zero on $(0, \frac{19}{100})$, denoting by x_2. Since $p_{12}(1-\varepsilon)|_{\varepsilon=0} < 0$ and $p_{12}(1-\varepsilon)|_{\varepsilon=x_1} < 0$, we have $x_2 > x_1$, and $p_{12}(1-\varepsilon) < 0$ on $(0, x_2)$, $p_{12}(1-\varepsilon) > 0$ on $(x_2, \frac{19}{100})$. Combining with (2), we obtain that on $(0, 1-\varepsilon)$, $p_{12}(u)$ has no zero for $\varepsilon \in (x_2, \frac{19}{100})$, exactly one zero for $\varepsilon \in (0, x_2)$. Since $p_{12}(1-\varepsilon)|_{\varepsilon=\frac{13}{100}} > 0$, it follows that $x_2 < \frac{13}{100}$.

(4) $p_{11}(0) = 3512678400\varepsilon^{10} - 164457216000\varepsilon^8 + 1726321766400\varepsilon^6 - 7600318387200\varepsilon^4 + 15286857062400\varepsilon^2 - 11588006707200 < -1.10360 \cdot 10^{13} < 0$ and $p_{11}(1-\varepsilon) = 479001600\varepsilon(640\varepsilon^9 - 5760\varepsilon^7 + 17280\varepsilon^5 - 17388\varepsilon^3 + 324\varepsilon + 27)$. By applying Sturm's Theorem we obtain that $p_{11}(1-\varepsilon)$ has one zero on $(0, \frac{19}{100})$, denoting it by x_3. Since $p_{11}(1-\varepsilon)|_{\varepsilon=\frac{19}{100}} < 0$ and $p_{11}(1-\varepsilon)|_{\varepsilon=\frac{13}{100}} > 0$, we get that $x_3 > \frac{13}{100} > x_2$, $p_{11}(1-\varepsilon) > 0$ on $(0, x_3)$, and $p_{11}(1-\varepsilon) < 0$ on $(x_3, \frac{19}{100})$. Combining them with (3), we obtain that on $(0, 1-\varepsilon)$, $p_{11}(u)$ has no zero for $\varepsilon \in (x_3, \frac{19}{100})$ and exactly one zero for $\varepsilon \in (0, x_3)$.

(5) $p_{10}(0) = -3512678400\varepsilon^{10} + 75953203200\varepsilon^8 - 580898304000\varepsilon^6 + 2071246464000\varepsilon^4 - 3517003929600\varepsilon^2 + 2291304153600 > 2.16431 \cdot 10^{12} > 0$ and $p_{10}(1-\varepsilon) = -2419200\varepsilon^2(19184\varepsilon^9 - 172656\varepsilon^7 - 24\varepsilon^6 + 517968\varepsilon^5 + 144\varepsilon^4 - 523908\varepsilon^3 - 216\varepsilon^2 + 17820\varepsilon + 2349)$. Since $p_{10}(1-\varepsilon)|_{\varepsilon=\frac{19}{100}} < 0$, by applying Sturm's Theorem we get that $p_{10}(1-\varepsilon) < 0$ on $(0, \frac{19}{100})$. Combining them with (4), we obtain that $p_{10}(u)$ has exactly one zero on $(0, 1-\varepsilon)$ for $\varepsilon \in (0, \frac{19}{100})$.

(6) $p_9(0) = (-3386880\varepsilon^{12} + 206115840\varepsilon^{10} - 2784983040\varepsilon^8 + 16266749184\varepsilon^6 - 47491702272\varepsilon^4 + 67927870080\varepsilon^2 - 37613600640) \cdot 10 < -3.51606 \cdot 10^{11} < 0$ and $p_9(1-\varepsilon) = 725760\varepsilon^3(7140\varepsilon^9 - 64260\varepsilon^7 - 80\varepsilon^6 + 192780\varepsilon^5 + 480\varepsilon^4 - 196491\varepsilon^3 - 720\varepsilon^2 + 11133\varepsilon + 1890) > 725760\varepsilon^3 \cdot 515.698065 > 0$. It follows from (5) that $p_9(u)$ has exactly one zero on $(0, 1-\varepsilon)$.

(7) $p_8(0) = 33868800\varepsilon^{12} - 883491840\varepsilon^{10} + 8385914880\varepsilon^8 - 38658332160\varepsilon^6 + 93269232000\varepsilon^4 - 112341116160\varepsilon^2 + 52702151040 > 48644817980 > 0$ and $p_8(1-\varepsilon) = -241920\varepsilon^4(1708\varepsilon^9 - 15372\varepsilon^7 - 148\varepsilon^6 + 46116\varepsilon^5 + 888\varepsilon^4 - 47745\varepsilon^3 - 1332\varepsilon^2 + 4887\varepsilon + 630) < -241920\varepsilon^4 \cdot 254.2874762 < 0$. Combining them with (6), we obtain that $p_8(u)$ has exactly one zero on $(0, 1-\varepsilon)$.

(8) $p_7(0) = -16934400\varepsilon^{12} + 291997440\varepsilon^{10} - 2112082560\varepsilon^8 + 7896026880\varepsilon^6 - 15884346240\varepsilon^4 + 16143442560\varepsilon^2 - 6430596480 < 291997440(\frac{19}{100})^{10} + 7896026880(\frac{19}{100})^6 + 16143442560(\frac{19}{100})^2 - 6430596480 = -5847446710 < 0$ and $p_7(1-\varepsilon) = 40320\varepsilon^5(493\varepsilon^9 - 4437\varepsilon^7 - 408\varepsilon^6 + 13311\varepsilon^5 + 2448\varepsilon^4 - 14920\varepsilon^3 - 3672\varepsilon^2 + 4827\varepsilon - 432)$. By applying Sturm's Theorem we get that $p_7(1-\varepsilon)$ has exactly one zero on $(0, \frac{19}{100})$. Combining them with (7), we obtain that $p_7(u)$ has at most two zeros on $(0, 1-\varepsilon)$.

(9) $p_6(0) = 5685120\varepsilon^{12} - 76930560\varepsilon^{10} + 450593280\varepsilon^8 - 1398677760\varepsilon^6 + 2366988480\varepsilon^4 - 2039022720\varepsilon^2 + 693696960 > 620022433.1 > 0$ and $p_6(1-\varepsilon) = 5760\varepsilon^6(45\varepsilon^9 - 405\varepsilon^7 + 1015\varepsilon^6 + 1215\varepsilon^5 - 6090\varepsilon^4 + 1498\varepsilon^3 + 9135\varepsilon^2 - 8139\varepsilon + 2131)$. Since $p_6(1-\varepsilon)|_{\varepsilon=\frac{19}{100}} > 0$, by applying Sturm's Theorem we obtain that and $p_6(1-\varepsilon) > 0$ on $(0, \frac{19}{100})$. Combining them with (8), we obtain that $p_6(u)$ has at most two zeros on $(0, 1-\varepsilon)$.

(10) $p_5(0) = 6480(1-\varepsilon^2)(224\varepsilon^{10} - 2349\varepsilon^8 + 10366\varepsilon^6 - 23022\varepsilon^4 + 24930\varepsilon^2 - 10329) < 6480(1-\varepsilon^2) \cdot (-9428.539309) < 0$ and $p_5(1-\varepsilon) = -2880\varepsilon^7(\varepsilon+2)(\varepsilon-1)^2(83\varepsilon^6 - 498\varepsilon^4 + 408\varepsilon^3 + 747\varepsilon^2 - 1224\varepsilon + 583) < -2880\varepsilon^7(\varepsilon+2)(\varepsilon-1)^2 \cdot 349.7910014 < 0$. Combining them with (9), we obtain that $p_5(u)$ has at most two zeros on $(0, 1-\varepsilon)$.

(11) $p_4(0) = 216(\varepsilon-1)^2(\varepsilon+1)^2(1400\varepsilon^8 - 11350\varepsilon^6 + 36341\varepsilon^4 - 51960\varepsilon^2 + 26985) > 216(\varepsilon-1)^2(\varepsilon+1)^2 \cdot 25108.71003 > 0$ and $p_4(1-\varepsilon) = 64\varepsilon^8(\varepsilon+2)(\varepsilon-1)^2(865\varepsilon^6 - 5190\varepsilon^4 + 4255\varepsilon^3 + 7785\varepsilon^2 - 12765\varepsilon + 5347) > 64\varepsilon^8(\varepsilon+2)(\varepsilon-1)^2 \cdot 2914.886340 > 0$. Combining them with (10), we obtain that $p_4(u)$ has at most two zeros on $(0, 1-\varepsilon)$.

(12) $p_3(0) = 648(1-\varepsilon)^3(1+\varepsilon)^3(83\varepsilon^6 - 499\varepsilon^4 + 1042\varepsilon^2 - 712) < 648(1-\varepsilon)^3(1+\varepsilon)^3 \cdot (-674.3798952) < 0$ and $p_3(1-\varepsilon) = -32\varepsilon^9(\varepsilon+2)^2(\varepsilon-1)^4(353\varepsilon^3 - 1059\varepsilon + 922) < -32\varepsilon^9(\varepsilon+2)^2(\varepsilon-1)^4 \cdot 720.79 < 0$. Combining them with (11), we obtain that $p_3(u)$ has at most two zeros on $(0, 1-\varepsilon)$.

(13) $p_2(0) = 324(\varepsilon-1)^4(\varepsilon+1)^4(26\varepsilon^4 - 104\varepsilon^2 + 103) > 324(\varepsilon-1)^4(\varepsilon+1)^4 \cdot 99.2456 > 0$ and $p_2(1-\varepsilon) = 144\varepsilon^{10}(\varepsilon+2)^2(\varepsilon-1)^4(14\varepsilon^3 - 42\varepsilon + 31) > 144\varepsilon^{10}(\varepsilon+2)^2(\varepsilon-1)^4 \cdot 23.02 > 0$. Combining them with (12), we obtain that $p_2(u)$ has at most two zeros on $(0, 1-\varepsilon)$.

(14) $p_1(0) = 54(1-\varepsilon)^5(1+\varepsilon)^5(22\varepsilon^2 - 41) < 54(1-\varepsilon)^5(1+\varepsilon)^5 \cdot (-40.2058) < 0$ and $p_1(1-\varepsilon) = -\frac{920}{3}\varepsilon^{11}(\varepsilon+2)^3(\varepsilon-1)^6 < 0$. Combining them with (13), we obtain that $p_1(u)$ has at most two zeros on $(0, 1-\varepsilon)$.

(15) $p(0) = 135(\varepsilon-1)^6(\varepsilon+1)^6 > 0$ and $p(1-\varepsilon) = 40\varepsilon^{12}(\varepsilon+2)^3(\varepsilon-1)^6 > 0$. Combining them with (14), we obtain that $p(u)$ has at most two zeros on $(0, 1-\varepsilon)$. That is, $p(u)$ changes sign at most twice on $(0, 1-\varepsilon)$, which completes the proof. $\square$

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